Codiscrete cofibrations *vs* iterated discrete fibrations for (∞, ℓ)-congruences

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The exactness properties putatively characterising (∞, ℓ)-topoi should be phrased in terms of effectivity of higher congruences. In the 2-dimensional case, it is known that 2-congruences are internal categories whose underlying graph is a discrete two-sided fibration, but for higher values of ℓ this admits two natural generalisations: internal categories whose underlying graph is an $(\ell-2)$-categorical two-sided fibration (recently studied by Loubaton), or internal $(\ell-1)$-categories whose underlying $(\ell-1)$-graph is an iterated discrete fibration.

I will explain how to compare both to a third notion suggested by formal enriched category theory: codiscrete two-sided cofibrations. While $(\ell-2)$-categorical fibrations require lax limits, the iterated discrete fibrations can be studied with very expressive weighted limits: the lost Australian folklore of generalised kernels, taking full advantage of the enrichment over (∞, ℓ)-$\mathfrak{Cat}$. If time allows, I will then use these tools to elucidate the structure of the fibration classifiers.

1. Motivation: exactness properties of allegories

Let $\mathfrak{E}$ be a regular category (for example a pretopos), so that we may form the 2-category $\mathfrak{Rel}(\mathfrak{E})$ of relations in $\mathfrak{E}$. A right-adjoint $\mathscr{P}$ to the natural inclusion $\iota\colon \mathfrak{E} \to \mathfrak{Rel}(\mathfrak{E})$ satisfies

$$\mathfrak{E}(E, \mathscr{P}F) \simeq \mathfrak{Rel}(\mathfrak{E})(E, F) = \tau_{\leq -1}(\mathfrak{E}_{/E\times F})$$

so that $\mathscr{P}$ exists if and only if $\mathfrak{E}$ admits local subobject classifiers, *i.e.* if and only if it is an elementary topos.

There is even more on topoi we can say from the perspective of their categories of spans[1], in particular on their exactness properties: the congruences (internal equival-

[1] In fact the situation can be axiomatised by studying dagger 2-categories behaving enough like $\mathfrak{Span}(\mathfrak{E})$, called allegories, but I will not pursue the details further here.

ence relations) whose effectivity is at play are exactly the symmetric monads in $\mathfrak{Rel}(\mathfrak{E})$ (and effectivity corresponds to their splitting as idempotent endomorphisms).

For (from now one, always presentable) $(n,1)$-topoi (with $1 \leq n \leq \infty$), we have a similar story where relations—aka (-1)-truncated spans—are replaced by $(n-2)$-truncated spans: an $(n,1)$-topos $\mathfrak{E}$ must have a classifier for $(n-2)$-truncated morphisms (up to size restrictions which I will always leave implicit), and its existence along with local cartesian closedness is equivalent to the existence of a right-adjoint to $\mathfrak{E} \hookrightarrow \mathfrak{Span}^{\tau_{\leq(n-2)}}(\mathfrak{E})$.

In addition, the effectivity condition for $(n,1)$-topoi is the effectivity of "n-efficient" groupoids, groupoid objects whose underlying graph is $(n-2)$-truncated. These can again be seen as symmetric monads in $\mathfrak{Span}^{\tau_{\leq(n-2)}}(\mathfrak{E})$.

Remark 1.1. As a small digression, the role of the category of spans becomes especially enlightening when we fix $n = \infty$. Indeed, an alternate characterisation of $(\infty,1)$-topoi is that they are the presentable $(\infty,1)$-categories in which all colimits are van Kampen. But, as observed by [SH11] in the 1-categorical case, van Kampen colimits are precisely those preserved by the inclusion $\mathfrak{E} \to \mathfrak{Span}(\mathfrak{E})$. Thus $\mathfrak{E}$ is an $(\infty,1)$-topos when $\mathfrak{E} \to \mathfrak{Span}(\mathfrak{E})$ preserves all colimits, which thanks to presentability means it admits a right adjoint (again up to size restrictions, so in fact it only has "enough right adjoints"), explaining the equivalence between the two characterisations.

As we moved from 1-topoi to $(\infty,1)$-topoi, the monomorphism condition gradually weakened to ∞-truncatedness, which is no condition at all. Unfortunately, there is little hope of this pattern carrying over to higher categorical levels. Indeed, the 2-category of categories, which is the prototypical 2-topos, fails to be locally cartesian closed as the only exponentiable morphisms are the Conduché fibrations, among which the Grothendieck fibrations and opfibrations, suggesting that slices should be replaced by fibrational slices (as is also done in the ∞-categorical setting by [AM24]). Likewise, as integrated by [Web07], we cannot expect a classifier for arbitrary morphisms, but only for discrete[2] fibrations and opfibrations.

