# Categorical spectra as pointed $(\infty, \mathbb{Z})$-categories

David Kern

Kunliga Tekniska Högskolan

TexasTech and Wichita State University Topology and Geometry seminar
19th November 2024

# Contents

# Contents - Section 0: Motivation

# Higher modules

$\Bbbk$ a ring (spectrum)

# Higher modules

$\Bbbk$ a ring (spectrum) categorifies to

$$\rightsquigarrow \mathfrak{Mod}(\Bbbk) = \mathfrak{Mod}_{\mathfrak{Sp}}(\Bbbk) \text{ symmetric monoidal stable } (\infty, 1)\text{-category}$$

# Higher modules

$\Bbbk$ a ring (spectrum) categorifies to

- $\rightsquigarrow$ $\mathfrak{Mod}(\Bbbk) = \mathfrak{Mod}_{\mathfrak{Sp}}(\Bbbk)$ symmetric monoidal stable $(\infty, 1)$-category
- $\rightsquigarrow$ $\mathfrak{Mod}^2(\Bbbk) \coloneqq \mathfrak{Mod}_{\mathfrak{St}_1}(\mathfrak{Mod}(\Bbbk))$ symm. mon. stable $(\infty, 2)$-category
- $\rightsquigarrow$ ...
- $\rightsquigarrow$ $\mathfrak{Mod}^{n+1}(\Bbbk) \coloneqq \mathfrak{Mod}_{\mathfrak{St}_n}(\mathfrak{Mod}^n(\Bbbk))$ symm. mon. stable $(\infty, n+1)$-category
- $\rightsquigarrow$ ...

# Higher modules

$\Bbbk$ a ring (spectrum) categorifies to

- $\rightsquigarrow$ $\mathfrak{Mod}(\Bbbk) = \mathfrak{Mod}_{\mathfrak{Sp}}(\Bbbk)$ symmetric monoidal stable $(\infty, 1)$-category
- $\rightsquigarrow$ $\mathfrak{Mod}^2(\Bbbk) \coloneqq \mathfrak{Mod}_{\mathfrak{St}_1}(\mathfrak{Mod}(\Bbbk))$ symm. mon. stable $(\infty, 2)$-category
- $\rightsquigarrow$ ...
- $\rightsquigarrow$ $\mathfrak{Mod}^{n+1}(\Bbbk) \coloneqq \mathfrak{Mod}_{\mathfrak{St}_n}(\mathfrak{Mod}^n(\Bbbk))$ symm. mon. stable $(\infty, n+1)$-category
- $\rightsquigarrow$ ...

**Stabilisation phenomenon [Stefanich]**

$\mathfrak{Mod}^{n-1}(\Bbbk)$ is the unit of $\mathfrak{Mod}^n(\Bbbk)$, and $\mathrm{End}_{\mathfrak{Mod}^n(\Bbbk)}\big(\mathfrak{Mod}^{n-1}(\Bbbk)\big) \simeq \mathfrak{Mod}^{n-1}(\Bbbk)$

(Convention: $\mathfrak{Mod}^0(\Bbbk) \coloneqq \Bbbk$, and $\mathfrak{St}_0 = \mathfrak{Sp}$)

# Higher modules

$\Bbbk$ a ring (spectrum) categorifies to

- $\rightsquigarrow$ $\mathfrak{Mod}(\Bbbk) = \mathfrak{Mod}_{\mathfrak{Sp}}(\Bbbk)$ symmetric monoidal stable $(\infty, 1)$-category
- $\rightsquigarrow$ $\mathfrak{Mod}^2(\Bbbk) \coloneqq \mathfrak{Mod}_{\mathfrak{St}_1}(\mathfrak{Mod}(\Bbbk))$ symm. mon. stable $(\infty, 2)$-category
- $\rightsquigarrow$ ...
- $\rightsquigarrow$ $\mathfrak{Mod}^{n+1}(\Bbbk) \coloneqq \mathfrak{Mod}_{\mathfrak{St}_n}(\mathfrak{Mod}^n(\Bbbk))$ symm. mon. stable $(\infty, n+1)$-category
- $\rightsquigarrow$ ...

**Stabilisation phenomenon [Stefanich]**

$\mathfrak{Mod}^{n-1}(\Bbbk)$ is the unit of $\mathfrak{Mod}^n(\Bbbk)$, and $\mathrm{End}_{\mathfrak{Mod}^n(\Bbbk)}\big(\mathfrak{Mod}^{n-1}(\Bbbk)\big) \simeq \mathfrak{Mod}^{n-1}(\Bbbk)$
(Convention: $\mathfrak{Mod}^0(\Bbbk) \coloneqq \Bbbk$, and $\mathfrak{St}_0 = \mathfrak{Sp}$)

What happens as $n \to \infty$?

# First, what are higher categories again?

**Definition**

An $(\infty, 0)$-category is an $\infty$-groupoid (aka space, anima, ...)
An $(\infty, n+1)$-category is an $(\infty, n)$-$\mathfrak{Cat}$-enriched $\infty$-category.

$\Longrightarrow$ For any $C, D \in \mathfrak{C}$, an $(\infty, n)$-category $\hom_{\mathfrak{C}}(C, D)$ of 1-cells and higher cells between them

# First, what are higher categories again?

**Definition**

An $(\infty, 0)$-category is an $\infty$-groupoid (aka space, anima, ...)
An $(\infty, n+1)$-category is an $(\infty, n)$-$\mathfrak{Cat}$-enriched $\infty$-category.

$\implies$ For any $C, D \in \mathfrak{C}$, an $(\infty, n)$-category $\hom_{\mathfrak{C}}(C, D)$ of 1-cells and higher cells between them

$+$ compositions, *e.g.*

$$\begin{tikzcd} C \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & D \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & E \end{tikzcd} \mapsto \begin{tikzcd} C \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & E \end{tikzcd}$$

# First, what are higher categories again?

**Definition**

An $(\infty, 0)$-category is an $\infty$-groupoid (aka space, anima, ...)
An $(\infty, n+1)$-category is an $(\infty, n)$-$\mathfrak{Cat}$-enriched $\infty$-category.

$\implies$ For any $C, D \in \mathfrak{C}$, an $(\infty, n)$-category $\hom_{\mathfrak{C}}(C, D)$ of 1-cells and higher cells between them

$+$ compositions, *e.g.*

$$\begin{tikzcd} C \arrow[r, bend left=70, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=70, ""{name=B, above}] & D \arrow[r, bend left=50, ""{name=U2, below}] \arrow[r, bend right=50, ""{name=B2, above}] & E \arrow[Rightarrow, from=U, to=M] \arrow[Rightarrow, from=M, to=B] \arrow[Rightarrow, from=U2, to=B2] \end{tikzcd} \mapsto \begin{tikzcd} C \arrow[r, bend left, ""{name=U3, below}] \arrow[r, bend right, ""{name=B3, above}] & E \arrow[Rightarrow, from=U3, to=B3] \end{tikzcd}$$

**“Definition” (Interpreted properly)**

An $(\infty, \omega)$-category is an $(\infty, \omega)$-$\mathfrak{Cat}$-enriched $\infty$-category.

