Brane actions for operads of (stacky) curves

David Kern

Institut Montpelliérain Alexandre Grothendieck

1st March 2023
Homotopical Methods in Algebraic Geometry at IHP

(Categorified) Gromov–Witten classes

$$\overline{\mathcal{M}}_{g,n+1} \xleftarrow{\text{Stab}} \coprod_{\beta \in \mathrm{Eff}(X)} \overline{\mathcal{M}}_{g,n+1}(X,\beta) \xrightarrow{(\mathrm{ev}_1,\dots,\mathrm{ev}_{n+1})} X^{n+1}$$

$$G_0\overline{\mathcal{M}}_{g,n+1} \xleftarrow{\text{Stab}_*} G_0\left(\coprod_{\beta} \overline{\mathcal{M}}_{g,n+1}(X,\beta)\right) \xleftarrow{(\mathrm{ev}_1,\dots,\mathrm{ev}_{n+1})^*} (G_0 X)^{\otimes n+1}$$

$$\circlearrowright \otimes \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,n+1}(X)}\right]$$

(Categorified) Gromov–Witten classes

$$\overline{\mathcal{M}}_{g,n+1} \xleftarrow{\text{Stab}} \coprod_{\beta \in \text{Eff}(X)} \overline{\mathcal{M}}_{g,n+1}(X,\beta) \xrightarrow{(\text{ev}_1,\dots,\text{ev}_{n+1})} X^{n+1}$$

$$G_0\overline{\mathcal{M}}_{g,n+1} \xleftarrow{\text{Stab}_*} G_0\left(\coprod_\beta \overline{\mathcal{M}}_{g,n+1}(X,\beta)\right) \xleftarrow{(\text{ev}_1,\dots,\text{ev}_{n+1})^*} (G_0X)^{\otimes n+1}$$

$$\circlearrowleft \otimes \left[\mathcal{O}^{\text{vir}}_{\overline{\mathcal{M}}_{g,n+1}(X)}\right]$$

Lemma [Schürg–Toën–Vezzosi, Mann–Robalo]

$\left[\mathcal{O}^{\text{vir}}_{\overline{\mathcal{M}}_{g,n+1}(X)}\right]$ "is" the structure sheaf of derived thickening $\mathbb{R}\overline{\mathcal{M}}_{g,n+1}(X)$

(Categorified) Gromov–Witten classes

$$\overline{\mathcal{M}}_{g,n+1} \xleftarrow{\ \mathrm{Stab}\ } \coprod_{\beta \in \mathrm{Eff}(X)} \mathbb{R}\overline{\mathcal{M}}_{g,n+1}(X,\beta) \xrightarrow{(\mathrm{ev}_1,\dots,\mathrm{ev}_{n+1})} X^{n+1}$$

$$\mathfrak{Coh}^{\mathrm{b}}\overline{\mathcal{M}}_{g,n+1} \xleftarrow{\ \mathrm{Stab}_*\ } \mathfrak{Coh}^{\mathrm{b}}\Big(\coprod_\beta \mathbb{R}\overline{\mathcal{M}}_{g,n+1}(X,\beta)\Big) \xleftarrow{(\mathrm{ev}_1,\dots,\mathrm{ev}_{n+1})^*} (\mathfrak{Coh}^{\mathrm{b}} X)^{\otimes n+1}$$

Lemma [Schürg–Toën–Vezzosi, Mann–Robalo]

$\left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,n+1}(X)}\right]$ "is" the structure sheaf of derived thickening $\mathbb{R}\overline{\mathcal{M}}_{g,n+1}(X)$

(Categorified) Gromov–Witten classes

$$\begin{array}{ccccc} & & \coprod_{\beta \in \mathrm{Eff}(X)} \mathbb{R}\overline{\mathscr{M}}_{g,n+1}(X,\beta) & & \\ & \swarrow^{\mathrm{Stab}} & & \searrow^{(\mathrm{ev}_1,\dots,\mathrm{ev}_{n+1})} & \\ \overline{\mathscr{M}}_{g,n+1} & & & & X^{n+1} \end{array}$$

$$\mathfrak{Coh}^{\mathrm{b}}\overline{\mathscr{M}}_{g,n+1} \xleftarrow{\mathrm{Stab}_*} \mathfrak{Coh}^{\mathrm{b}}\left(\coprod_\beta \mathbb{R}\overline{\mathscr{M}}_{g,n+1}(X,\beta)\right) \xleftarrow{(\mathrm{ev}_1,\dots,\mathrm{ev}_{n+1})^*} (\mathfrak{Coh}^{\mathrm{b}} X)^{\otimes n+1}$$