The importance of discrete fibrations for 2-topos theory was also recognised by Street's work [Str82] on the 2-categorical Giraud theorem, as furnishing the right categorification of congruences.

Definition 1.2. *A 2-**congruence**, or **catead**, in an $(\infty,2)$-category $\mathfrak{K}$, is an internal category $X_\bullet\colon \Delta^{op} \to \mathfrak{K}$ such that the span $X_0 \xleftarrow{d_1} X_1 \xrightarrow{d_0} X_0$ is a two-sided discrete fibration.*

The prototypical example of a catead is the **simplicial kernel** of an arrow $f\colon A \to B$, which is defined as the simplicial object

$$\cdots \qquad f \downarrow f \downarrow f \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; f \downarrow f \;\substack{\longrightarrow\\ \longrightarrow}\; A$$

where $f \downarrow \cdots \downarrow f$ is the iterated comma $(f \downarrow f) \times_A \cdots \times_A (f \downarrow f)$. The functor $\ker\colon \mathfrak{K}^2 \to \mathfrak{K}^{\Delta^{op}}$ has a right adjoint computing the 2-categorical quotient of a simplicial object, known as a **codescent object** from [Str04], and which can be characterised

[2] I follow the terminology of Riehl–Verity, in which "discrete" does not mean truncated, but 0-categorical, so in the homotopical setting $(\infty,0)$-categorical *i.e.* groupoidal.

by an explicit (but complicated) universal property or defined as a weighted colimit as we will see later.

Having these definitions, we can say that an $(\infty, 2)$-category is **effective** if every catead is the simplicial kernel of its codescent object[3].

If we now wish to consider the matter for (∞, ℓ)-categories, with now $\ell \in \overline{\mathbb{N}}$ greater than 2, there are two natural further categorifications of the notion of congruence:

- In keeping with the pattern of the "$(n-2)$ dimensionality condition" for $(n, 1)$-topoi, we may define an **ℓ-congruence** to be an internal category whose underlying graph is an $(\ell-2)$-categorical two-sided fibration.
- Making instead the *shape* higher-categorical, and from the observation from [Ker24] that internal $(\ell-1)$-categories[4] are (globular) monads in $(\ell-1)$-iterated spans, we may define an **$(\ell-1)$-catead** to be an internal $(\ell-1)$-category $X_\bullet \colon \Theta_{\ell-1} \to \mathfrak{K}$ whose underlying $(\ell-1)$-graph is an iterated discrete two-sided fibration.

For defining effectivity of ℓ-congruences, the same approach as for $\ell = 2$ can be used, replacing the limits and colimits by their lax (that is, Gray-enriched) variants, as done in [Mes24] for $\ell = 3$ and in [Lou25] for all values of ℓ. For $(\ell-1)$-cateads, we can remain in the realm of strong limits by making a good use of $(\infty, \ell-1)$-$\mathfrak{Cat}$-weighted limits.

2. Cellular kernels and codescent objects

Let $\mathfrak{V}$ be a closed monoidal $(\infty, 1)$-category with limits and colimits. (In practice we will only use today the case $\mathfrak{V} = (\infty, \ell-1)$-$\mathfrak{Cat}$ with its cartesian monoidal structure, although the case of the Gray tensor product can be useful to recover the setting of Loubaton's ℓ-congruences.)

Recall that if $\mathfrak{V}$ is semi-cartesian, a (conical) limit $\lim \mathscr{D}$ of a $\mathfrak{V}$-functor $\mathscr{D} \colon I \to \mathfrak{C}$ is determined by the requirement that its universal cone $\mathscr{const}\,\mathbb{1} \Rightarrow \mathfrak{C}(\lim \mathscr{D}, \mathscr{D}-)$ induce an equivalence $\mathfrak{C}(C, \lim \mathscr{D}) \simeq \mathfrak{V}^I(\mathscr{const}\,\mathbb{1}, \mathfrak{C}(C, \mathscr{D}))$. The idea of weighted limits is to replace the presehaf $\mathscr{const}\,\mathbb{1}$ by a shape to be designed to the user's specifications.

