SAG *v.* DGA

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1 Smoothness vs smoothness

Definition 1.1 (Strong modules). *Let A be a connective $\mathscr{E}_\infty$-ring spectrum. An A-module M is **strong** if for every $n \in \mathbb{Z}$, the canonical map $\pi_0(M) \otimes_{\pi_0 R} \pi_n(R) \to \pi_n(M)$ is an isomorphism of abelian groups.*

*An A-algebra $A \to B$ is **strong** if B is strong as an A-module.*

Recall that we have the following:

Lemma 1.2 ([HA, Remark 7.2.4.22]). *A connective A-module M has Tor-amplitude 0 if and only if M is strong and $\pi_0 M$ is (classically) flat.*

Lemma 1.3 ([SAG, Lemma B.1.3.3]). *Let B be an A-algebra of finite presentation. Then $\mathbb{L}_{B/A} = 0$ if and only if B is strong and $\pi_0 A \to \pi_0 B$ is (classically) étale.*

This suggests a first definition of smoothness.

Definition 1.4 (Fibre smoothness). *A morphism $\varphi \colon A \to B$ of connective $\mathscr{E}_\infty$-ring spectra is **fibre smooth** if it is strong and $\pi_0\varphi$ is classically smooth.*

The name is justified by the following result, which shows that fibre smoothness can be expressed as a condition on geometric fibres.

Proposition 1.5 ([SAG, Proposition 11.2.3.6]). *A morphism $\varphi \colon A \to B$ almost of finite presentation is fibre smooth if and only if for every geometric κ-point $A \to \kappa$, with κ an algebraically closed field, $\kappa \otimes_A B$ is a truncated ring which is (classically) regular.*

The classical theory of smoothness suggests that we should understand this definition in terms of lifting against nilpotent closed immersions.

Proposition 1.6 ([SAG, Corollary 11.2.4.2]). *Let $\varphi \colon A \to B$ be a flat morphism of connective $\mathscr{E}_\infty$-ring spectra such that $\pi_0 B$ is finitely presented over $\pi_0 A$. The following are equivalent:*

- *φ is fibre smooth,*

- *for every surjection of (truncated) commutative rings* $R \to \overline{R}$ *with nilpotent kernel, every solid diagram*

$$\begin{array}{ccc} A & \longrightarrow & R \\ \downarrow & \nearrow & \twoheaddownarrow \\ B & \longrightarrow & \overline{R} \end{array} \tag{1}$$

  admits a dashed lift.

It thus appears that the notion of fibre-smoothness is not fully satisfying, in that it is not derived enough to capture non-truncated nilpotent quotients. In fact, we can quantify more precisely the failure to see non-truncated structure.

Lemma 1.7 ([Stacks, Tag `07BU` and Tag `00TC`] or [SAG, Proposition 11.2.4.1])**.** *Let* A *and* B *be classical commutative rings, and* $A \xrightarrow{\varphi} B$ *a map of finite presentation. The following are equivalent:*

- φ *is smooth*
- *for every prime* $\mathfrak{p}$ *of* B, $\pi_1(\mathbb{L}_{B/A})_{\mathfrak{p}} = 0$ *and* $\Omega^1_{B/A,\mathfrak{p}} \simeq \pi_0(\mathbb{L}_{B/A})_{\mathfrak{p}}$ *is a (classically) projective (iff finite free, iff flat)* $B_{\mathfrak{p}}$*-module,*
- $\tau_{\leq 1}\mathbb{L}_{B/A}$ *is equivalent to a finite (classically) projective* B*-module in degree* 0*.*

Thus, the notion of fibre smoothness only considers the 1-truncation of the cotangent complex, not all of its degrees. To obtain a fully "derived" or spectral notion, it thus seems natural to replace the projectivity hypothesis on $\Omega^1_{B/A}$ by one on $\mathbb{L}_{B/A}$.

Lemma 1.8 ([SAG])**.** *For* $A \to B$ *be a morphism of connective* $\mathscr{E}_\infty$*-ring spectra, the following are equivalent:*

- *the cotangent complex* $\mathbb{L}_{B/A}$ *is projective (as a* connective B*-module),*
- *every lifting problem such as in eq.* (1) *but with* $R \to \overline{R}$ *any map of connective* $\mathscr{E}_\infty$*-ring spectra inducing on* π_0 *a surjection with nilpotent kernel admits a solution.*

Definition 1.9 (Differential smoothness)**.** *A map satisfying the equivalent conditions above is said to be* ***formally differentially smooth****. A map* f *is* ***differentially smooth*** *if it is formally differentially smooth and* $\pi_0 f$ *is finitely presented.*

Remark 1.10. If φ is differentially smooth, its cotangent complex (which, by definition, is projective) has finite rank. This is because $\pi_0\mathbb{L}_\varphi$ is $\Omega^1_{\pi_0\varphi}$, which is finitely presented.