$\implies$ For any $C, D \in \mathfrak{C}$, an $(\infty, \omega)$-category $\hom_{\mathfrak{C}}(C, D)$

# Infinitely iterated modules

- The top-dimensional cells in iterated $\mathbb{k}$-modules are points of $\mathbb{k}$
- Codimension-1 cells are $\mathbb{k}$-modules
- ...
- 0-cells are $\mathfrak{Mod}^{n-1}(\mathbb{k})$-modules

# Infinitely iterated modules

- The top-dimensional cells in iterated $\mathbb{k}$-modules are points of $\mathbb{k}$
- Codimension-1 cells are $\mathbb{k}$-modules
- ...
- 0-cells are $\mathfrak{Mod}^{n-1}(\mathbb{k})$-modules

**Upshot for $n = \omega$**

- We know the "$\infty$-dimensional" (or infinitely shifted) cells
- The 0-cells are "$\mathfrak{Mod}^{\infty}(\mathbb{k})$": unclear

# Infinitely iterated modules

- The top-dimensional cells in iterated $\Bbbk$-modules are points of $\Bbbk$
- Codimension-1 cells are $\Bbbk$-modules
- ...
- 0-cells are $\mathfrak{Mod}^{n-1}(\Bbbk)$-modules

**Upshot for $n = \omega$**

- We know the "$\infty$-dimensional" (or infinitely shifted) cells
- The 0-cells are "$\mathfrak{Mod}^{\infty}(\Bbbk)$": unclear

Idea: Put the "top cells" in dimension 0, and the rest in $< 0$ dimensions

$$\implies \text{“}\mathfrak{Mod}^{\infty}(\Bbbk)\text{” is pushed to dimension } -\infty$$

# Delooping $\mathcal{E}_\infty$ (commutative) monoids

The operad $\mathcal{E}_1$ is the associative $\infty$-operad
$\mathcal{E}_n$ is the little $n$-disks operad:

$$\mathcal{E}_n\text{-}\mathfrak{Alg} \simeq \mathcal{E}_1\text{-}\mathfrak{Alg}(\mathcal{E}_{n-1}\text{-}\mathfrak{Alg}) \qquad \text{[Dunn, Lurie]}$$

**Delooping hypothesis [Baez–Shulman, Gepner–Haugseng]**

$\mathcal{E}_n$-monoids are the same as $n$-uply degenerate $(\infty, n)$-categories

# Delooping $\mathcal{E}_\infty$ (commutative) monoids

The operad $\mathcal{E}_1$ is the associative $\infty$-operad
$\mathcal{E}_n$ is the little $n$-disks operad:

$$\mathcal{E}_n\text{-}\mathfrak{Alg} \simeq \mathcal{E}_1\text{-}\mathfrak{Alg}(\mathcal{E}_{n-1}\text{-}\mathfrak{Alg}) \qquad \text{[Dunn, Lurie]}$$

**Delooping hypothesis [Baez–Shulman, Gepner–Haugseng]**

$\mathcal{E}_n$-monoids are the same as $n$-uply degenerate $(\infty, n)$-categories

Commutative $\infty$-operad $\mathcal{E}_\infty$:

$$\mathcal{E}_\infty\text{-}\mathfrak{Alg} \simeq \lim_n \mathcal{E}_n\text{-}\mathfrak{Alg}$$

$\mathcal{E}_\infty$-algebras are “infinitely degenerate” $(\infty, \omega)$-categories?

# Delooping $\mathcal{E}_\infty$ (commutative) monoids

The operad $\mathcal{E}_1$ is the associative $\infty$-operad
$\mathcal{E}_n$ is the little $n$-disks operad:

$$\mathcal{E}_n\text{-}\mathfrak{Alg} \simeq \mathcal{E}_1\text{-}\mathfrak{Alg}(\mathcal{E}_{n-1}\text{-}\mathfrak{Alg}) \qquad \text{[Dunn, Lurie]}$$

**Delooping hypothesis [Baez–Shulman, Gepner–Haugseng]**

$\mathcal{E}_n$-monoids are the same as $n$-uply degenerate $(\infty, n)$-categories

Commutative $\infty$-operad $\mathcal{E}_\infty$:

$$\mathcal{E}_\infty\text{-}\mathfrak{Alg} \simeq \lim_n \mathcal{E}_n\text{-}\mathfrak{Alg}$$

$\mathcal{E}_\infty$-algebras are "infinitely degenerate" $(\infty, \omega)$-categories?

- Have the degeneracies in infinitely many negative dimensions

# Contents – Section 1: Categorical spectra

# Contents – Section 1: Categorical spectra

# Recollections on spectra

Suspension/loop space adjunction
$$\begin{tikzcd} \infty\text{-}\mathfrak{Grpd}_* \arrow[rr, bend left, "\Sigma"] \arrow[rr, phantom, "\perp"] & & \infty\text{-}\mathfrak{Grpd}_* \arrow[ll, bend left, "\Omega"] \end{tikzcd}$$

## Definition

$$\mathfrak{Sp} = \lim\left(\cdots \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_* \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_* \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_*\right)$$

# Recollections on spectra

Suspension/loop space adjunction
$$\begin{tikzcd} \infty\text{-}\mathfrak{Grpd}_* \arrow[r, bend left, "\Sigma"] \arrow[r, phantom, "\perp"] & \infty\text{-}\mathfrak{Grpd}_* \arrow[l, bend left, "\Omega"] \end{tikzcd}$$

## Definition

$$\mathfrak{Sp} = \lim\left(\cdots \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_* \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_* \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_*\right)$$

- Universal way to turn $\Sigma \dashv \Omega$ into an equivalence $\Sigma\colon \mathfrak{Sp} \simeq \mathfrak{Sp}\colon \Omega = \Sigma^{-1}$

# Recollections on spectra

Suspension/loop space adjunction
$$\begin{tikzcd} \infty\text{-}\mathfrak{Grpd}_* \arrow[r, bend left, "\Sigma"] \arrow[r, phantom, "\perp"] & \infty\text{-}\mathfrak{Grpd}_* \arrow[l, bend left, "\Omega"] \end{tikzcd}$$

## Definition

$$\mathfrak{Sp} = \lim\left(\cdots \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_* \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_* \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_*\right)$$

- Universal way to turn $\Sigma \dashv \Omega$ into an equivalence $\Sigma\colon \mathfrak{Sp} \simeq \mathfrak{Sp}\colon \Omega = \Sigma^{-1}$
- Objects $((X_n, x_n) \in \infty\text{-}\mathfrak{Grpd}_*)_{n \geqslant 0}$ with $X_n \xrightarrow{\simeq} \Omega_{x_{n+1}} X_{n+1}$

# Recollections on spectra

Suspension/loop space adjunction
$$\begin{tikzcd} \infty\text{-}\mathfrak{Grpd}_* \arrow[rr, bend left, "\Sigma"] \arrow[rr, phantom, "\perp"] & & \infty\text{-}\mathfrak{Grpd}_* \arrow[ll, bend left, "\Omega"] \end{tikzcd}$$