Lemma [Schürg–Toën–Vezzosi, Mann–Robalo]

$\left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathscr{M}}_{g,n+1}(X)}\right]$ “is” the structure sheaf of derived thickening $\mathbb{R}\overline{\mathscr{M}}_{g,n+1}(X)$

Idea: Algebra (in spans) over the operad $\overline{\mathscr{M}}_{0,\bullet+1}$

Case of a stacky target

For $\overline{\mathcal{M}}_{g,n+1}(X)$ to be proper: need stacky curves [Abramovich–Graber–Vistoli]

At a marking locally of the form $\mathrm{Spec}(\kappa[x])/\mu_r$

At a node $\mathrm{Spec}\big(\kappa[x,y]/\langle xy\rangle\big)/\mu_s$ with *balanced* action $(x,y)\mapsto(\zeta\cdot x,\zeta^{-1}\cdot y)$

Evaluation maps

$$\mathrm{ev}_i\colon \mathbb{R}\overline{\mathcal{M}}_{g,n+1}(X)\to$$
$$(C,\Sigma_1,\ldots,\Sigma_{n+1},f)\mapsto f(\Sigma_i)$$

Case of a stacky target

For $\overline{\mathscr{M}}_{g,n+1}(X)$ to be proper: need stacky curves [Abramovich–Graber–Vistoli]

At a marking locally of the form $\mathrm{Spec}(\kappa[x])/\mu_r$

At a node $\mathrm{Spec}\big(\kappa[x,y]/\langle xy\rangle\big)/\mu_s$ with *balanced* action $(x,y)\mapsto(\zeta\cdot x,\zeta^{-1}\cdot y)$

Evaluation maps

$$\mathrm{ev}_i\colon \mathbb{R}\overline{\mathscr{M}}_{g,n+1}(X)\to\{\mu_{r_i}\text{-gerbes in } X\}$$
$$(C,\Sigma_1,\ldots,\Sigma_{n+1},f)\mapsto f(\Sigma_i)$$

Cyclotomic loop stack

$$\overline{\mathscr{L}}_{\mu}X := \coprod_{r\geqslant 1}\mathscr{M}\!or^{\mathrm{rep}}(\mathcal{B}\,\mu_r,X)/\,\mathcal{B}\,\mu_r$$

Case of a stacky target

For $\overline{\mathscr{M}}_{g,n+1}(X)$ to be proper: need stacky curves [Abramovich–Graber–Vistoli]

At a marking locally of the form $\operatorname{Spec}(\kappa[x])/\mu_r$

At a node $\operatorname{Spec}\big(\kappa[x,y]/\langle xy\rangle\big)/\mu_s$ with *balanced* action $(x,y)\mapsto(\zeta\cdot x,\zeta^{-1}\cdot y)$

Evaluation maps

$$\begin{aligned}\mathrm{ev}_i\colon \mathbb{R}\overline{\mathscr{M}}_{g,n+1}(X) &\to \{\mu_{r_i}\text{-gerbes in } X\}\\ (C,\Sigma_1,\ldots,\Sigma_{n+1},f) &\mapsto f(\Sigma_i)\end{aligned}$$

Cyclotomic loop stack

$$\overline{\mathscr{L}}_{\mu}X := \coprod_{r\geqslant 1}\mathscr{M}or^{\mathrm{rep}}(\mathcal{B}\,\mu_r,X)/\mathcal{B}\,\mu_r$$

Remark: $\mathbb{T}_{\mathscr{L}_\mu X,(x,\mathcal{G})}\simeq\Gamma(\mathcal{B}\,\mu_r,\mathbb{T}_{X,x}|_{\mathcal{G}})\simeq(\mathbb{T}_{X,x}|_{\mathcal{G}})^{\mu_r}$. Over $\mathbb{Q}$, trivial derived structure

Why care about stacky targets?