Definition 2.1. *Let $\mathscr{D} \colon I \to \mathfrak{C}$ a $\mathfrak{V}$-functor.*

*A **limit** of $\mathscr{D}$ weighted by a $\mathfrak{V}$-copresheaf $\mathscr{W} \colon I \to \mathfrak{V}$ is an object $\{\mathscr{W}, \mathscr{D}\}$ equiped with a universal $\mathscr{W}$-weighted cone $\mathscr{W} \Rightarrow \mathfrak{C}(\{\mathscr{W}, \mathscr{D}\}, \mathscr{D}-)$,* i.e. *inducing an equivalence*

$$\mathfrak{C}(C, \{\mathscr{W}, \mathscr{D}\}) \simeq \mathfrak{V}^I(\mathscr{W}, \mathfrak{C}(C, \mathscr{D})). \tag{1}$$

*A **colimit** of $\mathscr{D}$ weighted by a $\mathfrak{V}$-presheaf $\mathscr{P} \colon I^{op} \to \mathfrak{V}$ is an object $\mathscr{P} \star \mathscr{D}$ equipped with a universal $\mathscr{P}$-weighted cocone,* i.e. *inducing an equivalence*

$$\mathfrak{C}(\mathscr{P} \star \mathscr{D}) \simeq \mathfrak{V}^{I^{op}}(\mathscr{P}, \mathfrak{C}(\mathscr{D}, C)). \tag{2}$$

[3] The definition of regularity is a bit subtler, *cf.* [BG14], and I will focus only on effectivity.

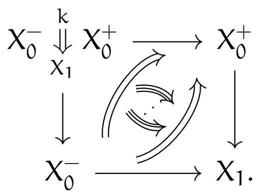
[4] We formally define internal n-categories as functors from the opposite of Joyal's cell category Θ_n of pasting diagrams satisfying the Segal and univalence-completeness condtions.

Example 2.2. In the classical enrichment over $\mathcal{V} = \mathfrak{Set}$, take $\mathbb{I} = \mathbb{D}_1$ the walking arrow and $\mathscr{W}$ the functor with values $(2 \to 1)$. Then a $\mathscr{W}$-weighted limit of a diagram $A \xrightarrow{f} B$ is a kernel pair $\ker(f) = A \times_B A$, while a $\mathscr{W}$-weighted colimit of a parallel pair is its coequaliser.

Example 2.3. Take $\mathcal{V} = \mathfrak{Cat}$ and $\mathbb{I} = \vee$ the walking cospan. Then for the weight $\mathscr{W}$ with values $\mathbb{D}_0 \xrightarrow{\ulcorner 0 \urcorner} \mathbb{D}_1 \xleftarrow{\ulcorner 1 \urcorner} \mathbb{D}_0$, a $\mathscr{W}$-limit of $X_0^- \to X_1 \leftarrow X_0^+$ is a comma

$$\begin{array}{ccc} X_0^- \downarrow_{X_1} X_0^+ & \longrightarrow & X_0^+ \\ \downarrow & \nearrow & \downarrow \\ X_0^- & \longrightarrow & X_1. \end{array} \tag{3}$$

We can more generally define the k-**comma** of $X_0^- \to X_1 \leftarrow X_0^+$ as its limit weighted by $\mathbb{D}_0 \xrightarrow{\ulcorner 0 \urcorner} \mathbb{D}_k \xleftarrow{\ulcorner 1 \urcorner} \mathbb{D}_0$. More explicitly, it is the universal way to complete the cospan to a square filled with a $(1+k)$-cell (and its supporting source and target boundaries) as in



We will now generalise weighted limits further to endow them with better functoriality properties.

Recall that a $\mathcal{V}$-**profunctor** from $\mathbb{I}$ to $\mathfrak{B}$ is a $\mathcal{V}$-functor $\mathfrak{B}^{op} \otimes \mathbb{I} \to \mathcal{V}$. For example, every $\mathcal{V}$-functor $f\colon \mathbb{I} \to \mathfrak{B}$ gives a profunctor $f_* = \mathfrak{B}(\mathrm{id}, f)\colon \mathbb{I} \nrightarrow \mathfrak{B}$ and $f^* = \mathfrak{B}(f, \mathrm{id})\colon \mathfrak{B} \nrightarrow \mathbb{I}$.