## Definition

$\mathfrak{Sp} = \lim\left(\cdots \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_* \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_* \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_*\right)$

- Universal way to turn $\Sigma \dashv \Omega$ into an equivalence $\Sigma\colon \mathfrak{Sp} \simeq \mathfrak{Sp}\colon \Omega = \Sigma^{-1}$
- Objects $((X_n, x_n) \in \infty\text{-}\mathfrak{Grpd}_*)_{n \geqslant 0}$ with $X_n \xrightarrow{\simeq} \Omega_{x_{n+1}} X_{n+1}$
- $k$th projection $\Omega^{\infty-k}\colon \mathfrak{Sp} \to \infty\text{-}\mathfrak{Grpd}_*$ such that $\Omega^{\infty-k} \circ \Omega \simeq \Omega^{\infty-k+1}$

# Recollections on spectra

Suspension/loop space adjunction
$$\begin{tikzcd} \infty\text{-}\mathfrak{Grpd}_* \arrow[r, bend left, "\Sigma"] \arrow[r, phantom, "\perp"] & \infty\text{-}\mathfrak{Grpd}_* \arrow[l, bend left, "\Omega"] \end{tikzcd}$$

## Definition

$$\mathfrak{Sp} = \lim\left(\cdots \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_* \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_* \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_*\right)$$

- Universal way to turn $\Sigma \dashv \Omega$ into an equivalence $\Sigma\colon \mathfrak{Sp} \simeq \mathfrak{Sp}\colon \Omega = \Sigma^{-1}$
- Objects $((X_n, x_n) \in \infty\text{-}\mathfrak{Grpd}_*)_{n\geqslant 0}$ with $X_n \xrightarrow{\simeq} \Omega_{x_{n+1}} X_{n+1}$
- $k$th projection $\Omega^{\infty-k}\colon \mathfrak{Sp} \to \infty\text{-}\mathfrak{Grpd}_*$ such that $\Omega^{\infty-k} \circ \Omega \simeq \Omega^{\infty-k+1}$

  $\implies$ For $n < 0$, $\Omega^{\infty-n} = \Omega^\infty \circ \Omega^{-n}$

# Recollections on spectra

Suspension/loop space adjunction
$$\begin{tikzcd} \infty\text{-}\mathfrak{Grpd}_* \arrow[rr, bend left, "\Sigma"] \arrow[rr, phantom, "\perp"] & & \infty\text{-}\mathfrak{Grpd}_* \arrow[ll, bend left, "\Omega"] \end{tikzcd}$$

## Definition

$$\mathfrak{Sp} = \lim\left(\cdots \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_* \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_* \xrightarrow{\Omega} \infty\text{-}\mathfrak{Grpd}_*\right)$$

- Universal way to turn $\Sigma \dashv \Omega$ into an equivalence $\Sigma\colon \mathfrak{Sp} \simeq \mathfrak{Sp}\colon \Omega = \Sigma^{-1}$
- Objects $\left((X_n, x_n) \in \infty\text{-}\mathfrak{Grpd}_*\right)_{n\geqslant 0}$ with $X_n \xrightarrow{\simeq} \Omega_{x_{n+1}} X_{n+1}$
- $k$th projection $\Omega^{\infty-k}\colon \mathfrak{Sp} \to \infty\text{-}\mathfrak{Grpd}_*$ such that $\Omega^{\infty-k} \circ \Omega \simeq \Omega^{\infty-k+1}$

  $\implies$ For $n < 0$, $\Omega^{\infty-n} = \Omega^{\infty} \circ \Omega^{-n}$

  $\implies$ Left-adjoints $\Sigma^{\infty-k} = \Sigma^{-k} \circ \Sigma^{\infty}\colon \infty\text{-}\mathfrak{Grpd}_* \to \mathfrak{Sp}$

# Categorical spectra

$$\text{Adjunction} \quad \begin{tikzcd} (\infty,\omega)\text{-}\mathfrak{Cat}_* \arrow[r, bend left, "\Sigma"] \arrow[r, phantom, "\perp"] & (\infty,\omega)\text{-}\mathfrak{Cat}_* \arrow[l, bend left, "\Omega"] \end{tikzcd} \quad \text{with } \Omega_X\mathfrak{X} = \hom_{\mathfrak{X}}(X,X)$$

## Definition [Stefanich]

$$\mathfrak{CatSp} = \lim\left(\cdots \xrightarrow{\Omega} (\infty,\omega)\text{-}\mathfrak{Cat}_* \xrightarrow{\Omega} (\infty,\omega)\text{-}\mathfrak{Cat}_* \xrightarrow{\Omega} (\infty,\omega)\text{-}\mathfrak{Cat}_*\right)$$

- Universal way to turn $\Sigma \dashv \Omega$ into an equivalence $\Sigma\colon \mathfrak{CatSp} \simeq \mathfrak{CatSp}\colon \Omega = \Sigma^{-1}$

# Categorical spectra

Adjunction
$$\begin{tikzcd} (\infty,\omega)\text{-}\mathfrak{Cat}_* \arrow[r, bend left, "\Sigma"] \arrow[r, phantom, "\perp"] & (\infty,\omega)\text{-}\mathfrak{Cat}_* \arrow[l, bend left, "\Omega"] \end{tikzcd}$$
with $\Omega_X\mathfrak{X} = \hom_{\mathfrak{X}}(X,X)$

**Definition** [Stefanich]

$$\mathfrak{CatSp} = \lim\left(\cdots \xrightarrow{\Omega} (\infty,\omega)\text{-}\mathfrak{Cat}_* \xrightarrow{\Omega} (\infty,\omega)\text{-}\mathfrak{Cat}_* \xrightarrow{\Omega} (\infty,\omega)\text{-}\mathfrak{Cat}_*\right)$$

- Universal way to turn $\Sigma \dashv \Omega$ into an equivalence $\Sigma\colon \mathfrak{CatSp} \simeq \mathfrak{CatSp}\colon \Omega = \Sigma^{-1}$
- Objects $((\mathfrak{X}_n, X_n) \in (\infty,\omega)\text{-}\mathfrak{Cat}_*)_{n\geqslant 0}$ with $\mathfrak{X}_n \xrightarrow{\simeq} \Omega_{X_{n+1}}\mathfrak{X}_{n+1}$

# Categorical spectra

Adjunction
$$\begin{tikzcd} (\infty,\omega)\text{-}\mathfrak{Cat}_* \arrow[r, bend left, "\Sigma"] \arrow[r, phantom, "\perp"] & (\infty,\omega)\text{-}\mathfrak{Cat}_* \arrow[l, bend left, "\Omega"] \end{tikzcd}$$
with $\Omega_X \mathfrak{X} = \hom_{\mathfrak{X}}(X,X)$

**Definition** [Stefanich]

$$\mathfrak{CatSp} = \lim\left(\cdots \xrightarrow{\Omega} (\infty,\omega)\text{-}\mathfrak{Cat}_* \xrightarrow{\Omega} (\infty,\omega)\text{-}\mathfrak{Cat}_* \xrightarrow{\Omega} (\infty,\omega)\text{-}\mathfrak{Cat}_*\right)$$