- X a stack, $\mathcal{P}_0 \in \mathrm{Pic}(X)$ line bundle: stable locus $X^{\mathcal{P}_0\text{-st}}$ [Heinloth, Halpern-Leistner]
  - $x \in X^{\mathcal{P}_0\text{-st}}$ iff $\mathrm{wt}_{\mathbb{G}_m}(\lambda(0)^*\mathcal{P}) < 0$ for any $\mathbb{A}^1/\mathbb{G}_m \xrightarrow{\lambda} X$ such that $\lambda(1) = x$

Why care about stacky targets?

- X a stack, $\mathcal{P}_0 \in \mathrm{Pic}(X)$ line bundle: stable locus $X^{\mathcal{P}_0\text{-st}}$ [Heinloth, Halpern-Leistner]
  - $x \in X^{\mathcal{P}_0\text{-st}}$ iff $\mathrm{wt}_{\mathbb{G}_m}(\lambda(0)^*\mathcal{P}) < 0$ for any $\mathbb{A}^1/\mathbb{G}_m \xrightarrow{\lambda} X$ such that $\lambda(1) = x$

- $\varepsilon \in \mathbb{Q}_{>0}$, $\mathcal{P} = \mathcal{P}_0 \otimes \varepsilon$: quasi-$\mathcal{P}$-stable maps to $X^{\mathcal{P}\text{-st}} = X^{\mathcal{P}_0\text{-st}}$ [Cheong–Ciocan-Fontanine–Kim–Maulik]
  - Destabilising components traded for basepoints

  Example: $\varepsilon > 2$: usual stable maps

Why care about stacky targets?

- X a stack, $\mathcal{P}_0 \in \mathrm{Pic}(X)$ line bundle: stable locus $X^{\mathcal{P}_0\text{-st}}$ [Heinloth, Halpern-Leistner]
  - $x \in X^{\mathcal{P}_0\text{-st}}$ iff $\mathrm{wt}_{\mathbb{G}_m}(\lambda(0)^*\mathcal{P}) < 0$ for any $\mathbb{A}^1/\mathbb{G}_m \xrightarrow{\lambda} X$ such that $\lambda(1) = x$

- $\varepsilon \in \mathbb{Q}_{>0}$, $\mathcal{P} = \mathcal{P}_0 \otimes \varepsilon$: quasi-$\mathcal{P}$-stable maps to $X^{\mathcal{P}\text{-st}} = X^{\mathcal{P}_0\text{-st}}$ [Cheong–Ciocan-Fontanine–Kim–Maulik]
  - Destabilising components traded for basepoints

  Example: $\varepsilon > 2$: usual stable maps

Parameter space $\mathrm{Pic}(X) \otimes \mathbb{Q}_{>0}$ for the stability condition, with walls-and-chambers structure

$\Longrightarrow$ Wall-crossing formulae between the virtual classes, and the induced CohFTs

Constructing the derived moduli stack of stable maps

$\mathfrak{M}_{g,(r_1,\cdots,r_n)}$ moduli stack of curves with marked gerbes of orders $r_1, \cdots, r_n$ [Olsson, Costello]

Constructing the derived moduli stack of stable maps

$\mathfrak{M}_{g,(r_1,\cdots,r_n)}$ moduli stack of curves with marked gerbes of orders $r_1, \cdots, r_n$ [Olsson, Costello]

Moduli of maps

$\overline{\mathscr{M}}_{g,n}(X,\beta)$ open in $t_0\,\mathscr{M}or_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$

$\Longrightarrow$ [Schürg–Toën–Vezzosi] Lifts uniquely to an open

$$\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta) \subset \mathscr{M}or_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$$

Proposition [Ciocan-Fontanine–Kapranov, Schürg–Toën–Vezzosi]

If X is smooth, $\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)$ is quasi-smooth ($\mathbb{L}_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)}$ perfect Tor-amplitude in $[-1,0]$)

Constructing the derived moduli stack of stable maps

$\mathfrak{M}_{g,(r_1,\cdots,r_n)}$ moduli stack of curves with marked gerbes of orders $r_1, \cdots, r_n$ [Olsson, Costello]

Remark: Universal curve $\mathfrak{C}_{g,(r_1,\cdots,r_n)} \simeq \mathfrak{M}_{g,(r_1,\cdots,r_n,1)} \to \mathfrak{M}_{g,(r_1,\cdots,r_n)}$