Definition 2.4. *A $\mathcal{V}$-profunctor $\mathscr{W}\colon \mathbb{I} \nrightarrow \mathfrak{B}$ can be curried to $\mathfrak{B}^{op} \to \mathcal{V}^{\mathbb{I}}$, and thus seen as a weight in the previous sense parametrised by $\mathfrak{B}^{op}$. A limit of $\mathscr{D}\colon \mathbb{I} \to \mathfrak{C}$ weighted by $\mathscr{W}$ is a $\mathcal{V}$-functor*

$$\mathfrak{B} \xrightarrow{b \mapsto \mathscr{W}(b,-)} \left(\mathcal{V}^{\mathbb{I}}\right)^{op} \xrightarrow{\{-,\mathscr{D}\}} \mathfrak{C}. \tag{4}$$

Remark 2.5. While I have chosen here this presentation for the sake of exposition, limits and colimits weighted by profunctors are more abstractly defined as giving lax extensions and lifts in the proarrow equipment $\mathcal{V}$-$\mathbb{P}\mathrm{rof}$, which thanks to formal nonsense endows them with good functoriality and duality properties. In particular, one can easily obtain the following observation:

Proposition 2.6. *Let $\mathscr{W}\colon \mathbb{I} \nrightarrow \mathfrak{B}$ be a profunctor, and $\mathfrak{C}$ be an object admitting limits and colimits. The functor $\{\mathscr{W}, -\}\colon \mathfrak{C}^{\mathbb{I}} \to \mathfrak{C}^{\mathfrak{B}}$ is right-adjoint to $\mathscr{W} \star -\colon \mathfrak{C}^{\mathfrak{B}} \to \mathfrak{C}^{\mathbb{I}}$.*

Corollary 2.7. *For each* $I \in \mathfrak{I}$*, the "evaluated colimit" functor* $(\mathscr{W} \star -)I\colon \mathfrak{C}^{\mathfrak{B}} \to \mathfrak{C}$ *admits a right adjoint* $\Delta_{\mathscr{W}_I}$ *which sends* $C \in \mathfrak{C}$ *to the functor* $\Delta_{w_I}(C)\colon \mathfrak{B} \to \mathfrak{C}$ *such that, for every* $B \in \mathfrak{B}$*, the object* $\Delta_{\mathscr{W}_I}(C)B$ *of* $\mathfrak{C}$ *is the power* $C^{\mathscr{W}(B,I)}$*.*

The main example of this construction is a general notion of kernel/quotient systems, which is due to [BSS99].

Construction 2.8. Since $\mathfrak{V}$ admits all colimits, free $\mathfrak{V}$-categories generated by any $(\infty, 1)$-categories exist. In particular, we will still denote by $\mathbb{2}$ the $\mathfrak{V}$-category freely generated by the category $\mathbb{2}$, so that the power $\mathfrak{V}^{\mathbb{2}}$ coincides with the $\mathfrak{V}$-category of $\mathfrak{V}$-functors $\mathbb{2} \to \mathfrak{V}$.

Let $\mathfrak{X}$ be a sub-$\mathfrak{V}$-category of $\mathfrak{V}^{\mathbb{2}}$ containing id_I, where I is the monoidal unit of $\mathfrak{V}$. The restricted evaluation $\mathfrak{V}$-functor

$$\mathfrak{X} \times \mathbb{2} \hookrightarrow \mathfrak{V}^{\mathbb{2}} \times \mathbb{2} \to \mathfrak{V} \tag{5}$$

defines a $\mathfrak{V}$-profunctor $\mathscr{E}v\colon \mathbb{2} \nrightarrow \mathfrak{X}^{\mathrm{op}}$.

Definition 2.9. *Let* $\mathfrak{C}$ *be a* $\mathfrak{V}$*-enriched* ∞*-category.*

The $\mathfrak{X}$***-kernel*** *of an arrow* f *of* $\mathfrak{C}$*, given by a* $\mathfrak{V}$*-functor* $\ulcorner f \urcorner\colon \mathbb{2} \to \mathfrak{C}$*, is the weighted limit*

$$\ker_{\mathfrak{X}} f \coloneqq \{\mathscr{E}v, \ulcorner f \urcorner\} : \mathfrak{X}^{\mathrm{op}} \to \mathfrak{C}. \tag{6}$$

The $\mathfrak{X}$***-quotient*** *of a* $\mathfrak{V}$*-diagram* $D\colon \mathfrak{X}^{\mathrm{op}} \to \mathfrak{C}$ *is the weighted colimit*

$$\mathrm{quot}_{\mathfrak{X}} D \coloneqq \mathscr{E}v \star D\colon \mathbb{2} \to \mathfrak{C}. \tag{7}$$

As a direct application of proposition 2.6, if $\mathfrak{C}$ admits all enriched limits and colimits, the $\mathfrak{V}$-functor $\ker_{\mathfrak{X}}$ is right-adjoint to $\mathrm{quot}\colon \mathfrak{C}^{\mathfrak{X}^{\mathrm{op}}} \to \mathfrak{C}^{\mathbb{2}}$.