- Universal way to turn $\Sigma \dashv \Omega$ into an equivalence $\Sigma\colon \mathfrak{CatSp} \simeq \mathfrak{CatSp}\colon \Omega = \Sigma^{-1}$
- Objects $((\mathfrak{X}_n, X_n) \in (\infty,\omega)\text{-}\mathfrak{Cat}_*)_{n\geqslant 0}$ with $\mathfrak{X}_n \xrightarrow{\simeq} \Omega_{X_{n+1}}\mathfrak{X}_{n+1}$
- $k$th projection $\Omega^{\infty-k}\colon \mathfrak{CatSp} \to (\infty,\omega)\text{-}\mathfrak{Cat}_*$ such that $\Omega^{\infty-k} \circ \Omega \simeq \Omega^{\infty-k+1}$

  $\Longrightarrow$ For $n<0$, $\Omega^{\infty-n} = \Omega^\infty \circ \Omega^{-n}$

  $\Longrightarrow$ Left-adjoints $\Sigma^{\infty-k} = \Sigma^{-k} \circ \Sigma^\infty\colon (\infty,\omega)\text{-}\mathfrak{Cat}_* \to \mathfrak{CatSp}$

# Examples

- The sequence $(\mathfrak{Mod}^n(\Bbbk))_{n\geqslant 0}$ with $\Omega_{\mathfrak{Mod}^n(\Bbbk)}\mathfrak{Mod}^{n+1}(\Bbbk) \simeq \mathfrak{Mod}^n(\Bbbk)$ defines a categorical spectrum $\mathfrak{mod}(\Bbbk)$
  - $\Sigma\mathfrak{mod}(\Bbbk) \simeq \mathfrak{mod}(\mathfrak{Mod}(\Bbbk))$

# Examples

- The sequence $(\mathfrak{Mod}^n(\Bbbk))_{n\geqslant 0}$ with $\Omega_{\mathfrak{Mod}^n(\Bbbk)}\mathfrak{Mod}^{n+1}(\Bbbk) \simeq \mathfrak{Mod}^n(\Bbbk)$ defines a categorical spectrum $\mathfrak{mod}(\Bbbk)$
  - $\Sigma\mathfrak{mod}(\Bbbk) \simeq \mathfrak{mod}(\mathfrak{Mod}(\Bbbk))$
- $(\mathfrak{V}, \otimes)$ an $\mathcal{E}_\infty$-monoidal $(\infty, \omega)$-category
  The sequence of deloopings $(\mathscr{B}^n\mathfrak{V})$ defines a categorical spectrum $\mathscr{B}\mathfrak{V}$

# Examples

- The sequence $(\mathfrak{Mod}^n(\Bbbk))_{n\geqslant 0}$ with $\Omega_{\mathfrak{Mod}^n(\Bbbk)}\mathfrak{Mod}^{n+1}(\Bbbk) \simeq \mathfrak{Mod}^n(\Bbbk)$ defines a categorical spectrum $\mathfrak{mod}(\Bbbk)$
  - $\Sigma\mathfrak{mod}(\Bbbk) \simeq \mathfrak{mod}(\mathfrak{Mod}(\Bbbk))$
- $(\mathfrak{V}, \otimes)$ an $\mathcal{E}_\infty$-monoidal $(\infty, \omega)$-category
  The sequence of deloopings $(\mathscr{B}^n\mathfrak{V})$ defines a categorical spectrum $\mathscr{B}\mathfrak{V}$

**Morita categorical spectrum**

$\mathfrak{V}$ symmetric monoidal $(\infty, 1)$-category. $\rightsquigarrow$ Morita $(\infty, n+1)$-category $\mathfrak{Mor}_n(\mathfrak{V})$ with

objects: $\mathcal{E}_n$-algebras in $\mathfrak{V}$

1-arrows: $\mathcal{E}_{n-1}$-algebras in $\mathcal{E}_n$-bimodules

... and so on...

# Examples

- The sequence $(\mathfrak{Mod}^n(\Bbbk))_{n\geqslant 0}$ with $\Omega_{\mathfrak{Mod}^n(\Bbbk)}\mathfrak{Mod}^{n+1}(\Bbbk) \simeq \mathfrak{Mod}^n(\Bbbk)$ defines a categorical spectrum $\mathfrak{mod}(\Bbbk)$
  - $\Sigma\mathfrak{mod}(\Bbbk) \simeq \mathfrak{mod}(\mathfrak{Mod}(\Bbbk))$
- $(\mathfrak{V}, \otimes)$ an $\mathcal{E}_\infty$-monoidal $(\infty, \omega)$-category
  The sequence of deloopings $(\mathscr{B}^n\mathfrak{V})$ defines a categorical spectrum $\mathscr{B}\mathfrak{V}$

**Morita categorical spectrum**

$\mathfrak{V}$ symmetric monoidal $(\infty, 1)$-category. $\rightsquigarrow$ Morita $(\infty, n+1)$-category $\mathfrak{Mor}_n(\mathfrak{V})$ with

objects: $\mathcal{E}_n$-algebras in $\mathfrak{V}$

1-arrows: $\mathcal{E}_{n-1}$-algebras in $\mathcal{E}_n$-bimodules

... and so on...

Non-linear version: Iterated spans

# Contents – Section 1: Categorical spectra

# Stable cells

**Recollection: stable homotopy groups of spectra**

$X = (X_n)_n$ spectrum. $\pi_k^s(X) = \operatorname{colim}_n \pi_{n+k}(X_n)$

# Stable cells

**Recollection: stable homotopy groups of spectra**

$X = (X_n)_n$ spectrum. $\pi_k^s(X) = \operatorname{colim}_n \pi_{n+k}(X_n)$

$\mathfrak{D}_n$ walking $n$-cell: one $n$-cell, and two $k$-cells for $k < n$

$$\mathfrak{D}_0 = * \qquad \mathfrak{D}_1 = \cdot \to \cdot \qquad \mathfrak{D}_2 = \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd} \qquad \mathfrak{D}_3 = \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \Rrightarrow \Downarrow"] & \cdot \end{tikzcd}$$

For $\mathfrak{C}$ any $(\infty, \omega)$-category, $\hom(\mathfrak{D}_k, \mathfrak{C}) = \{k\text{-cells in } \mathfrak{C}\}$

# Stable cells

## Recollection: stable homotopy groups of spectra

$X = (X_n)_n$ spectrum. $\qquad \pi_k^s(X) = \operatorname{colim}_n \pi_{n+k}(X_n)$

$\mathfrak{D}_n$ walking $n$-cell: one $n$-cell, and two $k$-cells for $k < n$

$$\mathfrak{D}_0 = * \qquad \mathfrak{D}_1 = \cdot \to \cdot \qquad \mathfrak{D}_2 = \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd} \qquad \mathfrak{D}_3 = \begin{tikzcd} \cdot \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow \Rrightarrow \Downarrow"] & \cdot \end{tikzcd}$$

For $\mathfrak{C}$ any $(\infty, \omega)$-category, $\hom(\mathfrak{D}_k, \mathfrak{C}) = \{k\text{-cells in } \mathfrak{C}\}$