Moduli of maps

$\overline{\mathcal{M}}_{g,n}(X,\beta)$ open in $t_0\,\mathcal{M}or_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$

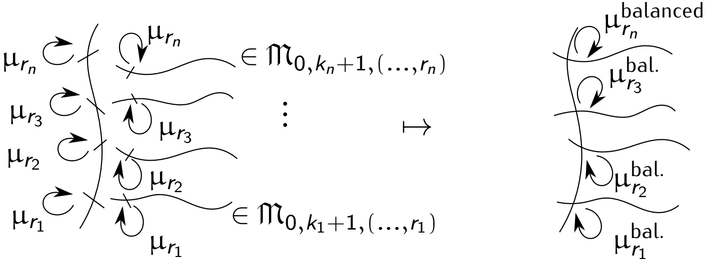
$\Longrightarrow$ [Schürg–Toën–Vezzosi] Lifts uniquely to an open

$$\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta) \subset \mathcal{M}or_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$$

Proposition [Ciocan-Fontanine–Kapranov, Schürg–Toën–Vezzosi]

If X is smooth, $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)$ is quasi-smooth ($\mathbb{L}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)}$ perfect Tor-amplitude in $[-1,0]$)

The operad of stacky curves

$\mathfrak{M}_{0,n,\mathbf{r}} \ni$  $\in \mathfrak{M}_{0,\sum_i k_i,(\dots)}$

Gluing maps

$$\mathfrak{M}_{g,n+1,(r_1,\dots,r_n,s)} \underset{\mathcal{B}^2\mu_s}{\times} \mathfrak{M}_{h,p+1,(s,t_1,\dots,t_p)} \to \mathfrak{M}_{g+h,n+p,(r_1,\dots,r_n,t_1,\dots,t_p)}$$

where $\mathfrak{M}_{g,n+1,(r_1,\dots,r_n,s)} \xrightarrow{\ulcorner \Sigma_{n+1} \urcorner} \mathcal{B}^2\mu_s$ and $\mathfrak{M}_{h,p+1,(s,t_1,\dots,t_p)} \xrightarrow{\ulcorner -T_i \urcorner} \mathcal{B}^2\mu_s$

The operad of stacky curves II

$\mathfrak{M}_{g,n+1,(r_1,\ldots,r_n,s)} \underset{\mathcal{B}^2\mu_s}{\times} \mathfrak{M}_{h,p+1,(s,t_1,\ldots,t_p)} \to \mathfrak{M}_{g+h,n+p,(r_1,\ldots,r_n,t_1,\ldots,t_p)}$ composition law for (modular) operad in stacks $\mathfrak{M} = (\mathfrak{M}_{\star,\bullet+1})$, with stack of colours $\mathcal{B}^2\mu := \coprod_{r\geqslant 1}\mathcal{B}^2\mu_r$

Unitality

$\mathfrak{M}_0(\emptyset; r) := \mathrm{Mult}_{\mathfrak{M}_0}(\emptyset; r) = \mathcal{B}\,\mathcal{B}\,\mu_r$: the nullary morphism has automorphisms $\mathcal{B}\,\mu_r$

The operad of stacky curves II

$\mathfrak{M}_{g,n+1,(r_1,\dots,r_n,s)} \underset{\mathcal{B}^2\mu_s}{\times} \mathfrak{M}_{h,p+1,(s,t_1,\dots,t_p)} \to \mathfrak{M}_{g+h,n+p,(r_1,\dots,r_n,t_1,\dots,t_p)}$ composition law for (modular) operad in stacks $\mathfrak{M} = (\mathfrak{M}_{\star,\bullet+1})$, with stack of colours $\mathcal{B}^2\mu := \coprod_{r\geqslant 1}\mathcal{B}^2\mu_r$

Unitality

$\mathfrak{M}_0(\emptyset; r) := \mathrm{Mult}_{\mathfrak{M}_0}(\emptyset; r) = \mathcal{B}\,\mathcal{B}\,\mu_r$: the nullary morphism has automorphisms $\mathcal{B}\,\mu_r$

$\Longrightarrow$ Only the (schematic) colour 1 is unital: $(\mathfrak{M}_0; 1)$ *hapaxunital* operad