Example 2.10. The most traditional application of this paradigm is the case of the enrichment in $\mathfrak{V} = (\infty, 1)$-$\mathfrak{Cat}$, so that $\mathfrak{V}$-categories are $(\infty, 2)$-categories. There we start from the eso/ff factorisation system, which is densely generated by the inclusions $\mathfrak{n} = \mathrm{obj}\,\mathbb{m} \to \mathbb{m}$. These then form a category $\mathfrak{X}$ equivalent to $\mathbb{\Delta}$, so that the kernel-quotient adjunction becomes an adjunction $\mathfrak{C}^{\mathbb{\Delta}^{\mathrm{op}}} \rightleftarrows \mathfrak{C}^{\mathbb{2}}$.

By the decomposition $\mathbb{m} \simeq \mathbb{2} \amalg_{\mathbb{1}} \cdots \amalg_{\mathbb{1}} \mathbb{2}$ and the fact that taking weighted limits preserves limits, we recover the earlier adjunction of simplicial kernel and codescent objects.

Example 2.11. Applying to the basic enrichment in $\mathfrak{V} = \infty$-$\mathfrak{Grpd}$, so that $\mathfrak{V}$-categories are simply $(\infty, 1)$-categories, also produces interesting results. We pick here the epi/mono factorisation system, which can be seen as induced from the previous bo/ff factorisation system under the inclusion ∞-$\mathfrak{Grpd} \hookrightarrow (\infty, 1)$-$\mathfrak{Cat}$. In particular, noting that this inclusion has a right-adjoint $\mathfrak{C} \mapsto \mathfrak{C}[\mathrm{all}^{-1}]$ inverting all arrows of an $(\infty, 1)$-category, it is densely generated by the maps $\mathfrak{n} \to \mathbb{m}[\mathrm{all}^{-1}] \simeq \mathbb{1}$, giving a category $\mathfrak{X}$ equivalent to that of finite sets.

Another way of seeing this category is as the crossed simplicial group $\mathbb{\Delta}\mathbb{S}$ obtained by adding the symmetric group $\mathbb{S}_n$ as automorphism group of the object $\mathbb{m}$ of $\mathbb{\Delta}$. We then see that the $\mathfrak{X}$-kernel of an arrow f is its Čech nerve, natural extension of the kernel pair to an internal category, equipped with the symmetric structure permuting the factors, while on the $\mathfrak{X}$-quotient side we see that geometric realisation is naturally defined not for plain simplicial objects but for symmetric simplicial objects.

Following from these examples, we get a natural definition of cellular kernels as ℓ-cateads and their higher codescent objects.

Construction 2.12. The eso/ff factorisation system on $\mathfrak{V} = (\infty, \ell-1)\text{-}\mathfrak{Cat}$ is densely generated by the maps $\mathrm{obj}(T) \to T$ for T any object of $\Theta_{\ell-1}$: this gives a shape category equivalent to $\Theta_{\ell-1}$. The $\mathscr{W}$-weighted limit of an arrow $f \colon A \to B$ in an (∞, ℓ)-category, called its **cellular kernel**, is then the cellular object whose value at $\mathfrak{D}_k \in \Theta_{\ell-1}$ is the self-k-comma $f \Downarrow^k f$ (and the value on an arbitrary pasting diagram is computed by expressing it as a amalgamated sum of globes and turning that into a fibre product of commas).

The $\mathscr{W}$-weighted colimit of a cellular object, which we might call its **cellular codescent object**, is more difficult to describe but the intuition is that the weight allows us to understand the objects in "dimension" k as defining k-arrows. In other words, we can think of an $(\ell-1)$-cellular object as prescribing a level-by-level description of the $(\ell-1)$-categorical structure of its codescent object.

We will see later techniques for computing some codescent objects, but let us mention for now a special example that demonstrates the above intuition:

Example 2.13 ([Bou10]). When $\mathfrak{K} = (\infty, \ell-1)\text{-}\mathfrak{Cat}$ and $X_\bullet$ is a complete Segal object all of whose terms are ∞-groupoids, its codescent object is the $(\infty, \ell-1)$-category it defines.

3. Bipartite kernels of codiscrete cofibrations

I have argued that, just as ℓ-congruences are known to, $(\ell-1)$-cateads provide a sensible approach to (∞, ℓ)-categorical exactness. There now remains the questions of how the two relate to each other. We will now see that they are in a sense equivalent.