Remark 1.11 ([SAG, Remark 11.2.2.3]). A map φ is differentially smooth if and only if it is locally (or equivalently, just almost) of finite presentation and $\mathbb{L}_\varphi$ is a flat module (*i.e.*, of Tor-amplitude 0). This justifies the mild generalisation of **quasismooth** maps as those whose cotangent complex has Tor-amplitude concentrated in $[0, 1]$.

Example 1.12 ([SAG, Proposition 11.2.4.4.]). If A is a $\mathbb{Q}$-algebra (which is equivalent to $\pi_0 A$ being a $\mathbb{Q}$-algebra), an A-algebra is fibre smooth if and only if it is differentially smooth.

Example 1.13 (Standard smooth maps). The canonical map $A \to A\{x_1, \dots, x_n\} = \mathrm{Sym}_A(A^n)$ is differentially smooth; indeed $\mathbb{L}_{A\{x_1,\dots,x_n\}/A} \simeq A^n$.

Proposition 1.14 ([SAG, Proposition 11.2.2.1]). *A finitely presented morphism $A \to B$ of connective $\mathscr{E}_\infty$-ring spectra is differentially smooth if and only if there exists a collection of elements $b_1, \dots, b_k \in \pi_0 B$ generating its unit ideal, and étale maps $A\{x_1, \dots, x_{n_i}\} \to B[b_i^{-1}]$ (so a factorisation of $A \to B[b_i^{-1}]$ as a composition of a standard smooth map and an étale map).*

We now globalise these notions.

Lemma 1.15 ([SAG, Propositions 11.2.5.1–11.2.5.4]). *The conditions of being fibre smooth and differentially smooth are étale-local on the (geometric) source and fppf-local on the (geometric) target.*

Definition 1.16. *A morphism $f\colon X \to Y$ of spectral Deligne–Mumford stack is **differentially smooth** (resp. **fibre smooth**) if for any commutative square*

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t}\, B & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Sp\acute{e}t}\, A & \longrightarrow & Y \end{array} \tag{2}$$

in which the horizontal maps are étale, the map $g^\sharp\colon A \to B$ is a differentially smooth (resp. fibre smooth) map of connective $\mathscr{E}_\infty$-ring spectra.

Finally, we can see that the globalised notions of smoothness are compatible with the global cotangent complex.

Theorem 1.17 ([SAG, Proposition 17.1.5.1, Proposition 17.3.9.4]). *A morphism $f\colon X \to Y$ of spectral Deligne–Mumford stack such that $\pi_0 f$ is (classically) locally of finite presentation is **differentially smooth** (resp. **fibre smooth**) if and only if it satisfies any (equivalently, all) of the following equivalent conditions:*

1. *the global cotangent complex $\mathbb{L}_{X/Y}$ is locally free of finite rank (resp., f is flat and $\pi_1(\mathbb{L}_{X/Y}|_{\mathrm{Sp\acute{e}t}\,\kappa}) \simeq 0$ for any κ-point of X),*
2. *for any connective $\mathscr{E}_\infty$-ring spectrum (resp., any classical commutative ring) A and any A point $\xi\colon \mathrm{Sp\acute{e}t}\, A \to X$, $\xi^*\mathbb{L}_{X/Y}$ is a projective A-module (resp., $\tau_{\leq 1}\xi^*\mathbb{L}_{X/Y}$ is projective),*
3. *for any connective $\mathscr{E}_\infty$-ring spectrum (resp., any classical commutative ring) A and any square-zero extension A^η of A by a connective (respr., truncated) A-module, any solid*

lifting problem

$$\begin{array}{ccc} \operatorname{Spét} A & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow \\ \operatorname{Spét} A^{\eta} & \longrightarrow & Y \end{array} \tag{3}$$

admits a dashed solution.

2 Affine spaces and projectives spaces

Definition 2.1. *Let* M *be an* $\mathscr{E}_\infty$*-monoid in types. For any* $\mathscr{E}_\infty$*-ring spectrum* R*, we set* $R^{[M]} := R \otimes_{\mathbb{S}} \Sigma^\infty M$*, endowed with its structure of cogebra in* R*-algebras, which is cogrouplike if and only if* M *is grouplike.*

Example 2.2. Let $\mathbb{F}$ be the free monoid on one generator, and $\mathbb{F} \to \daleth$ its group completion.