Extensions of $C \in \mathfrak{M}_0(r_1,\dots,r_n; r_{n+1})$

$$\begin{array}{ccc} \mathrm{Ext}(C) & \longrightarrow & \mathfrak{M}_0(r_1,\dots,r_n,1;r_{n+1}) \\ \downarrow & \lrcorner & \downarrow \\ * & \xrightarrow[\ulcorner C \urcorner]{} & \mathfrak{M}_0(r_1,\dots,r_n;r_{n+1}) \end{array}$$

The operad of stacky curves II

$\mathfrak{M}_{g,n+1,(r_1,\dots,r_n,s)} \underset{\mathcal{B}^2\mu_s}{\times} \mathfrak{M}_{h,p+1,(s,t_1,\dots,t_p)} \to \mathfrak{M}_{g+h,n+p,(r_1,\dots,r_n,t_1,\dots,t_p)}$ composition law for (modular) operad in stacks $\mathfrak{M} = (\mathfrak{M}_{\star,\bullet+1})$, with stack of colours $\mathcal{B}^2\mu := \coprod_{r\geqslant 1}\mathcal{B}^2\mu_r$

Unitality

$\mathfrak{M}_0(\emptyset; r) := \mathrm{Mult}_{\mathfrak{M}_0}(\emptyset; r) = \mathcal{B}\mathcal{B}\mu_r$: the nullary morphism has automorphisms $\mathcal{B}\mu_r$

$\Longrightarrow$ Only the (schematic) colour 1 is unital: $(\mathfrak{M}_0; 1)$ *hapaxunital* operad

Extensions of $C \in \mathfrak{M}_0(r_1,\dots,r_n; r_{n+1})$

$$\begin{array}{ccc} \mathrm{Ext}(C) & \longrightarrow & \mathfrak{M}_0(r_1,\dots,r_n,1;r_{n+1}) \\ \downarrow & \lrcorner & \downarrow \\ * & \xrightarrow[\ulcorner C \urcorner]{} & \mathfrak{M}_0(r_1,\dots,r_n;r_{n+1}) \end{array}$$

Geometry

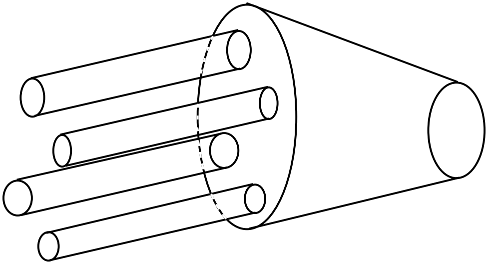
$$\mathrm{Ext}(C) \simeq \mathfrak{C}_{0,(r_1,\dots,r_n,r_{n+1})} \underset{\mathfrak{M}_{0,(r_1,\dots,r_n,r_{n+1})}}{\times} \{C\}$$

$$\simeq C$$

in $\mathbb{R}\overline{\mathcal{M}}_{0,n+1}(X,\beta)_C = \mathcal{M}or(\mathrm{Ext}(C), X)$

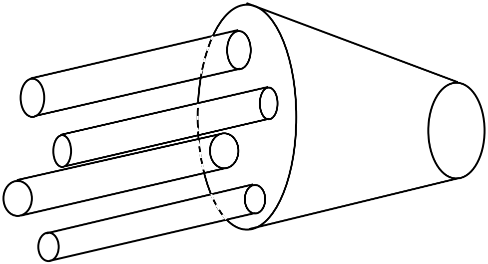
Brane action for little disks

[Chas–Sullivan] Loop product: $\mathscr{E}_2$-algebra structure on $\mathrm{H}_\bullet(LX)$, $LX = \mathscr{M}or(S^1, X)$



Brane action for little disks

[Chas–Sullivan] Loop product: $\mathscr{E}_2$–algebra structure on $\mathrm{H}_\bullet(LX)$, $LX = \mathscr{M}or(S^1, X)$



In $(\mathscr{E}_2, *)$, for $\sigma \in \mathscr{E}_2(n)$: $\mathrm{Ext}(\sigma) \simeq \bigvee^n S^1$. In particular, $\mathrm{Ext}(\mathrm{id}) \simeq S^1$

$$\coprod^n \mathrm{Ext}(\mathrm{id}) \longrightarrow \mathrm{Ext}(\sigma) \longleftarrow \mathrm{Ext}(\mathrm{id})$$

