Symmetric 2-Segal conditions for shifted cotangent groupoids

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Contents – Section 1: Shifted cotangent bundles

1. Shifted cotangent bundles
2. Shifted symplectic groupoids
3. Calabi–Yau monads and correspondences

Algebraic symplectic structures and the cotangent complex

"Smooth" symplectic structure: non-degenerate section of vector bundle $\mathscr{A}^{2,\mathrm{cl}}(X) \subset \wedge^2 \Omega^1_X$

If X is not smooth, Ω^1_X is not a vector bundle.

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Orbisingularity: $X = [V/G]$

$\mathbb{L}_X = [\overset{0}{\Omega^1_V} \xrightarrow{\mathrm{d}} \overset{1}{\mathfrak{g}^\vee} \otimes \mathcal{O}_V]$ (Remark: $\mathfrak{QCoh}(X) = \mathfrak{QCoh}(V)^G$)

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Note: In both cases, $\mathbb{L}_X$ is a finite complex of vector bundles, aka a perfect complex.

Derived schemes

Upshot: Symplectic forms in singular settings should "live in" $\Gamma(\wedge^2\mathbb{L}_X)$.
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- $X \underset{\text{loc.}}{=} \operatorname{Spec} R$ with R an algebra in $\mathfrak{Mod}_{\mathbb{C}}$, and $\Omega^1_{X,x} \in \mathfrak{Mod}_{\mathbb{C}}$ for any $x\colon \operatorname{Spec} \mathbb{C} \to X$
- $\mathbb{L}_{X,x} \in \mathfrak{Ch}(\mathbb{C}) =: \mathfrak{dMod}_{\mathbb{C}}$: derived ($\infty$-)category (In fact $\mathfrak{dMod} = \mathfrak{Ch}[\mathrm{qis}^{-1}]$)

How to reconcile the two?

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How to reconcile the two?

$\Longrightarrow$ View R as a particular case of algebra in $\mathfrak{dMod}_{\mathbb{C}}$

Definition

$\mathfrak{dAlg}_{\mathbb{C}} \coloneqq \mathfrak{cdga}^{\leqslant 0}_{\mathbb{C}}[\mathrm{qis}^{-1}]$: ∞-category of commutative algebras in $\mathfrak{dMod}^{\leqslant 0}_{\mathbb{C}}$
$\mathfrak{dAff}_{\mathbb{C}} = {\mathfrak{dAlg}_{\mathbb{C}}}^{\mathrm{op}}$ and derived schemes are locally derived affines

Symplectic forms redux

X a derived scheme (or stack). $\mathscr{A}^2(X,0) \coloneqq \Gamma(X, \wedge^2 \mathbb{L}_X)$

Remark

$\wedge^2 \mathbb{L}_X = \mathrm{Sym}^2(\mathbb{L}_X[1])[-2]$

Indeed, by antisymmetry of odd degrees, $\mathrm{Sym}^\bullet(M[1]) = \bigoplus_{n\geqslant 0} (\wedge^n M)[n]$

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New phenomenon for derived modules: we can shift them!

Definition (n-shifted 2-forms)

$\mathscr{A}^2(X,n) = \Gamma\left(X, (\wedge^2 \mathbb{L}_X)[n]\right) = \Gamma(X, \mathrm{Sym}^2(\mathbb{L}_X[1])[n-2])$

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Closed n-shifted 2-forms $\eqqcolon$ n-shifted presymplectic forms

$\mathscr{A}^{2,\mathrm{cl}}(X,n) = \left\{\omega_0 \in \mathscr{A}^2(X,n) + \text{ key } \mathrm{d}_{\mathrm{dR}}\,\omega_0 = \mathrm{d}\,\omega_1, \mathrm{d}_{\mathrm{dR}}\,\omega_1 = \mathrm{d}\,\omega_2, \dots\right\} \to \mathscr{A}^2(X,n)$

Examples

$\omega_0 \colon \mathcal{O}_X \to \mathbb{L}_X \wedge \mathbb{L}_X[n]$ an n-shifted 2-form is **non-degenerate** if $\omega_0^\flat \colon \mathbb{T}_X \coloneqq \mathbb{L}_X^\vee \xrightarrow{\simeq} \mathbb{L}_X[n]$: exhibit symmetry of the cotangent complex