We have

$$\mathbb{F} \simeq \mathfrak{FinSet}^{\simeq} \simeq \coprod_{n \in \mathbb{N}} \mathcal{B}\,\mathfrak{S}_n \tag{4}$$

and, by the Barratt–Priddy–Quillen theorem,

$$\daleth \simeq \coprod_{n \in \mathbb{Z}} \mathcal{B}\,\mathfrak{S}_\infty. \tag{5}$$

Note that, by formal nonsense, we also have $\daleth \simeq \Omega^\infty \mathbb{S}$.

By combining universal properties, it is clear that $R^{[\mathbb{F}]} \simeq R\{t\} = \mathrm{Sym}_R(R)$ is the **free** R-algebra on one generator. Likewise, $R^{[\mathbb{F}^n]} \simeq R\{t_1, \dots, t_n\} = \mathrm{Sym}_R(R^{\oplus n})$. Concomitantly, one can see (*cf.* [Gre17, Proposition 2.1.7]) that $R^{[\daleth^n]} =: R\{t_1^{\pm 1}, \dots, t_n^{\pm 1}\}$ is a localisation of $R\{t_1, \dots, t_n\}$ at the elements $t_1, \dots, t_n \in \pi_0(R\{t_1, \dots, t_n\})$.

Example 2.3. Let $\mathbb{N}$ be the free 0-truncated monoid on one generator, and $\mathbb{N} \to \mathbb{Z}$ its group completion. We have $\mathbb{N} = \tau_{\leq 0}\mathbb{F} = \pi_0\mathbb{F}$ and $\mathbb{Z} = \tau_{\leq 0}\daleth = \pi_0\daleth$ so that eq. (4) and eq. (5) give $\mathbb{N} \simeq \coprod_{n \in \mathbb{N}} *$ and $\mathbb{Z} \simeq \coprod_{n \in \mathbb{Z}} *$.

We then define, for any $\mathscr{E}_\infty$-ring spectrum R, the **polynomial** R-algebra on n generators as $R[t_1, \dots, t_n] := R^{[\mathbb{N}^n]}$, and likewise $R[t_1^{\pm}, \dots, t_n^{\pm}] := R^{[\mathbb{Z}^n]}$.

Definition 2.4. *Let* R *be an* $\mathscr{E}_\infty$*-ring spectrum. The* ***spectral affine*** n***-space*** *over* R *is* $\mathbb{A}^n_{R,\ghost} = \operatorname{Spét} R\{t_1, \dots, t_n\}$*. The* ***flat affine*** n***-space*** *over* R *is* $\mathbb{A}^n_{R,\flat} = \operatorname{Spét} R[t_1, \dots, t_n]$*.*

Remark 2.5. Expanding out the definition, we have that for any R-algebra A,

$$\mathbb{A}^n_{R,\ghost}(T) = \hom_{R\text{-}\mathfrak{Alg}}(R\{t_1, \dots, t_n\}, A) \simeq \hom_{R\text{-}\mathfrak{Mod}}(R^n, A) \simeq (\Omega^\infty A)^n, \tag{6}$$

recovering the expected definition of the functor of points of affine n-space. Meanwhile, $\mathbb{A}^1_{R,\flat}(A) \simeq \hom_{\mathscr{E}_\infty\text{-}\mathfrak{Alg}(\infty\text{-}\mathfrak{Grpd})}(\mathbb{N}, (\Omega^\infty A, \times))$ will be interpreted as the type of **strictly commutative** elements of $\Omega^\infty A$.

Proposition 2.6. *$\mathbb{A}^n_{R,\text{👻}}$ is differentially smooth over* $\operatorname{Spét} R$*, but not flat unless* R *is a* $\mathbb{Q}$*-algebra, and* $\mathbb{A}^n_{R,\flat}$ *is fibre smooth over* $\operatorname{Spét} R$ *but not differentially smooth unless* R *is a* $\mathbb{Q}$*-algebra.*

Proof. Differential smoothness of $\mathbb{A}^n_{R,\text{👻}}$ is established easily from the fact (proved in [HA, Proposition 7.4.3.14] by comparing the universal properties) that the cotangent complex of $R \to \operatorname{Sym}_R(M)$ is $M \otimes_R \operatorname{Sym}_R(M)$.