## Stable cells

$\mathfrak{X} = (\mathfrak{X}_n)_n$ categorical spectrum. $\qquad \operatorname{cell}_k(\mathfrak{X}) = \operatorname*{colim}_{n \geqslant 0} \hom(\mathfrak{D}_{k+n}, \mathfrak{X}_n)$

- $\operatorname{cell}_k(\Sigma \mathfrak{X}) \simeq \operatorname{cell}_{k-1}(\mathfrak{X})$

# Invertibility

$\forall n \in \mathbb{Z}$, composition maps $\mathrm{cell}_n(\mathfrak{X}) \underset{\mathrm{cell}_{n-1}(\mathfrak{X})}{\times} \mathrm{cell}_n(\mathfrak{X}) \to \mathrm{cell}_n(\mathfrak{X})$

Univalence/Rezk-completeness: Invertible $n$-cells are the image of $\mathrm{cell}_{n-1}(\mathfrak{X}) \hookrightarrow \mathrm{cell}_n(\mathfrak{X})$

# Invertibility

$\forall n \in \mathbb{Z}$, composition maps $\mathrm{cell}_n(\mathfrak{X}) \underset{\mathrm{cell}_{n-1}(\mathfrak{X})}{\times} \mathrm{cell}_n(\mathfrak{X}) \to \mathrm{cell}_n(\mathfrak{X})$

Univalence/Rezk-completeness: Invertible $n$-cells are the image of $\mathrm{cell}_{n-1}(\mathfrak{X}) \hookrightarrow \mathrm{cell}_n(\mathfrak{X})$

**Definition**

A categorical spectrum is $n$-categorical if its $k$-cells are invertible $\forall k > n$

- $(-\infty)$-categorical categorical spectra recover spectra

# Invertibility

$\forall n \in \mathbb{Z}$, composition maps $\mathrm{cell}_n(\mathfrak{X}) \underset{\mathrm{cell}_{n-1}(\mathfrak{X})}{\times} \mathrm{cell}_n(\mathfrak{X}) \to \mathrm{cell}_n(\mathfrak{X})$

Univalence/Rezk-completeness: Invertible $n$-cells are the image of $\mathrm{cell}_{n-1}(\mathfrak{X}) \hookrightarrow \mathrm{cell}_n(\mathfrak{X})$

**Definition**

A categorical spectrum is $n$-categorical if its $k$-cells are invertible $\forall k > n$

- $(-\infty)$-categorical categorical spectra recover spectra
- Ex.: the core $\mathfrak{Mor}(\mathfrak{Mod}(\Bbbk))^{\simeq}$ is the Brauer spectrum

# Invertibility

$\forall n \in \mathbb{Z}$, composition maps $\mathrm{cell}_n(\mathfrak{X}) \underset{\mathrm{cell}_{n-1}(\mathfrak{X})}{\times} \mathrm{cell}_n(\mathfrak{X}) \to \mathrm{cell}_n(\mathfrak{X})$

Univalence/Rezk-completeness: Invertible $n$-cells are the image of $\mathrm{cell}_{n-1}(\mathfrak{X}) \hookrightarrow \mathrm{cell}_n(\mathfrak{X})$

**Definition**

A categorical spectrum is $n$-categorical if its $k$-cells are invertible $\forall k > n$

- $(-\infty)$-categorical categorical spectra recover spectra
- Ex.: the core $\mathfrak{Mor}(\mathfrak{Mod}(\Bbbk))^{\simeq}$ is the Brauer spectrum

**Definition**

A categorical spectrum $\mathfrak{X}$ is connective if $\mathrm{cell}_k(\mathfrak{X}) \simeq * \, \forall k < 0$

- $\mathscr{B}\colon \mathcal{E}_\infty\text{-}\mathfrak{Alg}((\infty, \omega)\text{-}\mathfrak{Cat}) \xrightarrow{\simeq} \mathfrak{CatSp}^{\mathrm{cn}}$

# Contents - Section 2: $\mathbb{Z}$-categories

# Contents - Section 2: $\mathbb{Z}$-categories

# $(\infty, \mathbb{Z})$-categories

Adjunction $\begin{tikzcd} (\infty,\omega)\text{-}\mathfrak{Cat} \arrow[rr, bend left, "\Xi"] \arrow[rr, phantom, "\perp"] & & (\infty,\omega)\text{-}\mathfrak{Cat} \arrow[ll, bend left, "\mathrm{H}"] \end{tikzcd}$ with:

$$\Xi\mathfrak{C} = \left(0 \xrightarrow{\hom(0,1)=\mathfrak{C}} 1\right) \qquad \text{and} \qquad \mathrm{H}\mathfrak{C} = \operatorname*{colim}_{C,D\in \mathrm{obj}(\mathfrak{C})} \hom_{\mathfrak{C}}(C,D)$$

Ex.: $\Xi\mathfrak{D}_n = \mathfrak{D}_{n+1}$ (so $\mathfrak{D}_n = \Xi^n *$)

# $(\infty, \mathbb{Z})$-categories

Adjunction
$$\begin{tikzcd} (\infty,\omega)\text{-}\mathfrak{Cat} \arrow[rr, bend left, "\Xi"] \arrow[rr, phantom, "\perp"] & & (\infty,\omega)\text{-}\mathfrak{Cat} \arrow[ll, bend left, "\mathrm{H}"] \end{tikzcd}$$
with:

$$\Xi\mathfrak{C} = \left(0 \xrightarrow{\hom(0,1)=\mathfrak{C}} 1\right) \qquad \text{and} \qquad \mathrm{H}\mathfrak{C} = \operatorname*{colim}_{C,D\in \mathrm{obj}(\mathfrak{C})} \hom_{\mathfrak{C}}(C,D)$$

Ex.: $\Xi\mathfrak{D}_n = \mathfrak{D}_{n+1}$ (so $\mathfrak{D}_n = \Xi^n *$)

## Definition

$$(\infty,\mathbb{Z})\text{-}\mathfrak{Cat} = \lim\left(\cdots \xrightarrow{\mathrm{H}} (\infty,\omega)\text{-}\mathfrak{Cat} \xrightarrow{\mathrm{H}} (\infty,\omega)\text{-}\mathfrak{Cat}\right)$$

# $(\infty, \mathbb{Z})$-categories

Adjunction
$$\begin{tikzcd} (\infty,\omega)\text{-}\mathfrak{Cat} \arrow[rr, bend left, "\Xi"] \arrow[rr, phantom, "\perp"] & & (\infty,\omega)\text{-}\mathfrak{Cat} \arrow[ll, bend left, "\mathrm{H}"] \end{tikzcd}$$
with:

$$\Xi\mathfrak{C} = \left(0 \xrightarrow{\hom(0,1)=\mathfrak{C}} 1\right) \qquad \text{and} \qquad \mathrm{H}\mathfrak{C} = \operatorname*{colim}_{C,D\in\mathrm{obj}(\mathfrak{C})} \hom_{\mathfrak{C}}(C,D)$$

Ex.: $\Xi\mathfrak{D}_n = \mathfrak{D}_{n+1}$ (so $\mathfrak{D}_n = \Xi^n *$)