Brane actions for hapaxunital operads

Theorem [Toën, Mann–Robalo, K., Pourcelot]

Let $(\mathbb{O}, O_0)$ be a hapaxunital ∞-operad in an $(\infty,1)$-topos $\mathfrak{T}$. There is a lax morphism of internal $(\infty,2)$-operads

$$\begin{aligned} \mathbb{O} &\xrightarrow{\mathscr{B}_{\mathbb{O}}} \mathscr{C}ospan(\mathfrak{T}_{/-})^{\amalg} \\ C &\mapsto \mathrm{Ext}(\mathrm{id}_C) \end{aligned} \quad \text{inducing for each } X \quad \begin{aligned} \mathbb{O} &\xrightarrow{\mathscr{B}_{\mathbb{O},X}} \mathscr{S}pan(\mathfrak{T}_{/-})^{\times} \\ C &\mapsto \mathscr{M}or(\mathrm{Ext}(\mathrm{id}_C), X) \end{aligned}$$

The action of $\sigma \in \mathbb{O}(C_1, \ldots, C_n; C_{n+1})$ is given by

$$\coprod_{i=1}^n \mathrm{Ext}(\mathrm{id}_{C_i}) \xrightarrow{(\sigma \circ_i -)_{i=1}^n} \mathrm{Ext}(\sigma) \xleftarrow{- \circ_1 \sigma} \mathrm{Ext}(\mathrm{id}_{C_{n+1}})$$

Sketch of construction [Mann–Robalo]

By descent: construct for $\mathfrak{T} = \infty\text{-}\mathfrak{Grpd}$

1. $\mathcal{O} \to \mathscr{Cospan}(\infty\text{-}\mathfrak{Grpd})^{\amalg}$ of $(\infty, 2)$-operads
   $\iff \mathscr{Env}(\mathcal{O}) \to \mathscr{Cospan}(\infty\text{-}\mathfrak{Grpd})^{\amalg}$ of monoidal $(\infty, 2)$-categories

Sketch of construction [Mann–Robalo]

By descent: construct for $\mathfrak{T} = \infty\text{-}\mathfrak{Grpd}$

1. $\mathfrak{O} \to \mathscr{Cospan}(\infty\text{-}\mathfrak{Grpd})^{\amalg}$ of $(\infty, 2)$-operads
$\iff \mathscr{Env}(\mathfrak{O}) \to \mathscr{Cospan}(\infty\text{-}\mathfrak{Grpd})^{\amalg}$ of monoidal $(\infty, 2)$-categories

2. $\mathscr{Env}(\mathfrak{O}) \to \mathscr{Cospan}(\infty\text{-}\mathfrak{Grpd})^{\amalg} \iff \mathfrak{Tw}(\mathscr{Env}(\mathfrak{O})) \to \infty\text{-}\mathfrak{Grpd}^{\mathrm{op}\,\amalg}$ [Barwick]

Sketch of construction [Mann–Robalo]

By descent: construct for $\mathfrak{T} = \infty\text{-}\mathfrak{Grpd}$

1. $\mathfrak{O} \to \mathscr{Cospan}(\infty\text{-}\mathfrak{Grpd})^{\amalg}$ of $(\infty, 2)$-operads
   $\iff$ $\mathscr{Env}(\mathfrak{O}) \to \mathscr{Cospan}(\infty\text{-}\mathfrak{Grpd})^{\amalg}$ of monoidal $(\infty, 2)$-categories

2. $\mathscr{Env}(\mathfrak{O}) \to \mathscr{Cospan}(\infty\text{-}\mathfrak{Grpd})^{\amalg} \iff \mathfrak{Tw}(\mathscr{Env}(\mathfrak{O})) \to \infty\text{-}\mathfrak{Grpd}^{\mathrm{op}\,\amalg}$ [Barwick]

3. $\mathfrak{Tw}(\mathscr{Env}(\mathfrak{O})) \to \infty\text{-}\mathfrak{Grpd}^{\mathrm{op}\,\amalg}$
   $\iff$ discrete cocartesian fibration of $(\infty, 1)$-operads $\widetilde{\mathfrak{B}(\mathfrak{O})} \to \mathfrak{Tw}(\mathscr{Env}(\mathfrak{O}))^{\mathrm{op}}$