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Derived critical loci are (-1)-shifted symplectic

$$\begin{array}{ccc} \mathbb{R}\operatorname{Crit}(f) & \longrightarrow & Y \\ \downarrow & & \downarrow \mathrm{d}_{\mathrm{dR}} f \\ Y & \xrightarrow{\quad 0 \quad} & T^\vee Y \end{array}$$

$$\mathbb{L}_X = \left[T_Y \xrightarrow{\operatorname{Hess}(f)} T_Y^\vee \right]$$

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$\mathcal{B}\, G$ is 2-shifted symplectic

$\mathbb{L}_{\mathcal{B}\, G} = \mathfrak{g}^\vee[-1]$, whence $\omega_0 \in \Gamma(\mathcal{B}\, G, \wedge^2 \mathbb{L}_{\mathcal{B}\, G})[2] = \operatorname{Sym}^2(\mathfrak{g}^\vee)^G$ is the Killing form

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Shifted cotangent stacks are shifted symplectic (Calaque)

$T^\vee[n]Y = \mathbb{V}_Y(\mathbb{L}_Y[n])$ total space of $\mathbb{L}_Y[n]^\vee$, with $\omega_0 = d_{dR}\,\theta$, θ soldering form

Yonedark magic

Lemma (Pantev–Toën–Vaquié–Vezzosi)

${\mathfrak{dAff}_{\mathbb{C}}}^{\mathrm{op}} \ni R \mapsto \mathscr{A}^{2,\mathrm{cl}}(\operatorname{Spec} R, n)$ satisfies étale descent, *i.e.* it is a sheaf/stack: "moduli stack of n-shifted presymplectic forms" $\mathscr{A}^{2,\mathrm{cl}}(-, n)$.

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Corollary (Pantev–Toën–Vaquié–Vezzosi)

For any derived stack X,

$$\mathscr{A}^{2,\mathrm{cl}}(X, n) \simeq \hom(X, \mathscr{A}^{2,\mathrm{cl}}(-, n))$$

Consequence: The ∞-category of n-shifted presymplectic derived stacks is a (slice) ∞-topos

$$\mathfrak{PrSymp}(n) = \mathfrak{dSt}_{/\mathscr{A}^{2,\mathrm{cl}}(-,n)}$$

$\mathfrak{Symp}(n)$ is the full subcategory of $\mathfrak{dSt}_{/\mathscr{A}^{2,\mathrm{cl}}(-,n)}$ on the non-degenerate forms: in practice, work in $\mathfrak{dSt}_{/\mathscr{A}^{2,\mathrm{cl}}(-,n)}$ and then check non-degeneracy.

Lagrangian correspondences

(Pre-)Lagrangian structures

Isotropic structure on $f\colon Y \to X$
relative to $\omega \in \mathscr{A}^{2,\mathrm{cl}}(X, n)$:
trivialisation $f^*\omega \xrightarrow{\simeq} 0$ in $\mathscr{A}^{2,\mathrm{cl}}(Y, n)$

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Pulling back presymplectic forms

$\ulcorner\omega\urcorner \colon X \to \mathscr{A}^{2,\mathrm{cl}}(-,n)$ (pre)symplectic, $f \colon Y \to X$. Then $\ulcorner f^*\omega\urcorner \colon Y \xrightarrow{f} X \xrightarrow{\ulcorner\omega\urcorner} \mathscr{A}^{2,\mathrm{cl}}(-,n)$

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$=$ correspondence $(*,0) \to (X,\omega)$ in $\mathfrak{Span}(\mathfrak{dSt}_{/\mathcal{A}^{2,\mathrm{cl}}(n)})$

$\Longrightarrow$ Lagrangian corresp. $(Y,\psi) \to (X,\omega)$ is

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \downarrow & \searrow & \\ Y & & & & X \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{A}^{2,\mathrm{cl}}(-,n) & & \end{array}$$

nondegen.