In order to relate congruences and cateads, it will prove useful to understand them both in relation to a third notion. Indeed, the essential characteristic of the $(\ell-2)$-categorical two-sided fibrations at the heart of ℓ-congruences in $(\infty, \ell-1)\text{-}\mathfrak{Cat}$ is that they encode $(\infty, \ell-1)$-profunctors. This is however very reliant on the enrichment in $(\infty, \ell-2)\text{-}\mathfrak{Cat}$, and for general enrichment in an arbitrary monoidal category $\mathfrak{V}$, [Str80] noticed that $\mathfrak{V}$-profunctors are instead always recovered by the codiscrete cofibrations in the 2-category $\mathfrak{V}\text{-}\mathfrak{Cat}$.

Definition 3.1. *A* ***codiscrete two-sided cofibration*** *in an* $(\infty, 2)$*-category* $\mathfrak{K}$ *is a discrete two-sided fibration in* $\mathfrak{K}^{\mathrm{op}}$*.*

Remark 3.2. As codiscrete cofibrations are a purely 2-categorical notion, we may as well define them in $\mathfrak{K}$ an (∞, ℓ)-category as codiscrete cofibrations in the underlying $(\infty, 2)$-category.

As it turns out, codiscrete cofibrations are a much more rigid kind of object than discrete fibrations.

Proposition 3.3 ([Str80]). *Consider a cospan* $E \to B \leftarrow F$ *in* $\mathfrak{V}$-$\mathfrak{Cat}$.

- *It is codiscrete if and only if the objects of* B *are the disjoint union of those of* E *and* F.
- *It is a two-sided cofibration from* E *to* F *if and only if there are no morphisms from the objects of* E *to those of* F *i* B.

We now have three structures, $(\ell - 2)$-categorical fibrations, codiscrete cofibrations, and iterated discrete fibrations, living in three slightly different yet related shapes: spans, cospans, and iterated spans. To compare them, we will use our weighted limits machinery—in fact a two-sided or "bipartite" version of the cellular kernels/quotients—to obtain adjunctions between these span-shaped categories.

Construction 3.4. Let $\vee_\ell$ be the locally full sub-$(\infty, \ell+1)$-category of ${\mathbb{G}_{/\mathfrak{D}_\ell}}^{\vee}$ (where $\mathbb{G}$ is the category of globes and $\vee$ the walking cospan) spanned by

objects: the cospans

$$\begin{array}{ccc} (\{0^-\} \to \mathfrak{D}_\ell) & & (\{0^+\} \to \mathfrak{D}_\ell) \\ & \searrow \qquad \swarrow & \\ & (T \to \mathfrak{D}_\ell) & \end{array} \tag{8}$$

where T is a globe (so with objects 0^- and 0^+) and $T \to \mathfrak{D}_\ell$ is inert,

morphisms: the component-wise inert transformations.

We see again that it is equivalent to $\mathbb{G}_{/\mathfrak{D}_\ell}$, which as observed in [Str00] corepresents ℓ-iterated spans, that we may interpret as **bipartite ℓ-graphs**.

The restricted evaluation functor $\vee \times \vee_\ell \to \vee \times (\infty, \ell)\text{-}\mathfrak{Cat}^{\vee} \xrightarrow{\text{ev}} (\infty, \ell)\text{-}\mathfrak{Cat}$ defines a profunctor $\mathscr{E}v \colon \vee \nrightarrow \wedge_\ell := {\vee_\ell}^{\text{op}}$.

Definition 3.5. *Let* $\mathfrak{K}$ *be an* $(\infty, \ell+1)$*-category.*

The ***bipartite kernel*** *of a cospan* $X \colon \vee \to \mathfrak{K}$ *is the weighted limit*

$$\overleftrightarrow{\ker} X := \{\mathscr{E}v, X\} \colon \wedge_\ell \to \mathfrak{K}. \tag{9}$$

The ***bipartite quotient*** *of a bipartite* ℓ*-graph* $G \colon \wedge_\ell \to \mathfrak{K}$ *is the weighted colimit*

$$\overleftrightarrow{\text{quot}}\, G := \mathscr{E}v \star G \colon \vee \to \mathfrak{K}. \tag{10}$$

Remark 3.6. The bipartite kernels are very easy to describe: for any cospan $X_0^- \xrightarrow{f} X_1 \xleftarrow{g} X_0^+$, the value of $\overleftrightarrow{\ker}(f, g)$ at $(0^\pm \to \mathfrak{D}_k)$ is $f \overset{k}{\Downarrow} g$, so that we can think of $\overleftrightarrow{\ker}(f, g)$ as being simply $f \overset{\bullet}{\Downarrow} g$.