**Definition**

$(\infty,\mathbb{Z})\text{-}\mathfrak{Cat} = \lim\left(\cdots \xrightarrow{\mathrm{H}} (\infty,\omega)\text{-}\mathfrak{Cat} \xrightarrow{\mathrm{H}} (\infty,\omega)\text{-}\mathfrak{Cat}\right)$

**Theorem [K.]**

Equivalence of $\infty$-categories $\mathfrak{CatSp} \simeq (\infty,\mathbb{Z})\text{-}\mathfrak{Cat}_*$

# Now what are higher categories again, again?

Joyal's cell category $\Theta$: category of $\omega$-categories free on pasting diagrams

E.g. $$\begin{tikzcd} \cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd} \in \Theta_2$$

# Now what are higher categories again, again?

Joyal's cell category $\Theta$: category of $\omega$-categories free on pasting diagrams

E.g. $$\begin{tikzcd} \cdot \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=D, above}] \arrow[Rightarrow, from=U, to=M] \arrow[Rightarrow, from=M, to=D] & \cdot \arrow[r, bend left=50, ""{name=A, below}] \arrow[r, bend right=50, ""{name=B, above}] \arrow[Rightarrow, from=A, to=B] & \cdot \end{tikzcd} \in \Theta_2$$

**Lemma [Ara]**

Any $T \in \Theta$ is a gluing of globes: $T \simeq \mathfrak{D}_{n_1} \coprod_{\mathfrak{D}_{m_1}} \cdots \coprod_{\mathfrak{D}_{m_{p-1}}} \mathfrak{D}_{n_p}$,

for $n_1, \ldots, n_p, m_1, \ldots, m_{p-1} \in \mathbb{N}$ with $m_i < n_i, n_{i+1}$

# Now what are higher categories again, again?

Joyal's cell category $\Theta$: category of $\omega$-categories free on pasting diagrams

E.g. $\begin{tikzcd} \cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd} \in \Theta_2$

**Lemma** [Ara]

Any $T \in \Theta$ is a gluing of globes: $T \simeq \mathfrak{D}_{n_1} \coprod_{\mathfrak{D}_{m_1}} \cdots \coprod_{\mathfrak{D}_{m_{p-1}}} \mathfrak{D}_{n_p}$,

for $n_1, \ldots, n_p, m_1, \ldots, m_{p-1} \in \mathbb{N}$ with $m_i < n_i, n_{i+1}$

**Proposition** [Berger]

$(\infty, \omega)\text{-}\mathfrak{Cat} \subseteq \mathfrak{Fun}(\Theta^{\mathrm{op}}, \infty\text{-}\mathfrak{Grpd})$ full subcat

# Now what are higher categories again, again?

Joyal's cell category $\Theta$: category of $\omega$-categories free on pasting diagrams

E.g. $$\begin{tikzcd} \cdot \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=D, above}] \arrow[Rightarrow, from=U, to=M] \arrow[Rightarrow, from=M, to=D] & \cdot \arrow[r, bend left=50, ""{name=A, below}] \arrow[r, bend right=50, ""{name=B, above}] \arrow[Rightarrow, from=A, to=B] & \cdot \end{tikzcd} \in \Theta_2$$

## Lemma [Ara]

Any $T \in \Theta$ is a gluing of globes: $T \simeq \mathfrak{D}_{n_1} \coprod_{\mathfrak{D}_{m_1}} \cdots \coprod_{\mathfrak{D}_{m_{p-1}}} \mathfrak{D}_{n_p}$,

for $n_1, \ldots, n_p, m_1, \ldots, m_{p-1} \in \mathbb{N}$ with $m_i < n_i, n_{i+1}$

## Proposition [Berger]

$(\infty, \omega)$-$\mathfrak{Cat} \subseteq \mathfrak{Fun}(\Theta^{\mathrm{op}}, \infty\text{-}\mathfrak{Grpd})$ full subcat on $\mathfrak{X} \colon \Theta^{\mathrm{op}} \to \infty\text{-}\mathfrak{Grpd}$ such that

$$\mathfrak{X}(T) \xrightarrow{\simeq} \mathfrak{X}(\mathfrak{D}_1) \underset{\mathfrak{X}(\mathfrak{D}_{m_1})}{\times} \cdots \underset{\mathfrak{X}(\mathfrak{D}_{m_{p-1}})}{\times} \mathfrak{X}(\mathfrak{D}_p) \qquad \text{(plus Rezk-completeness)}$$

# The stable cells category

- $\Xi$ restricts to an endofunctor of $\Theta$ (but $\mathrm{H}$ doesn't)

**Definition [Lessard]**

$$\Theta_{\mathbb{Z}} \coloneqq \operatorname{colim}\left(\Theta \xrightarrow{\Xi} \Theta \xrightarrow{\Xi} \Theta \xrightarrow{\Xi} \cdots\right)$$

# The stable cells category

- $\Xi$ restricts to an endofunctor of $\Theta$ (but $\mathrm{H}$ doesn't)

**Definition** [Lessard]

$$\Theta_{\mathbb{Z}} \coloneqq \operatorname{colim}\left(\Theta \xrightarrow{\Xi} \Theta \xrightarrow{\Xi} \Theta \xrightarrow{\Xi} \cdots\right)$$

- $\mathbb{G} \subset \Theta$ subcat on globes and $\omega$-graphs morphisms: $\mathfrak{D}_0 \underset{i_0^-}{\overset{i_0^+}{\rightrightarrows}} \cdots \underset{i_{n-1}^-}{\overset{i_{n-1}^+}{\rightrightarrows}} \mathfrak{D}_n \underset{i_n^-}{\overset{i_n^+}{\rightrightarrows}} \cdots$

# The stable cells category

- $\Xi$ restricts to an endofunctor of $\Theta$ (but $\mathrm{H}$ doesn't)

**Definition** [Lessard]

$$\Theta_{\mathbb{Z}} \coloneqq \operatorname{colim}\left(\Theta \xrightarrow{\Xi} \Theta \xrightarrow{\Xi} \Theta \xrightarrow{\Xi} \cdots\right)$$

- $\mathbb{G} \subset \Theta$ subcat on globes and $\omega$-graphs morphisms: $\mathfrak{D}_0 \underset{i_0^-}{\overset{i_0^+}{\rightrightarrows}} \cdots \underset{i_{n-1}^-}{\overset{i_{n-1}^+}{\rightrightarrows}} \mathfrak{D}_n \underset{i_n^-}{\overset{i_n^+}{\rightrightarrows}} \cdots$
- Stable globe category $\mathbb{G}_{\mathrm{st}}$: shape

$$\cdots \underset{i_{-m-1}^-}{\overset{i_{-m-1}^+}{\rightrightarrows}} \mathfrak{D}_{-m} \underset{i_{-m}^-}{\overset{i_{-m}^+}{\rightrightarrows}} \cdots \underset{i_{-2}^-}{\overset{i_{-2}^+}{\rightrightarrows}} \mathfrak{D}_{-1} \underset{i_{-1}^-}{\overset{i_{-1}^+}{\rightrightarrows}} \mathfrak{D}_0 \underset{i_0^-}{\overset{i_0^+}{\rightrightarrows}} \mathfrak{D}_1 \underset{i_1^-}{\overset{i_1^+}{\rightrightarrows}} \cdots \underset{i_{n-1}^-}{\overset{i_{n-1}^+}{\rightrightarrows}} \mathfrak{D}_n \underset{i_n^-}{\overset{i_n^+}{\rightrightarrows}} \cdots$$