Shifting phenomena in symplectic geometry

Delooping (Calaque)

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with X n-symplectic $\implies$ $L_1 \times_X L_2$ is $(n-1)$-symplectic

Ex.: Recover derived critical loci: $0, \mathrm{d_{dR}} f \colon V \to T^\vee[n]V$ Lagrangian

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Quotients of symplectic groupoids (Calaque–Safronov)

$G_\bullet$ n-shifted symplectic groupoid $\implies$ $|G_\bullet|$ $(n+1)$-symplectic stack

Contents - Section 2: Shifted symplectic groupoids

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Atlases and groupoids for algebraic derived stacks

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- U is a union of affine derived schemes,
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Taking the kernel (aka nerve) of the surjection ϖ: get a groupoid $G_\bullet$ (in $\mathfrak{dSt}$)

$$\cdots \qquad G_2 = U \times_X U \times_X U \underset{\longrightarrow}{\overset{\longrightarrow}{\underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}}}} G_1 = U \times_X U \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} G_0 = U$$

where:

- G_i is a union of $(n-1)$-Artin stacks
- $G_{i+1} \to G_i$ is smooth with $(n-1)$-Artin fibres

and $X = |G_\bullet| = \varinjlim G_\bullet$

Groupoids in general

Notation: For $f\colon [k] \to [n]$ in Δ, write $X_n \to X_{\{f(1),\ldots,f(k)\}} = X_k$

Internal categories

A category object in an ∞-category $\mathfrak{C}$ is a simplicial object $X_\bullet\colon \Delta^{\mathrm{op}} \to \mathfrak{C}$ such that the Segal cone $\left\{X_n \to X_{\{i,i+1\}} = X_1\right\}_{0\leqslant i\leqslant n}$ exhibits $X_n = X_1 \underset{d_0, X_0, d_1}{\times} \cdots \underset{d_0, X_0, d_1}{\times} X_1$

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$X_\bullet$ is further a groupoid object if, equivalently:

- Unordered Segal decomposition(s): $X_2 \xrightarrow{\simeq} X_{\{0,1\}} \underset{X_{\{0\}}}{\times} X_{\{0,2\}}$ and $X_2 \xrightarrow{\simeq} X_{\{1,2\}} \underset{X_{\{2\}}}{\times} X_{\{0,2\}}$
- Compatible $\mathfrak{S}_{\bullet+1}$-actions (in fact only need the sub-$C_{\bullet+1}$-actions)

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- Compatible $\mathfrak{S}_{\bullet+1}$-actions (in fact only need the sub-$C_{\bullet+1}$-actions)

A monoid in $\mathfrak{C} \rightsquigarrow$ internal category $\mathrm{Bar}_\bullet A$ with $\mathrm{Bar}_n A = A^n$.
Groupoid iff A is a group.

Shifted symplectic groupoids

$\mathscr{A}^{2,\mathrm{cl}}(n)$ abelian group $\implies$ groupoid $\mathrm{Bar}_\bullet \mathscr{A}^{2,\mathrm{cl}}(n)$ in $\mathfrak{dSt}$.

Definition (Shifted presymplectic groupoid)

An n-shifted symplectic groupoid is a groupoid $G_\bullet$ in $\mathfrak{dSt}$ over $\mathrm{Bar}_\bullet \mathscr{A}^{2,\mathrm{cl}}(n)$

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Consequences

- For any k, map $G_k \xrightarrow{(\ulcorner\theta_1\urcorner,\dots,\ulcorner\theta_k\urcorner)} \mathscr{A}^{2,\mathrm{cl}}(n)^k \leftrightsquigarrow k$ n-symplectic structures θ_i on G_k
- Isotropic correspondence $\gamma_k \colon G_1^k \leftarrow G_k \to G_1$

$G_\bullet$ is n-symplectic if the γ_k are Lagrangian correspondences

Symplectic presentations

Proposition (Calaque–Safronov)

An n-presymplectic structure $\omega_\bullet$ on $G_\bullet$ induces an $(n+1)$-shifted isotropic structure on $G_0 \twoheadrightarrow |G_\bullet|$. It is Lagrangian (in part. $|G_\bullet|$ is $(n+1)$-symplectic) iff $\omega_\bullet$ is symplectic.

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Theorem (Calaque–K.)