For $\mathbb{A}^n_{R,\flat}$, we use the fact that $R[t_1, \ldots, t_n] \simeq \bigoplus_{n\in\mathbb{N}} R^n$, from which strongness is easily established, and then notice that $\pi_0(R[t_1, \ldots, t_n]) \simeq (\pi_0 R)[t_1, \ldots, t_n]$. □

Remark 2.7. As shown in [BM04, Theorem 5] (*cf.* also [Gre17, Lemma 2.2.6] for a modern proof) For any $\mathscr{E}_\infty$-monoid M, and any $\mathscr{E}_\infty$-ring spectrum R, we have $\mathbb{L}_{R[M]/M} \simeq \mathcal{B}^\infty M^{\mathrm{grp}} \otimes_{\mathbb{S}} R^{[M]}$ where $\mathcal{B}^\infty \colon \mathscr{E}_\infty\text{-}\mathfrak{Alg}(\infty\text{-}\mathfrak{Grpd})^{\mathrm{gp}} \simeq \mathfrak{Sp}^{\mathrm{cn}}$ is inverse to Ω^∞.

Definition 2.8. *We set* $\mathbb{G}_{m,R,\text{👻}} = \operatorname{Spét}(R\{t^{\pm 1}\})$ *and* $\mathbb{G}_{m,R,\flat} = \operatorname{Spét}(R[t^{\pm 1}])$*.*

Remark 2.9. For any R-algebra A, we have $\mathbb{G}_{m,R,\text{👻}}(A) = \Omega^\infty A \times_{\pi_0 A} \pi_0 A^\times$.

Definition 2.10. *The* ***spectral projective*** n***-space*** *is* $\mathbb{P}^n_{R,\text{👻}} = (\mathbb{A}^{n+1}_{R,\text{👻}} \setminus \{0\})/\mathbb{G}_{m,R,\text{👻}}$*, and the* ***flat projective*** n***-space*** *is* $\mathbb{P}^n_{R,\flat} = (\mathbb{A}^{n+1}_{R,\flat} \setminus \{0\})/\mathbb{G}_{m,R,\flat}$*.*

Proposition 2.11 ([Gre17, Theorem 2.6.18, Proposition 2.8.16])**.** *The projective spaces admit the usual atlases by affine spaces.*

In particular, $\mathbb{P}^n_{R,\text{👻}}$ *and* $\mathbb{P}^n_{R,\flat}$ *are spectral algebraic spaces.*

Proposition 2.12 ([SAG, Theorem 19.2.6.2, Remark 19.2.6.4])**.** *For any* $\mathscr{E}_\infty$*-ring* A*,* $\mathbb{P}^n_{R,\text{👻}}(A)$ *can be described as the equivalent types:*

1. *the core of the subcategory of* $A\text{-}\mathfrak{Mod}_{/A^{n+1}}$ *on those* $L \to A^{n+1}$ *admitting a retraction and such that* $\operatorname{Sym}_A L$ *is a line bundle (equivalently,* L *projective of rank* 1*)*
2. *the type of (naturally* $\mathbb{T}$*-graded) line bundles* $E = \operatorname{Sym}_A L$ *(so locally equivalent to* $A\{t\}$*) on* $\operatorname{Spét} A$ *with an* $\mathbb{T}$*-equivariant map* $A\{t_0, \ldots, t_n\} \to E$ *which is surjective on* π_0*.*

Proposition 2.13 ([Gre17, Remark 2.8.18])**.** *For any* $\mathscr{E}_\infty$*-ring* A*,* $\mathbb{P}^n_{R,\flat}(A)$ *is the type of (naturally* $\mathbb{Z}$*-graded) flat line bundles* E *(locally equivalent to* $A[t]$*) with a* $\mathbb{Z}$*-equivariant map of algebras* $A[t_0, \ldots, t_n] \to E$ *which is surjective on* π_0*.*

3 Derived algebraic geometry

3.1 Animation

Definition 3.1.1 (Animation)**.** *Let* $\mathfrak{C}$ *be a cocomplete* 1*-category generated under* 1*-colimits (so, equivalently by [ČS19, (5.1.1.1)], under sifted* 1*-colimits) by its subcategory* $\mathfrak{C}^{\mathrm{sfp}}$ *of objects strongly of finite presentation (aka compact projective). The* ***animation*** *of* $\mathfrak{C}$*, denoted* $\mathfrak{Ani}(\mathfrak{C})$*, is the* ∞*-category freely generated under sifted colimits by* $\mathfrak{C}^{\mathrm{sfp}}$*.*

Explicitly, this means that, for any ∞-category $\mathfrak{D}$ with sifted colimits, an ∞-functor $\mathscr{F}\colon \mathfrak{C}^{\mathrm{sfp}} \to \mathfrak{D}$ determines an essentially unique sifted colimits-preserving ∞-functor $\mathbb{L}\mathscr{F}\colon \mathfrak{Ani}(\mathfrak{C}) \to \mathfrak{D}$, which restricts to $\mathscr{F}$ on $\mathfrak{C}^{\mathrm{sfp}} \subset \mathfrak{Ani}(\mathfrak{C})$.