**Lemma** [Lessard]

$$\mathbb{G}_{\mathbb{Z}} \coloneqq \operatorname{colim}\left(\mathbb{G} \xrightarrow{\Xi} \mathbb{G} \xrightarrow{\Xi} \mathbb{G} \xrightarrow{\Xi} \cdots\right) \simeq \mathbb{G}_{\mathrm{st}}$$

# Globular presentation for $(\infty, \mathbb{Z})$-categories

**Lemma [Lessard]**

Any $T \in \Theta_{\mathbb{Z}}$ is a gluing of stable globes: $T \simeq \mathfrak{D}_{n_1} \coprod_{\mathfrak{D}_{m_1}} \cdots \coprod_{\mathfrak{D}_{m_{p-1}}} \mathfrak{D}_{n_p}$,

for $n_1, \ldots, n_p, m_1, \ldots, m_{p-1} \in \mathbb{Z}$ with $m_i < n_i, n_{i+1}$

# Globular presentation for $(\infty, \mathbb{Z})$-categories

**Lemma** [Lessard]

Any $T \in \Theta_{\mathbb{Z}}$ is a gluing of stable globes: $T \simeq \mathfrak{D}_{n_1} \coprod_{\mathfrak{D}_{m_1}} \cdots \coprod_{\mathfrak{D}_{m_{p-1}}} \mathfrak{D}_{n_p}$,

for $n_1, \ldots, n_p, m_1, \ldots, m_{p-1} \in \mathbb{Z}$ with $m_i < n_i, n_{i+1}$

**Proposition** [Lessard, K.]

$(\infty, \mathbb{Z})$-$\mathfrak{Cat} \subseteq \mathfrak{Fun}(\Theta_{\mathbb{Z}}^{\mathrm{op}}, \infty\text{-}\mathfrak{Grpd})$ full subcat on $\mathfrak{X} \colon \Theta_{\mathbb{Z}}^{\mathrm{op}} \to \infty\text{-}\mathfrak{Grpd}$ such that

$$\mathfrak{X}(T) \xrightarrow{\simeq} \mathfrak{X}(\mathfrak{D}_1) \underset{\mathfrak{X}(\mathfrak{D}_{m_1})}{\times} \cdots \underset{\mathfrak{X}(\mathfrak{D}_{m_{p-1}})}{\times} \mathfrak{X}(\mathfrak{D}_p) \qquad \text{(plus Rezk-completeness)}$$

# Globular presentation for $(\infty, \mathbb{Z})$-categories

**Lemma** [Lessard]

Any $T \in \Theta_{\mathbb{Z}}$ is a gluing of stable globes: $T \simeq \mathfrak{D}_{n_1} \coprod_{\mathfrak{D}_{m_1}} \cdots \coprod_{\mathfrak{D}_{m_{p-1}}} \mathfrak{D}_{n_p}$,

for $n_1, \ldots, n_p, m_1, \ldots, m_{p-1} \in \mathbb{Z}$ with $m_i < n_i, n_{i+1}$

**Proposition** [Lessard, K.]

$(\infty, \mathbb{Z})$-$\mathfrak{Cat} \subseteq \mathfrak{Fun}(\Theta_{\mathbb{Z}}^{\mathrm{op}}, \infty\text{-}\mathfrak{Grpd})$ full subcat on $\mathfrak{X} \colon \Theta_{\mathbb{Z}}^{\mathrm{op}} \to \infty\text{-}\mathfrak{Grpd}$ such that

$$\mathfrak{X}(T) \xrightarrow{\simeq} \mathfrak{X}(\mathfrak{D}_1) \underset{\mathfrak{X}(\mathfrak{D}_{m_1})}{\times} \cdots \underset{\mathfrak{X}(\mathfrak{D}_{m_{p-1}})}{\times} \mathfrak{X}(\mathfrak{D}_p) \qquad \text{(plus Rezk-completeness)}$$

$\implies$ $\hom(\mathfrak{D}_n, \mathfrak{C})$ $n$-cells of $\mathfrak{C}$ $\forall n \in \mathbb{Z}$, plus composition operations

Remark: Segal condition comes from an “automatic” Segal condition determined by $\Theta_{\mathbb{Z}}$

# Contents - Section 2: $\mathbb{Z}$-categories

# Monoidal comparison

Define $\mathcal{E}_n\mathfrak{CatSp} \coloneqq \lim\left(\cdots \xrightarrow{\Omega} \mathcal{E}_n\text{-}\mathfrak{Alg}((\infty,\omega)\text{-}\mathfrak{Cat}) \xrightarrow{\Omega} \mathcal{E}_n\text{-}\mathfrak{Alg}((\infty,\omega)\text{-}\mathfrak{Cat})\right)$

**Theorem**

For any $0 \leqslant n \leqslant \infty$, equivalence $\mathcal{E}_n\mathfrak{CatSp} \simeq \mathcal{E}_n\text{-}\mathfrak{Alg}((\infty,\mathbb{Z})\text{-}\mathfrak{Cat})$

**Proof.**

Both $(\infty,\mathbb{Z})$-categories and $\mathcal{E}_n$-algebras are given as Segal objects □

# Monoidal comparison

Define $\mathcal{E}_n\mathfrak{CatSp} \coloneqq \lim\left(\cdots \xrightarrow{\Omega} \mathcal{E}_n\text{-}\mathfrak{Alg}((\infty,\omega)\text{-}\mathfrak{Cat}) \xrightarrow{\Omega} \mathcal{E}_n\text{-}\mathfrak{Alg}((\infty,\omega)\text{-}\mathfrak{Cat})\right)$

**Theorem**

For any $0 \leqslant n \leqslant \infty$, equivalence $\mathcal{E}_n\mathfrak{CatSp} \simeq \mathcal{E}_n\text{-}\mathfrak{Alg}((\infty,\mathbb{Z})\text{-}\mathfrak{Cat})$

**Proof.**

Both $(\infty,\mathbb{Z})$-categories and $\mathcal{E}_n$-algebras are given as Segal objects □

**Proposition [Stefanich]**

For any $n$, $\mathcal{E}_n\mathfrak{CatSp} \simeq \mathfrak{CatSp}$

$\Longrightarrow$ A pointing ($\mathcal{E}_0$-structure) on an $(\infty,\mathbb{Z})$-category is enough to infinitely deloop it to an $\mathcal{E}_\infty$-monoidal $(\infty,\mathbb{Z})$-category