$X = |G_\bullet|$ derived Artin stack with atlas $G_\bullet$. Then $T^\vee[n+1]X$ admits a presentation by a symplectic groupoid given in level k by

$$N^\vee[n]\left(G_k \to G_1^{k+1} = G_{\{0,1\}} \times \cdots \times G_{\{k-1,k\}} \times \overline{G_{\{0,k\}}}\right)$$

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For $\mathcal{B}\,G = |\mathrm{Bar}_\bullet\, G|$

$N^\vee[n](G^k \to G^k \times G) \simeq (T^\vee[n]G) \times_G G^k = \mathbb{V}_{G^k}(\mathfrak{g}^\vee[n] \otimes \mathcal{O}_{G^k})$: quotient by adjoint G-action

First ingredient: functoriality of cotangent bundles

$f\colon Y \to X$ morphism of Artin derived stacks: there is a Lagrangian correspondence

$$\begin{array}{ccc} & T^\vee[n]X \times_X Y & \\ \swarrow & & \searrow \\ T^\vee[n]Y & & T^\vee[n]X. \end{array}$$

Upshot: ∞-functor $\mathscr{T}$: $\mathfrak{dSt} \xrightarrow{T^\vee[n]} \mathfrak{LagCorr}(n) \subset \mathfrak{Span}(\mathfrak{dSt}_{/\mathscr{A}^{2,\mathrm{cl}}(n)})$

First ingredient: functoriality of cotangent bundles

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$$\begin{array}{ccc} & T^\vee[n]X \times_X Y \simeq N^\vee[n](Y/X \times Y) & \\ \swarrow & & \searrow \\ T^\vee[n]Y & & T^\vee[n]X. \end{array}$$

More generally, for any span $Y \xleftarrow{g} Z \xrightarrow{f} X$ of Artin derived stacks, Lagrangian

$$\begin{array}{ccc} & N^\vee[n](Z/X \times Y) & \\ \swarrow & & \searrow \\ T^\vee[n]Y & & T^\vee[n]X. \end{array}$$

Upshot: ∞-functor $\mathscr{T} : \mathfrak{Span}(\mathfrak{dSt}) \xrightarrow{(T^\vee[n], N^\vee[n])} \mathfrak{LagCorr}(n) \subset \mathfrak{Span}(\mathfrak{dSt}_{/\mathscr{A}^{2,\mathrm{cl}}(n)})$

Groupoids and algebras in spans

Problem: $\mathcal{T}$ has no reason to send a groupoid in $\mathbf{dSt}$ to a groupoid in $\mathbf{dSt}_{/\mathscr{A}^{2,\mathrm{cl}}(n)}$.

If $\mathcal{T}$ does not preserve groupoids in arrows, what does it preserve?

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Remark:

1. For any $\mathfrak{C}$ with limits, $\mathfrak{Span}(\mathfrak{C})$ has a monoidal structure "$\times$" by $C``\times"D = C \times D$

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Remark:

1. For any $\mathfrak{C}$ with limits, $\mathfrak{Span}(\mathfrak{C})$ has a monoidal structure "$\times$" by C"$\times$"$D = C \times D$
2. If $(M, +)$ is a monoid in $\mathfrak{C}$, monoidal structure on $\mathfrak{Span}(\mathfrak{C}_{/M})$ with

$$\begin{array}{c} C \\ {\scriptstyle f}\downarrow \\ M \end{array} \boxplus \begin{array}{c} D \\ \downarrow{\scriptstyle g} \\ M \end{array} = \begin{array}{ccc} C \times D & & \\ {\scriptstyle f \times g}\downarrow & \searrow^{f \boxplus g} & \\ M \times M & \xrightarrow[+]{} & M \end{array}$$

Our $\mathscr{T}$ is a monoidal functor $\mathfrak{Span}(\mathfrak{dSt})^{\text{“}\times\text{”}} \to \mathfrak{LagCorr}(n)^{\boxplus}$

Contents - Section 3: Calabi–Yau monads and correspondences

1 Shifted cotangent bundles

2 Shifted symplectic groupoids

3 Calabi–Yau monads and correspondences

Groupoids as algebras in spans

$X_\bullet$ category object in $\mathfrak{C}$

$\rightsquigarrow$ X_1 is an algebra in $\mathfrak{Span}(\mathfrak{C})$ with

- unit given by
$$* \longleftarrow X_0 \xrightarrow{s_0} X_1$$
- multiplication
$$X_1 \times X_1 \hookleftarrow X_1 \times_{X_0} X_1 \xleftarrow[\simeq]{(d_2, d_0)} X_2 \xrightarrow{d_1} X_{\{0,2\}} = X_1$$

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Problem: Not all (CY) algebras arise this way: (d_2, d_0) isn't always an iso