Sketch of construction [Mann–Robalo]

By descent: construct for $\mathfrak{T} = \infty\text{-}\mathfrak{Grpd}$

1. $\mathfrak{O} \to \mathscr{C}\!\mathit{ospan}(\infty\text{-}\mathfrak{Grpd})^{\amalg}$ of $(\infty,2)$-operads
   $\iff \mathscr{E}\!\mathit{nv}(\mathfrak{O}) \to \mathscr{C}\!\mathit{ospan}(\infty\text{-}\mathfrak{Grpd})^{\amalg}$ of monoidal $(\infty,2)$-categories

2. $\mathscr{E}\!\mathit{nv}(\mathfrak{O}) \to \mathscr{C}\!\mathit{ospan}(\infty\text{-}\mathfrak{Grpd})^{\amalg} \iff \mathfrak{Tw}(\mathscr{E}\!\mathit{nv}(\mathfrak{O})) \to \infty\text{-}\mathfrak{Grpd}^{\mathrm{op}\,\amalg}$ [Barwick]

3. $\mathfrak{Tw}(\mathscr{E}\!\mathit{nv}(\mathfrak{O})) \to \infty\text{-}\mathfrak{Grpd}^{\mathrm{op}\,\amalg}$
   $\iff$ discrete cocartesian fibration of $(\infty,1)$-operads $\widetilde{\mathfrak{B}(\mathfrak{O})} \to \mathfrak{Tw}(\mathscr{E}\!\mathit{nv}(\mathfrak{O}))^{\mathrm{op}}$

4. $\widetilde{\mathfrak{B}(\mathfrak{O})} \to \mathfrak{Tw}(\mathscr{E}\!\mathit{nv}(\mathfrak{O}))$ encoded by discrete cartesian fibration of $(\infty,1)$-categories
   $\mathfrak{B}(\mathfrak{O}) \to \mathfrak{Env}(\mathfrak{Tw}(\mathscr{E}\!\mathit{nv}(\mathfrak{O})))$ with weak cartesian structure [Lurie].

Recovering the Gromov–Witten action

Corollary

There is a lax morphism of $(\infty,2)$-operads in $\mathfrak{dSt}$

$$\overline{\mathscr{M}}_0 \xrightarrow{\mathscr{GW}} \mathscr{S}pan(\mathfrak{dSt}_{/\_})^{\times}$$
$$* \mapsto \overline{\mathscr{L}}_{\mu} X^{\mathcal{P}\text{-st}}$$

Recovering the Gromov–Witten action

Corollary

There is a lax morphism of $(\infty,2)$-operads in $\mathfrak{dSt}$
$$\overline{\mathcal{M}}_0 \xrightarrow{\mathscr{GW}} \mathcal{S}pan(\mathfrak{dSt}_{/-})^{\times}$$
$$* \mapsto \overline{\mathscr{L}}_{\mu} X^{\mathcal{P}\text{-st}}$$

Proof.

$$\begin{array}{ccc} \mathfrak{M}_0 & \xrightarrow{\mathscr{B}_{\mathfrak{M},X}} & \mathcal{S}pan(\mathfrak{dSt}_{/-})^{\times} \\ \text{Stab}\downarrow & \overset{\mathscr{GW}}{\dashrightarrow} & \\ \overline{\mathcal{M}}_0 & & \end{array}$$

Construct $\mathscr{GW}$ as oplax extension:
$\mathscr{GW} = \mathrm{Opex}_{\mathrm{Stab}} \mathscr{B}_{\mathfrak{M},X}$

Recovering the Gromov–Witten action

Corollary

There is a lax morphism of $(\infty,2)$-operads in $\mathfrak{dSt}$

$$\overline{\mathscr{M}}_0 \xrightarrow{\mathscr{GW}} \mathscr{Span}(\mathfrak{dSt}_{/\_})^{\times}$$
$$* \mapsto \overline{\mathscr{L}}_{\mu} X^{\mathcal{P}\text{-st}}$$

Proof.