Remark 3.1.2 (Why sifted colimits?). Recall that an **algebraic theory** (or **Lawvere theory**) is a 1-category $\mathfrak{T}$ with finite products, and a **model** of $\mathfrak{T}$ is a product-preserving functor $\mathfrak{T} \to \mathfrak{Set}$. Then, by [ARV10, Theorem 4.13], the category of models of $\mathfrak{T}$ is the sifted colimits completion of $\mathfrak{T}^{\mathrm{op}}$.

Thus, the idea of animation is that if we can present $\mathfrak{C}$ is the category of models of the algebraic theory $\mathfrak{C}^{\mathrm{sfp}}$, then $\mathfrak{Ani}(\mathfrak{C})$ is the ∞-category of models of $\mathfrak{C}^{\mathrm{sfp}}$ seen as an "algebraic ∞-theory".

Proposition 3.1.3 (Quillen, Bergner, [HTT, Corollary 5.5.9.3])**.** *Let $\mathfrak{C}$ be as above and such that $\mathfrak{C}^{\mathrm{sfp}}$ admits finite products. Then $\mathfrak{Ani}(\mathfrak{C})$ can be modelled (through an appropriate model structure) by the category of finite product-preserving functors $\mathfrak{C}^{\mathrm{sfp,op}} \to \mathfrak{sSet}$ (so, of simplicial objects in $\mathfrak{C}$).*

The construction of the animation of $\mathfrak{C}$ thus recovers Quillen's definition of the non-abelian derived ∞-category of $\mathfrak{C}$ (and the model structure on $\mathfrak{sC}$ is induced from the Kan–Quillen model structure on $\mathfrak{sSet}$ through the monadic functor $\mathfrak{C} \to \mathfrak{Set}$ from viewing $\mathfrak{C}$ as a category of models of a Lawvere theory).

Example 3.1.4.

- The category of sets is generated under sifted 1-colimits by the finite sets. Its animation is the ∞-category $\mathfrak{Ani}(\mathfrak{Set}) \simeq \infty\text{-}\mathfrak{Grpd}$ of ∞-groupoids or types (thus also known as animated sets, or simply "anima").
- The animation of the category of groups (whose strongly finitely presented objects are the free groups on finite sets) is equivalent to the ∞-category of grouplike $\mathscr{E}_1$-monoids in types. However, the animation of the category of *abelian* groups, which through the Dold–Kan correspondence is equivalent to the connective derived ∞-category of $\mathbb{Z}$, is *not* equivalent to grouplike $\mathscr{E}_\infty$-monoids in $\infty\text{-}\mathfrak{Grpd}$ (as the latter is the ∞-category of connective $\mathbb{S}$-modules).
- The ∞-category $\mathfrak{Ani}(\mathfrak{Ring})$ of **animated rings** is, by definition, the animation of the category of rings (whose strongly finitely presented objects are the retracts of finite type polynomial $\mathbb{Z}$-algebras). Since every retract of a polynomial algebra is in particular a quotient, and thus a sifted colimit, of polynomial algebras, we will generally see $\mathfrak{Ani}(\mathfrak{Ring})$ as the sifted colimits completion of the category $\mathfrak{Poly}$ of finite type polynomial $\mathbb{Z}$-algebras.

Lemma 3.1.5 ([SAG, Corollary 25.1.4.3])**.** *For any classical ring* R*, there is an equivalence of ∞-categories* $\mathfrak{Ani}(R\text{-}\mathfrak{Alg}^{\heartsuit}) \simeq \mathfrak{Ani}(\mathfrak{Ring})_{R/}$.

We may thus define the ∞-category of animated R-algebras, for any animated ring R, to be the slice of the ∞-category of animated rings under R.