2-Segal objects

Definition

A 2-Segal object in $\mathfrak{C}$ is $X_\bullet\colon \Delta^{\mathrm{op}} \to \mathfrak{C}$ such that for any $N \geqslant 3$,

$$X_n \xrightarrow{\simeq} X_{\{0,1,2\}} \underset{X_{\{0,2\}}}{\times} \cdots \underset{X_{\{0,n-2\}}}{\times} X_{\{0,n-2,n-1\}} \underset{X_{\{0,n-1\}}}{\times} X_{\{0,n-1,n\}}$$

and

$$X_n \xrightarrow{\simeq} X_{\{0,1,n\}} \underset{X_{\{1,n\}}}{\times} \cdots \underset{X_{\{n-3,n\}}}{\times} X_{\{n-3,n-2,n\}} \underset{X_{\{n-2,n\}}}{\times} X_{\{n-2,n-1,n\}}$$

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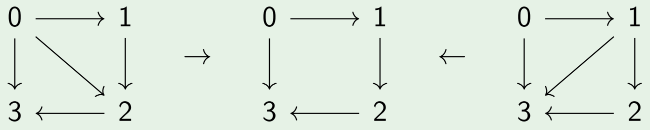
For $N = 3$



The algebra of 2-Segal objects

Theorem (Penney, Gal–Gal, Stern)

There is an equivalence of ∞-categories between (cyclic) 2-Segal objects in $\mathfrak{C}$ and (CY) algebras in $\mathfrak{Span}(\mathfrak{C})$.

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Lemma (Calaque–K.)

For any cyclic 2-Segal $X_\bullet$, the 1-Segal map $X_2 \xrightarrow{(d_2,d_0)} X_{\{0,1\}} \times_{X_{\{1\}}} X_{\{1,2\}}$ admits a section.

Proof.

$$\gamma\colon X_{\{0,1\}} \times_{X_{\{1\}}} X_{\{1,2\}} \xrightarrow{s_1 \times_{s_0} (\tau^2 \circ s_0)} X_{\{0,1,3\}} \times_{X_{\{1,3\}}} X_{\{1,2,3\}} \xrightarrow{(d_2,d_0)^{-1}} X_{\{0,1,2,3\}} \xrightarrow{d_3} X_{\{0,1,2\}}$$

□

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$$\left(0 \xrightarrow{f} 1, 1 \xrightarrow{g} 2\right) \mapsto \left(\begin{array}{ccc} 0 & \xrightarrow{f} & 1 \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \mathrm{id}} & \\ 3 & & \end{array} , \begin{array}{ccc} & & 1 \\ & \swarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle g} \\ 3 & \xleftarrow{g^{-1}} & 2 \end{array} \right) \mapsto \begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 3 & \longleftarrow & 2 \end{array} \mapsto \begin{array}{ccc} 0 & \xrightarrow{f} & 1 \\ \downarrow & \searrow{\scriptstyle gf} & \downarrow{\scriptstyle g} \\ 3 & \longleftarrow & 2 \end{array}$$

□

Refining the algebra structure

γ section of $(d_2, d_0) \implies X_\bullet$ is 1-Segal iff $\gamma \circ (d_2, d_0) = \mathrm{id}_{X_2}$

Problem: 1-Segal condition is $X_1 \times_{X_0} X_1 \xleftarrow{\simeq} X_2$, but algebra only knows $X_1 \times X_1 \leftarrow X_2$

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Requires: know that X_1 lives over X_1, in two different ways d_1, d_0

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$\rightsquigarrow$ Composition in an $(\infty, 2)$-category (rather, double ∞-category) where $X_0 \leftarrow X_1 \rightarrow X_0$ are 1-arrows

Double ∞-category $\mathbb{S}\mathrm{pan}_1^+(\mathfrak{dSt})$ of spans

$$\begin{array}{ccccc} X_0 & \longleftarrow & X_1 & \longrightarrow & X_0' \\ \downarrow & & \downarrow & & \downarrow \\ Y_0 & \longleftarrow & Y_1 & \longrightarrow & Y_0' \end{array}$$

Composition

$$\begin{array}{ccccccc} & & & X_1 \times_{X_0'} X_1' & & & \\ & & \swarrow & \lrcorner & \searrow & & \\ & X_1 & & & & X_1' & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ X_0 & & & X_0' & & & X_0'' \end{array}$$