In particular, two-sided kernels are always iterated discrete fibrations. The two-sided quotients are a more complicated combination of higher cocommas, but are still codiscrete cofibrations.

Without any additional effort, we immediately obtain:

Corollary 3.7. *There is a span of adjunctions*

$$\mathfrak{Span}(\mathfrak{K}) \underset{-\downarrow^{\mathrm{lax}}-}{\overset{-\uparrow^{\mathrm{lax}}-}{\rightleftarrows}} \mathfrak{Cospan}(\mathfrak{K}) \underset{\overleftrightarrow{\mathrm{ker}}}{\overset{\overleftrightarrow{\mathrm{quot}}}{\leftrightarrows}} \mathfrak{Span}_{\ell-1}(\mathfrak{K}) \tag{11}$$

I will start on the lax side since, due to lack of skill with lax limits, there I can only conjecture the equivalence:

Conjecture 3.8. *For $\mathfrak{K}$ nice enough (*i.e. *an (∞,ℓ)-topos, at least a category of internal $(\infty,\ell-1)$-categories in an $(\infty,1)$-topos), the adjunction on the LHS is idempotent, and so restricts to an equivalence between $(\ell-2)$-categorical two-sided fibrations and codiscrete two-sided cofibrations.*

While I cannot provide the proof myself, a good pointer is that I expect it to be simply a two-sided (or "bipartite") adaptation of the comparison between higher congruences and filtrations of [Lou25].

Now on the side of strong limits, I can actually state the result (expecting it to generalise to the same class of (∞,ℓ)-categories).

Theorem 3.9 (in progress). *For $\mathfrak{K} = (\infty,\ell-1)$-$\mathfrak{Cat}$, the adjunction $\overleftrightarrow{\mathrm{quot}} \dashv \overleftrightarrow{\mathrm{ker}}$ is idempotent, and so restricts to an equivalence between iterated discrete two-sided fibrations and codiscrete two-sided cofibrations.*

Idea of proof. The proof is essentially a two-sided (and ℓ-categorical) version of the effectivity of 1-cateads in [Bou10]: its main ingredient is the computation of the two-sided quotient of an iterated span which is an iterated discrete fibration, generalising example 2.13.

More precisely, I claim that if $X_\bullet$ is an iterated discrete two-sided fibration in $\mathfrak{K} = (\infty,\ell-1)$-$\mathfrak{Cat}$, then the apex $\overleftrightarrow{\mathrm{quot}}(X_\bullet)_1$ of the cospan $\overleftrightarrow{\mathrm{quot}}(X_\bullet)$ is the "horizontal" $(\infty,\ell-1)$-category of the double $(\infty,\ell-1)$-category freely generated by $X_\bullet$ seen as a double $(\ell-1)$-graph.

Indeed, the right-adjoint to $\overleftrightarrow{\mathrm{quot}}(-)_1$ described by corollary 2.7 can be seen to decompose as

$$\mathfrak{K} \xrightarrow{\mathcal{S}q} (\ell-1)\text{-}\mathfrak{Cat}(\mathfrak{K}) \to (\ell-1)\text{-}\mathfrak{Grph}(\mathfrak{K}) \to (\ell-1)\text{-}\mathfrak{BipartGrph}(\mathfrak{K}) \supset \mathfrak{DiscFib}_{\ell-1}(\mathfrak{K})$$

where $\mathcal{S}q$ is the ℓ-categorical version of the square (or quintet) construction, the next two arrows are forgetful functors, and the last (backwards) inclusion is the fact that, when starting from a codiscrete cofibration, this composite produces an iterated discrete fibration.

The two forgetful functors admit obvious left-adjoints, but the square functor does not admit a left-adjoint in general. However, when astricting it to the double $(\infty,\ell-1)$-categories that are $(\ell-1)$-cateads, the construction of horizontal categories does provide a left-adjoint. □

A. Epilogue: Globular fibration classifiers

If we argue that iterated fibrations should take the role of relations for higher exact categories, we also expect they should have a classifier in higher topoi.

For $(\infty, 2)$-topoi (where 2-congruences and 1-cateads), [Web07] introduced the natural categorification of the subobjects classifiers.