$$\mathscr{GW}(*) = \varinjlim_{\mathrm{Stab}(r)\to *} \mathscr{B}_{\mathfrak{M},X}(r) = \varinjlim_{\coprod_{r\in\mathbb{N}+1} \mathcal{B}(\mathcal{B}\,\mu_r)} \mathscr{Mor}(\mathscr{B}_{\mathfrak{M}}(r), X)$$

Recovering the Gromov–Witten action

Corollary

There is a lax morphism of $(\infty,2)$-operads in $\mathfrak{dSt}$
$$\overline{\mathscr{M}}_0 \xrightarrow{\mathscr{GW}} \mathscr{Span}(\mathfrak{dSt}_{/-})^{\times}$$
$$* \mapsto \overline{\mathscr{L}}_{\mu} X^{\mathcal{P}\text{-st}}$$

Proof.

$$\mathscr{GW}(*) = \varinjlim_{\mathrm{Stab}(r)\to *} \mathscr{B}_{\mathfrak{M},X}(r) = \coprod_{r\in\mathbb{N}+1} \varinjlim_{\mathcal{B}(\mathcal{B}\,\mu_r)} \mathscr{Mor}(\mathscr{B}_{\mathfrak{M}}(r), X)$$

Remark: $\mathscr{B}_{\mathfrak{M}}(r) = \mathrm{Ext}(\mathrm{id}_r) = \mathfrak{M}_{0,(r,r,1)} \underset{\mathfrak{M}_{0,(r,r)}}{\times} \{\mathrm{id}_r\} = * \underset{\mathcal{B}^2\,\mu_r}{\times} * = \Omega\,\mathcal{B}^2\,\mu_r = \mathcal{B}\,\mu_r$

Recovering the Gromov–Witten action

Corollary

There is a lax morphism of $(\infty,2)$-operads in $\mathfrak{dSt}$

$$\begin{aligned} \overline{\mathscr{M}}_0 &\xrightarrow{\mathscr{GW}} \mathscr{Span}(\mathfrak{dSt}_{/-})^{\times} \\ * &\mapsto \overline{\mathscr{L}}_{\mu} X^{\mathcal{P}\text{-st}} \end{aligned}$$

Proof.

$$\mathscr{GW}(*) \simeq \coprod_{r \in \mathbb{N}+1} \varinjlim_{\mathcal{B}(\mathcal{B}\,\mu_r)} \mathscr{Mor}(\mathcal{B}\,\mu_r, X)$$

Remark: $\mathscr{B}_{\mathfrak{M}}(r) = \mathrm{Ext}(\mathrm{id}_r) = \mathfrak{M}_{0,(r,r,1)} \underset{\mathfrak{M}_{0,(r,r)}}{\times} \{\mathrm{id}_r\} = * \underset{\mathcal{B}^2\,\mu_r}{\times} * = \Omega\,\mathcal{B}^2\,\mu_r = \mathcal{B}\,\mu_r$

Recovering the Gromov–Witten action

Corollary

There is a lax morphism of $(\infty,2)$-operads in $\mathfrak{dSt}$ $\quad \overline{\mathscr{M}}_0 \xrightarrow{\mathscr{GW}} \mathscr{Span}(\mathfrak{dSt}_{/-})^{\times}$

$$* \mapsto \overline{\mathscr{L}}_{\mu} X^{\mathcal{P}\text{-st}}$$

Proof.

$$\mathscr{GW}(*) \simeq \coprod_{r\in\mathbb{N}+1} \underset{\mathcal{B}(\mathcal{B}\,\mu_r)}{\underrightarrow{\mathrm{colim}}}\, \mathscr{Mor}(\mathcal{B}\,\mu_r, X) = \coprod_{r\in\mathbb{N}+1} \mathscr{Mor}(\mathcal{B}\,\mu_r, X)/\mathcal{B}\,\mu_r =: \overline{\mathscr{L}}_{\mu} X$$

Remark: $\mathscr{B}_{\mathfrak{M}}(r) = \mathrm{Ext}(\mathrm{id}_r) = \mathfrak{M}_{0,(r,r,1)} \underset{\mathfrak{M}_{0,(r,r)}}{\times} \{\mathrm{id}_r\} = * \underset{\mathcal{B}^2\,\mu_r}{\times} * = \Omega\,\mathcal{B}^2\,\mu_r = \mathcal{B}\,\mu_r$

□