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Double ∞-category $\mathbb{S}\mathrm{pan}_2(\mathfrak{dSt})$ of iterated spans

$$\begin{array}{ccccc} X_0 & \longleftarrow & X_1 & \longrightarrow & X_0' \\ \uparrow & & \uparrow & & \uparrow \\ Z_0 & \longleftarrow & Z_1 & \longrightarrow & Z_0' \\ \downarrow & & \downarrow & & \downarrow \\ Y_0 & \longleftarrow & Y_1 & \longrightarrow & Y_0' \end{array}$$

Composition

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Monads in double categories

A monad in a double ∞-category $\mathfrak{K}$ is: horizontal endomorphism $X_0 \xrightarrow{t} X_0$ with cells

$$\begin{array}{ccc} X_0 & \xrightarrow{\mathrm{id}_{X_0}} & X_0 \\ \| & \Downarrow \eta & \| \\ X_0 & \xrightarrow[t]{} & X_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} X_0 & \xrightarrow{t \circ t} & X_0 \\ \| & \Downarrow \mu & \| \\ X_0 & \xrightarrow[t]{} & X_0 \end{array} \quad \text{associative and unital}$$

$\Longrightarrow$ $=$ monoid in $\hom^{\mathrm{hor}}(X_0, X_0)$

Theorem (Dyckerhoff–Kapranov)

Every 2-Segal object $X_\bullet$ in $\mathfrak{C}$ gives rise to a monad $\mathcal{H}(X_\bullet)$ in $\mathbb{Span}_2(\mathfrak{C})$

Triple ∞-category of iterated spans

Questions

1. What about going back, from algebras in spans to 2-Segal objects?
2. How to understand the 1-Segal condition?
3. What about morphisms?

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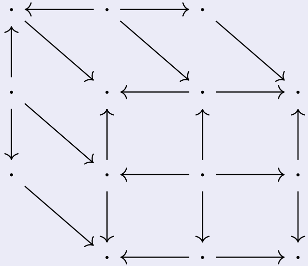
The triple ∞-category $\mathbb{S}\mathrm{pan}_2^+(\mathfrak{dSt})$

Now 3 directions for

arrows: horizontal, vertical, transversal

cells: horizontal, vertical, basic

$+$ cubes (maps between iterated spans)



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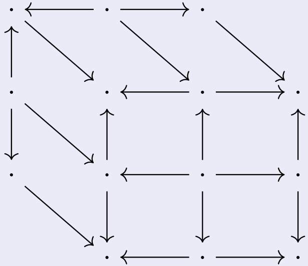
$\Longrightarrow$ Basic monads recover the Theorem



The joys of companionship

Companion pairs in $\mathbb{Span}_2^+(\mathfrak{dSt})$

The basic cell
$$\begin{array}{ccccc} A & \longleftarrow & C & \longrightarrow & B \\ \| & & \| & & \| \\ A & \longleftarrow & C & \longrightarrow & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ X & \longleftarrow & Z & \longrightarrow & Y \end{array}$$

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is characterised as companion to
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Universal characterisation (adjunction-style)

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Universal characterisation (adjunction-style)

First consequence

Unit cell of $\mathcal{H}(X_\bullet)$ is:
$$\begin{array}{ccccc} X_0 & = & X_0 & = & X_0 \\ \| & & \| & & \| \\ X_0 & = & X_0 & = & X_0 \\ \| & & \downarrow{\scriptstyle s_0} & & \| \\ X_0 & \longleftarrow & X_1 & \longrightarrow & X_0 \end{array}$$
$\rightsquigarrow$ must be a companion

Companion cells and the 1-Segal condition

Observation

Considering only the companion (basic) cells in $\mathbb{Span}_2^+(\mathfrak{C})$ recovers $\mathbb{Span}_1^+(\mathfrak{C})$

Companion cells and the 1-Segal condition

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Considering only the companion (basic) cells in $\mathbb{S}\mathrm{pan}_2^+(\mathfrak{C})$ recovers $\mathbb{S}\mathrm{pan}_1^+(\mathfrak{C})$