Definition A.1 (Weber). *A **classifying discrete opfibration** in an $(\infty, 2)$-category $\mathfrak{K}$ is a discrete opfibration $\Omega_\star \to \Omega$ such that for any object $\mathsf{K} \in \mathfrak{K}$, the functor*

$$\mathfrak{K}(\mathsf{K}, \Omega) \to \mathfrak{DiscOpFib}(\mathsf{K})$$

given by pulling back $\Omega_\star \to \Omega$ to a discrete opfibration over K is fully faithful.

Remark A.2. The fully faithful requirement, rather than an equivalence, is simply due to size issues preventing all fibrations from being classified. We can call the ones in the image of this functor the **classified** discrete opfibrations (thought of as the ones with small enough fibres), and so for every classified discrete opfibration $p \colon \mathsf{E} \to \mathsf{K}$ there is an essentially unique $\ulcorner p \urcorner \colon \mathsf{K} \to \Omega$ such that $\mathsf{E} \simeq \mathsf{K} \times_\Omega \Omega_\star$.

We can now simply mimic this definition with iterated fibrations (recalling that an opfibration over K is a two-sided fibration from $\mathbb{1}$ to K).

Definition A.3. *A **classifying iterated discrete opfibration** in an (∞, ℓ)-category $\mathfrak{K}$ is an iterated discrete two-sided fibration $\Omega_{\star,\bullet} \to \mathbb{1} \times \Omega$ such that for any classified iterated discrete fibration $\mathsf{E}_\bullet \to \mathbb{1} \times \mathsf{K}$, there is an essentially unique morphism $\mathsf{K} \to \Omega$ such that E is the pullback of $\Omega_\star$:*

$$\begin{array}{ccccc} \mathsf{E}_\bullet & \longrightarrow & \Omega_{\star,\bullet} & \longrightarrow & \mathbb{1} \\ \downarrow & \lrcorner & \downarrow & & \\ \mathsf{K} & \overset{\exists!}{\dashrightarrow} & \Omega & & \end{array}$$

However, $\Omega_{\star,\bullet}$ is *a priori* a complicated object, which we can only hope to understand better. For this, we will rely on a refinement of the fibration classifier, due to [Mes25].

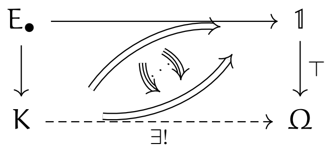
Definition A.4 (Mesiti). *A **good fibration classifier** in an $(\infty, 2)$-category $\mathfrak{K}$ is an object Ω equipped with a pointing $\top \colon \mathbb{1} \to \Omega$ such that for any "classified" discrete opfibration $p \colon \mathsf{E} \to \mathsf{K}$, there is an essentially unique morphism $\ulcorner p \urcorner \colon \mathsf{K} \to \Omega$ such that $\mathsf{E} \simeq \ulcorner p \urcorner \downarrow \top$.*

As observed by [Mes25], the pullback gluing property of comma squares implies that every good fibration classifier is in particular a fibration classifier, with classifying fibration $\Omega_\star = \mathrm{id}_\Omega \downarrow \top$.

We now generalise to iterated fibrations using higher commas.

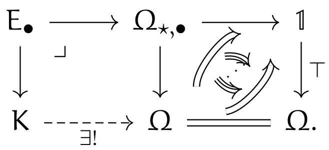
Definition A.5. *A **good iterated fibration classifier** in an (∞, ℓ)-category $\mathfrak{K}$ is an object Ω equipped with a pointing $\top \colon \mathbb{1} \to \Omega$ such that for any "classified" iterated discrete fibration*

$\mathsf{E}_\bullet \to \mathbb{1} \times \mathsf{K}$ *there is an essentially unique morphism* $\mathsf{K} \to \Omega$ *making* $\mathsf{E}_\bullet$ *fit in the higher comma square*



as the two-sided kernel $\mathsf{E}_\bullet = \mathsf{K} \overset{\bullet}{\underset{\Omega}{\Downarrow}} \mathbb{1}$.

As before, we find by gluing of higher commas that a good iterated fibration classifier Ω admits a classifying iterated fibration $\Omega_{\star,\bullet} = \mathrm{id}_\Omega \overset{\bullet}{\Downarrow} \top$:



Corollary A.6. *If an iterated fibration classifier* Ω *refines to a good iterated fibration classifier, its classifying iterated fibration* $\Omega_{\star,\bullet}$ *admits an explicit description as "higher-pointed objects" (where a* k***-pointing*** *means a* k*-arrow between two pointings) of* Ω.

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