Remark 3.1.6. The inclusion $\imath\colon \mathfrak{Poly} \to \mathscr{E}_\infty\text{-}\mathfrak{Ring}^{\mathrm{cn}}$ determines an ∞-functor $\theta \coloneqq \mathbb{L}\imath\colon \mathfrak{Ani}(\mathfrak{Ring}) \to \mathscr{E}_\infty\text{-}\mathfrak{Ring}^{\mathrm{cn}}$. The image by θ of an animated ring A will be denoted A° and called the **underlying $\mathscr{E}_\infty$-ring spectrum** of A.

Notation 3.1.7. For A an animated ring, we let $A\text{-}\mathfrak{Mod}$ be the ∞-category of A°-modules.

Remark 3.1.8. Any animated ring A can be seen in particular as an $\mathscr{E}_1$-algebra in animated abelian groups, and the connective part of $A\text{-}\mathfrak{Mod}$ coincides with the ∞-category of "animated" A-modules in this sense.

If A is a classical (truncated) ring, then $A\text{-}\mathfrak{Mod}^{cn}$ is equivalent to $\mathfrak{Ani}(A\text{-}\mathfrak{Mod}^{\heartsuit})$.

In fact, the following strenghtening of this Remark will be useful when discussing the algebraic cotangent complex (and, later, symmetric powers).

For this, let $\mathfrak{RingMod}^{\heartsuit}$ be the category of pairs of a ring and a (classical) module over it, and let $\mathfrak{Ani}(\mathfrak{Ring})\mathfrak{Mod}^{cn}$ be the ∞-category of pairs (A, M) where A is an animated ring and M is a connective A-module. Let finally $\mathfrak{RingMod}^{ssfp}$ be the full subcategory of $\mathfrak{RingMod}^{\heartsuit}$ on the pairs (A, M) where A is a polynomial ring of finite type and M a free A-module of finite rank.

Lemma 3.1.9 ([SAG, Proposition 25.2.1.2]). *The objects of* $\mathfrak{RingMod}^{ssfp}$ *provide a set of strongly finitely presented objects generating* $\mathfrak{Ani}(\mathfrak{Ring})\mathfrak{Mod}^{cn}$ *under sifted colimits.*

That is, $\mathfrak{Ani}(\mathfrak{Ring})\mathfrak{Mod}^{cn} \simeq \mathfrak{Ani}(\mathfrak{RingMod}^{\heartsuit})$.

We now turn back to animated rings themselves, and their comparison with $\mathscr{E}_\infty$-ring spectra.

Lemma 3.1.10 ([SAG, Proposition 25.1.5.2]). *Let* R *be a classical commutative ring. The functor* $\mathbb{Z}\text{-}\mathfrak{Mod}^{cn} \to \mathbb{S}\text{-}\mathfrak{Mod}^{cn} \simeq \mathscr{E}_\infty\text{-}\mathfrak{Alg}(\infty\text{-}\mathfrak{Grpd})^{gp} \xrightarrow{R^{[-]}} R\text{-}\mathfrak{Alg}$ *factors as*

$$\mathbb{Z}\text{-}\mathfrak{Mod}^{cn} \xrightarrow{R^{\mathbb{L}[-]}} \mathfrak{Ani}(R\text{-}\mathfrak{Alg}^{\heartsuit}) \xrightarrow{\theta} R\text{-}\mathfrak{Alg} \tag{7}$$

where the functor $R^{\mathbb{L}[-]}$ *commutes with sifted colimits (and in fact with all small colimits).*

This means that whenever a grouplike $\mathscr{E}_\infty$-monoid in types M, seen as a connective spectrum, carries a structure of $\mathbb{Z}$-module, its group R-algebra carries a structure of animated ring: $R^{[M]} \simeq (R^{\mathbb{L}[M]})^\circ$.

Example 3.1.11. Applying this to $\mathbb{Z}$, we find that $\mathbb{G}_{m,R,\flat}$ carries a structure of derived scheme.

Lemma 3.1.12 ([SAG, Remark 25.1.3.6]). *Let* A *be an animated* R*-algebra, for* R *a classical ring. There is an ismorphism* $\pi_\bullet A^\circ \simeq \pi_\bullet \hom(R[t], A)$.

This implies that we may think of $\hom(R[t], A)$ as the **underlying type** of the animated R-algebra A.