Corollary

Gregarious monads (structure cells are companions) in $\mathbb{S}\mathrm{pan}_2^+(\mathfrak{C})$ are monads in $\mathbb{S}\mathrm{pan}_1^+(\mathfrak{C})$ (Note: gregarious in $\mathbb{S}\mathrm{pan}_2^+(\mathfrak{C}) \iff$ 1-Segal condition):

$$\begin{array}{ccccc} X_0 & = & X_0 & = & X_0 \\ \| & & \| & & \| \\ X_0 & = & X_0 & = & X_0 \\ \| & & \downarrow s_0 & & \| \\ X_0 & \leftarrow & X_1 & \rightarrow & X_0 \end{array} \qquad \begin{array}{ccccc} X_0 & \leftarrow & X_1 \times_{X_0} X_1 & \rightarrow & X_0 \\ \| & & \uparrow & & \| \\ X_0 & \longleftarrow & X_2 & \longrightarrow & X_0 \\ \downarrow & & \downarrow d_1 & & \downarrow \\ X_0 & \longleftarrow & X_1 & \longrightarrow & X_0 \end{array}$$

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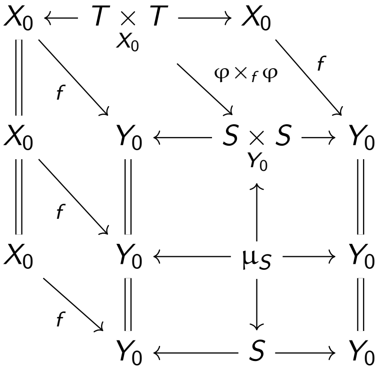
Lemma (Haugseng)

Monads in $\mathbb{S}\mathrm{pan}_1^+(\mathfrak{C})$ are categories in $\mathfrak{C}$

Likewise: CY monads in $\mathbb{S}\mathrm{pan}_2^+(\mathfrak{C})$ are cyclic 2-Segal objects, and groupoids iff the structure cells are companions (*i.e.* satisfy 1-Segal)

Orthomorphisms and isotropic structures

An orthomorphism of (basic) monads is a transversal morphism between them



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Proposition

Orthomorphisms between (CY) monads in $\mathbb{S}\mathrm{pan}_2^+(\mathfrak{C})$ correspond to morphisms between (cyclic) 2-Segal objects: equivalence $\mathfrak{Mnd}_\perp^{\mathrm{CY}}(\mathbb{S}\mathrm{pan}_2^+(\mathfrak{C})) \simeq 2\text{-}\mathfrak{Seg}^{S^1}(\mathfrak{C})$

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Lemma

For any monoid A in $\mathfrak{C}$, equivalence $\mathfrak{Mnd}_\perp^{\mathrm{CY}}(\mathbb{S}\mathrm{pan}_2^+(\mathfrak{C}_{/A})) \xrightarrow{\simeq} \mathfrak{Mnd}_\perp^{\mathrm{CY}}(\mathbb{S}\mathrm{pan}_2^+(\mathfrak{C}))_{/A}$

Taking $A = \mathscr{A}^{2,\mathrm{cl}}(n)$, get: n-shifted isotropic groupoids $\simeq \mathfrak{Mnd}_\perp^{\mathrm{CY,gr.}}(\mathbb{S}\mathrm{pan}_2^+(\mathfrak{dSt}_{/\mathscr{A}^{2,\mathrm{cl}}(n)}))$

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Conclusion: $\mathfrak{Grpd}(\mathfrak{dSt}) \simeq \mathfrak{Mnd}_\perp^{\mathrm{CY,gr.}}(\mathbb{S}\mathrm{pan}_2^+(\mathfrak{dSt})) \xrightarrow{(T^\vee[n], N^\vee[n])} \mathfrak{Mnd}_\perp^{\mathrm{CY,gr.}}(\mathbb{S}\mathrm{pan}_2^+(\mathfrak{dSt}_{/\mathscr{A}^{2,\mathrm{cl}}(n)})) \simeq \mathfrak{Grpd}(\mathfrak{dSt})_{/\mathrm{Bar}_\bullet \mathscr{A}^{2,\mathrm{cl}}(n)}$

Backup

Shifting and delooping for presymplectic forms

Lemma (looping-delooping)