# Proof of Stefanich's monoidal equivalence

$$
\begin{tikzcd}
\vdots \arrow[d, equal] & \vdots \arrow[d, equal] & \vdots \arrow[d] & \cdots \\
\mathcal{E}_k\text{-}\mathfrak{Alg}((\infty,\omega)\text{-}\mathfrak{Cat}) \arrow[d, equal] & \mathcal{E}_k\text{-}\mathfrak{Alg}((\infty,\omega)\text{-}\mathfrak{Cat}) \arrow[l, "\Omega"'] \arrow[d] & \mathcal{E}_{k-1}\text{-}\mathfrak{Alg}((\infty,\omega)\text{-}\mathfrak{Cat}) \arrow[l, "\Omega"'] \arrow[d] & \cdots \\
\mathcal{E}_k\text{-}\mathfrak{Alg}((\infty,\omega)\text{-}\mathfrak{Cat}) \arrow[d] & \mathcal{E}_{k-1}\text{-}\mathfrak{Alg}((\infty,\omega)\text{-}\mathfrak{Cat}) \arrow[l, "\Omega"] \arrow[d] & \mathcal{E}_{k-2}\text{-}\mathfrak{Alg}((\infty,\omega)\text{-}\mathfrak{Cat}) \arrow[l, "\Omega"] \arrow[d] & \cdots \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \cdots \\
\mathcal{E}_1\text{-}\mathfrak{Alg}((\infty,\omega)\text{-}\mathfrak{Cat}) \arrow[d] & \mathcal{E}_0\text{-}\mathfrak{Alg}((\infty,\omega)\text{-}\mathfrak{Cat})) \arrow[l, "\Omega"'] \arrow[d] & \mathcal{E}_0\text{-}\mathfrak{Alg}((\infty,\omega)\text{-}\mathfrak{Cat}) \arrow[l, "\Omega"'] \arrow[d] & \cdots \\
\mathcal{E}_0\text{-}\mathfrak{Alg}((\infty,\omega)\text{-}\mathfrak{Cat}) & \mathcal{E}_0\text{-}\mathfrak{Alg}((\infty,\omega)\text{-}\mathfrak{Cat})) \arrow[l, "\Omega"] & \mathcal{E}_0\text{-}\mathfrak{Alg}((\infty,\omega)\text{-}\mathfrak{Cat}) \arrow[l, "\Omega"] & \cdots
\end{tikzcd}
$$

# Cells and stable cells

**Proposition**

$\mathfrak{X}$ a categorical spectrum, $\kappa(\mathfrak{X})$ the corresponding pointed $(\infty, \mathbb{Z})$-category.
For any $n \in \mathbb{Z}$, equivalence

$$\mathrm{cell}_n(\mathfrak{X}) \simeq \hom(\mathfrak{D}_n, \kappa\mathfrak{X})$$

**Proof.**

$$\hom(\mathfrak{D}_{k+i}, (\kappa\mathfrak{X})_i) \simeq \hom(\Xi^i\mathfrak{D}_k, \mathrm{H}^{\infty - i}\mathfrak{C}) \simeq \hom(\mathfrak{D}_k, \mathrm{H}^{\infty}\mathfrak{C})$$

# Cells and stable cells

## Proposition

$\mathfrak{X}$ a categorical spectrum, $\kappa(\mathfrak{X})$ the corresponding pointed $(\infty, \mathbb{Z})$-category.
For any $n \in \mathbb{Z}$, equivalence

$$\mathrm{cell}_n(\mathfrak{X}) \simeq \hom(\mathfrak{D}_n, \kappa\mathfrak{X})$$

## Proof.

$\hom(\mathfrak{D}_{k+i}, (\kappa\mathfrak{X})_i) \simeq \hom(\Xi^i\mathfrak{D}_k, \mathrm{H}^{\infty-i}\mathfrak{C}) \simeq \hom(\mathfrak{D}_k, \mathrm{H}^{\infty}\mathfrak{C})$
$\implies$ Diagram $\hom(\mathfrak{D}_k, (\kappa\mathfrak{X})_0) \to \hom(\mathfrak{D}_{k+1}, (\kappa\mathfrak{X})_1) \to \cdots$ is constant. $\square$

# Cells and stable cells

## Proposition

$\mathfrak{X}$ a categorical spectrum, $\kappa(\mathfrak{X})$ the corresponding pointed $(\infty, \mathbb{Z})$-category.
For any $n \in \mathbb{Z}$, equivalence

$$\mathrm{cell}_n(\mathfrak{X}) \simeq \hom(\mathfrak{D}_n, \kappa\mathfrak{X})$$

## Proof.

$\hom(\mathfrak{D}_{k+i}, (\kappa\mathfrak{X})_i) \simeq \hom(\Xi^i \mathfrak{D}_k, \mathrm{H}^{\infty - i}\mathfrak{C}) \simeq \hom(\mathfrak{D}_k, \mathrm{H}^{\infty}\mathfrak{C})$
$\implies$ Diagram $\hom(\mathfrak{D}_k, (\kappa\mathfrak{X})_0) \to \hom(\mathfrak{D}_{k+1}, (\kappa\mathfrak{X})_1) \to \cdots$ is constant. $\square$

## Corollary [Lessard]

Equivalence $\mathfrak{Sp} \simeq (\infty, \mathbb{Z})\text{-}\mathfrak{Cat}_*^{\mathrm{grpd}}$

## Proof.

Restrict the equivalence $\mathfrak{CatSp} \simeq (\infty, \mathbb{Z})\text{-}\mathfrak{Cat}_*$ to objects with all cells invertible $\square$

# Backup

# Univalence for $(\infty, n)$-categories

**Theorem [Ayala–Francis]**

For *finite n*, functors $\Theta_n^{\mathrm{op}} \to \infty\text{-}\mathfrak{Grpd}$ with the Segal conditions are equivalent to *flagged* $(\infty, n)$-categories:

$$\mathfrak{C}_0 \to \mathfrak{C}_1 \to \cdots \to \mathfrak{C}_{n-1} \to \mathfrak{C}_n = \mathfrak{C}$$

Univalence: $\mathfrak{C}_i$ is the $(\infty, i)$-core of $\mathfrak{C}$ for all $i < n$

**Equivalent characterisation [Rezk]**

$\mathfrak{eq}$ the walking equivalence. A Segal sheaf $\mathfrak{X}$ is univalent iff

$$\hom(\Xi^k *, \mathfrak{X}) \xrightarrow{\simeq} \hom(\Xi^k \mathfrak{eq}, \mathfrak{X}) \text{ for all } k < n$$

For $n = \omega$: call a Segal $\Theta$-presheaf a flagged $(\infty, \omega)$-category

# Univalence for $(\infty, \mathbb{Z})$-categories

A Segal $\Theta_{\mathbb{Z}}$-presheaf corresponds to a sequence $(\mathfrak{C}_n)_{n\geqslant 0}$ of *flagged* $(\infty, \omega)$-categories with $\mathrm{H}\mathfrak{C}_{n+1} \simeq \mathfrak{C}_n$

**Lemma [K.]**

For $\mathscr{C}$ a Segal $\Theta_{\mathbb{Z}}$-presheaf, the following are equivalent:

- Each $\mathfrak{C}_i$ is univalent
- $\mathscr{C}$ is local for $\Xi^k \mathfrak{eq} \to \Xi^k *$ for each $k \in \mathbb{Z}$