Proposition 3.1.13 ([SAG, Propositions 25.1.2.4, 25.1.2.2]). *The* ∞*-functor* $\theta\colon \mathfrak{Ani}(\mathfrak{Ring}) \to \mathscr{E}_\infty\text{-}\mathfrak{Ring}^{cn}$ *is both monadic and comonadic. In particular, it is conservative and preserves both limits and colimits (such as tensor products).*

Over $\mathbb{Q}$*, it is even an equivalence of* ∞*-categories.*

Remark 3.1.14. Write $\mathscr{rect}$ for the right adjoint to θ. Looking through the adjunction, we find that for any $\mathscr{E}_\infty$-ring spectrum A, the underlying type of $\mathscr{rect}(A)$ is $\hom(Z[t], \mathscr{rect}\,A) \simeq \hom(Z[t], A)$, identified in remark 2.5 as the type of **strictly commutative elements** of A.

3.2 The algebraic cotangent complex

Construction 3.2.1. Consider the functor $\mathfrak{RingMod}^{\mathrm{ssfp}} \to \mathfrak{Ring} \hookrightarrow \mathfrak{Ani}(\mathfrak{Ring})$ sending a pair (A, M) to the trivial square-zero extension $A \oplus M$, seen as an animated ring. We denote its left derived ∞-functor $\mathfrak{Ani}(\mathfrak{Ring})\mathfrak{Mod}^{\mathrm{cn}} \to \mathfrak{Ani}(\mathfrak{Ring})$ as $(A, M) \mapsto A \oplus^{\mathbb{L}} M$.

As explained in [SAG, Remark 25.3.1.2], we have for any animated ring A and connective A-module M an equivalence $(A \oplus^{\mathbb{L}} M)^{\circ} \simeq A^{\circ} \oplus M$.

Note that the A-algebra $A \oplus^{\mathbb{L}} M$ is canonically augmented over A.

Definition 3.2.2. *Let* A *be an animated ring. For any connective* A*-module* M*, we denote* $\mathrm{Der}_A(A, M)$ *the type* $\hom_{/A}(A, A \oplus^{\mathbb{L}} M)$ *of* A***-derivations*** *of* A *into* M*.*

An ***algebraic cotangent complex*** *of* A *is an* A*-module* $\mathbb{L}\Omega^1_A$ *corepresenting the functor* $M \mapsto \mathrm{Der}_A(A, M)$*.*

Lemma 3.2.3. *For any animated ring* A*, the functor* $M \mapsto \mathrm{Der}_A(A, M)$ *is corepresentable, that is* A *admits an algebraic cotangent complex.*

Remark 3.2.4 (Explicit construction of the algebraic cotangent complex). Write the animated ring A as the quotient of a simplicial object $\widetilde{A}_{\bullet}$, each of whose term is a polynomial ring of possibly infinite type (in other words, take a quasi-free resolution of A). Then $\mathbb{L}\Omega^1_A$ is the quotient of the simplicial object $A \underset{\widetilde{A}_{\bullet}}{\otimes} \Omega^1_{\widetilde{A}_{\bullet}/\mathbb{Z}}$.

We now wish to relate the algebraic cotangent complex of an animated ring A to the (topological) cotangent complex of its underlying $\mathscr{E}_{\infty}$-ring spectrum A°.

Proposition 3.2.5 ([SAG, Proposition 25.3.5.1])**.** *Let* $\varphi \colon A \to B$ *be a morphism of animated rings, and suppose that there is* $m \geq -1$ *such that* $\mathrm{fib}\,\varphi$ *is* m*-connective. Then* $\mathrm{fib}(\mathbb{L}_{B^{\circ}/A^{\circ}} \to \mathbb{L}\Omega^1_{A/B})$ *is* $(m+3)$*-connective.*

Example 3.2.6. Any map φ is (-1)-connective. It follows that $\mathrm{fib}(\mathbb{L}_{\varphi^{\circ}} \to \mathbb{L}\Omega^1_{\varphi})$ is 2-connective, *i.e.* the difference between the cotangent complexes always lies outside of the "quasi-smooth domain" $[0, 1]$.

Corollary 3.2.7 ([SAG, Variant 25.3.5.2])**.** *For any animated ring* A*,* $\mathrm{fib}(\mathbb{L}_{A^{\circ}/\mathbb{S}} \to \mathbb{L}\Omega^1_{A/\mathbb{Z}})$ *is 2-connective.*

Proof. The comparison map $\mathbb{L}_{A^{\circ}/\mathbb{S}} \to \mathbb{L}\Omega^1_{A/\mathbb{Z}}$ factors as $\mathbb{L}_{A^{\circ}/\mathbb{S}} \xrightarrow{\kappa} \mathbb{L}_{A^{\circ}/\mathbb{Z}} \to \mathbb{L}\Omega^1_{A/\mathbb{Z}}$, and $\mathrm{fib}(\kappa) \simeq A \otimes_{\mathbb{Z}} \mathbb{L}_{\mathbb{Z}/\mathbb{S}}$, which is 2-connective because $\mathrm{cofib}(\mathbb{S} \to \mathbb{Z})$ is 2-connective. □