$\Omega_0\mathscr{A}^{2,\mathrm{cl}}(n+1) \coloneqq * \underset{\mathscr{A}^{2,\mathrm{cl}}(n+1)}{\times} * \simeq \mathscr{A}^{2,\mathrm{cl}}(n)$, and conversely $\mathrm{Bar}_\bullet\, \mathscr{A}^{2,\mathrm{cl}}(n)$ presents $\mathscr{A}^{2,\mathrm{cl}}(n+1)$

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$G_\bullet$ n-shifted presymplectic groupoid $\implies$ $(n+1)$-shifted isotropic $G_0 \to |G_\bullet|$

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Fact: "Quotient" $\varinjlim: \mathfrak{C}^{\Delta^{\mathrm{op}}} \to \mathfrak{Ar}(\mathfrak{C}) = \mathfrak{C}^{\to}$ is left-adjoint to $\ker_\bullet : \mathfrak{Ar}(\mathfrak{C}) \to \mathfrak{C}^{\Delta^{\mathrm{op}}}$

Observation: $\ker_\bullet(* \to \mathscr{A}^{2,\mathrm{cl}}(n+1)) \simeq \mathrm{Bar}_\bullet \mathscr{A}^{2,\mathrm{cl}}(n)$

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Observation: $\ker_\bullet(* \to \mathscr{A}^{2,\mathrm{cl}}(n+1)) \simeq \mathrm{Bar}_\bullet \mathscr{A}^{2,\mathrm{cl}}(n)$

$\implies$ $G_\bullet \to \mathrm{Bar}_\bullet \mathscr{A}^{2,\mathrm{cl}}(n)$ corresponds to morphism of quotient maps

$$\begin{array}{ccc} G_0 & \twoheadrightarrow & |G_\bullet| \\ \downarrow & & \downarrow \ulcorner\omega\urcorner \\ * & \twoheadrightarrow & \mathscr{A}^{2,\mathrm{cl}}(n+1) \end{array}$$

$\square$

Non-degeneracy for isotropic correspondences

Classical Lagrangians

$$\begin{array}{ccccccccc}
 & & \mathscr{T}_{L/X} & \rightarrowtail & \mathscr{T}_L & & & & \\
 & & {\scriptstyle \simeq}\,\Big\downarrow & & \Big\downarrow {\scriptstyle \omega|_{\mathscr{T}_L}} & \searrow^{\omega|_L = 0} & & & \\
0 & \longrightarrow & \mathscr{N}_L^{\vee} & \rightarrowtail & \mathscr{T}_X^{\vee}|_L & \twoheadrightarrow & \mathscr{T}_L^{\vee} & \longrightarrow & 0
\end{array}$$

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$$
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0 & \longrightarrow & \mathcal{N}_L^\vee & \rightarrowtail & \mathcal{T}_X^\vee|_L & \twoheadrightarrow & \mathcal{T}_L^\vee & \longrightarrow & 0
\end{array}
$$

For shifted isotropic correspondences

$$
\begin{array}{ccc}
L & \xrightarrow{f} & X \\
{\scriptstyle g}\downarrow & & \downarrow \\
Y & \longrightarrow & \mathcal{A}^{2,\mathrm{cl}}(n)
\end{array}
\quad \text{Lagrangian if} \quad
\begin{array}{ccc}
\mathbb{T}_L & \longrightarrow & f^*\mathbb{T}_X \simeq f^*\mathbb{L}_X[n] \\
\downarrow & & \downarrow \\
g^*\mathbb{T}_Y \simeq g^*\mathbb{L}_Y[n] & \longrightarrow & \mathbb{L}_L[n]
\end{array}
\quad \text{cocartesian}
$$

Link with classical symplectic groupoids

Lemma (Calaque–Safronov)

$G_\bullet$ n-shifted symplectic groupoid. Then the n-presymplectic structure on G_1 is symplectic.

Proof.

$G_0 \to |G_\bullet|$ n-Lagrangian $\implies$ $G_1 \simeq G_0 \times_{|G_\bullet|} G_0$ n-symplectic $\square$

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Proposition (Calaque–Safronov)

$G_\bullet$ n-presymplectic groupoid, and suppose we know G_1 is n-symplectic. TFAE:

1. $\gamma_0 \colon * \leftarrow G_0 \to G_1$ is non-degenerate (*i.e.* Lagrangian),
2. $\gamma_2 \colon G_1^2 \leftarrow G_2 \to G_1$ is non-degenerate,
3. all the γ_k are non-degenerate.