Theorem 3.2.8 ([Sch01, Theorem 4.4], [SAG, Corollary 25.3.3.3.])**.** *Let* A *be an animated ring. There is an* $\mathscr{E}_1$*-ring spectrum* A^+ *such that* $\mathfrak{Sp}(\mathfrak{Ani}(\mathfrak{Ring})_{/A}) \simeq A^+\text{-}\mathfrak{Mod}$*.*

In particular, $\mathbb{L}_{A^{\circ}/\mathbb{Z}}$ *carries a canonical structure of left* A^+*-module.*

Idea of proof. The functor $\mathfrak{Sp}(\mathfrak{Ani}(\mathfrak{Ring})_{/A}) \to \mathfrak{Sp}(\mathfrak{Sp}) \simeq \mathfrak{Sp}$ induced by the construction $(B \to A) \mapsto \mathrm{fib}(B^{\circ} \to A^{\circ})$ is monadic. We then take A^+ to be the image of $\mathbb{S}$ under the monad in question. □

Proposition 3.2.9 ([SAG, Remark 25.3.3.7]). *For any morphism of animated rings $A \to B$, we have $\mathbb{L}\Omega^1_{A/B} \simeq A^\circ \otimes_{A^+} \mathbb{L}_{A^\circ/B}$.*

Proposition 3.2.10 ([Sch01, §7.9], [SAG, Proposition 25.3.4.2]). *There is an equivalence of spectra $A^\circ \otimes_{\mathbb{S}} \mathbb{Z} \xrightarrow{\simeq} A^+$.*

Note however that it is only an equivalence of the underlying spectra, not of ring spectra (necessarily so, since $A^\circ \otimes_{\mathbb{S}} \mathbb{Z}$ is $\mathscr{E}_\infty$ while A^+ is only $\mathscr{E}_1$). Furthermore, the natural variant $\mathbb{Z} \otimes_{\mathbb{S}} A^\circ \to A^+$ is *not* an equivalence (not even of spectra).

4 Spectral and chromatic phenomena in less-commutative geometry

4.1 Symmetric algebras and shearing

Construction 4.1.1. For any $n \in \mathbb{N}$, consider the functor $\mathfrak{RingMod}^{\mathrm{sfp}} \to \mathfrak{RingMod}^{\mathrm{sfp}}$ given by $(A, M) \mapsto (A, \pi_0 \operatorname{Sym}^n_A M)$. We denote its left derived functor as $\mathfrak{Ani}(\mathfrak{Ring})\mathfrak{Mod}^{\mathrm{cn}} \to \mathfrak{Ani}(\mathfrak{Ring})\mathfrak{Mod}^{\mathrm{cn}}$, $(A, M) \mapsto (A, \mathbb{L}\operatorname{Sym}^n_A M)$.

Doing the same thing with the exterior powers $\bigwedge^n_A$ or the divided powers Γ^n_A instead of the symmetric powers gives functors $\mathbb{L}\bigwedge^n_A$ and $\mathbb{L}\Gamma^n_A$.

Proposition 4.1.2 (Illusie, [SAG, Proposition 25.2.4.2]). *Let A be an animated ring and M a connective A-module. For every $n \geq 0$, there are equivalences of A-modules*

$$\mathbb{L}\operatorname{Sym}^n_A(\Sigma M) \simeq \Sigma^n \mathbb{L}\textstyle\bigwedge^n_A M \qquad \textit{and} \qquad \mathbb{L}\textstyle\bigwedge^n_A(\Sigma m) \simeq \Sigma^n \mathbb{L}\Gamma^n_A(M). \tag{8}$$

Lemma 4.1.3 ([SAG, Construction 25.2.2.6]). *For any animated ring A and connective A-module M, the total symmetric power $\mathbb{L}\operatorname{Sym}^\bullet_A(M) := \bigoplus_{n\geq 0} \mathbb{L}\operatorname{Sym}^n_A(M)$ admits a canonical structure of animated ring.*

Furthermore, the functor

$$\begin{aligned} \mathfrak{Ani}(\mathfrak{Ring})\mathfrak{Mod}^{\mathrm{cn}} &\to \mathfrak{Fun}([1], \mathfrak{Ani}(\mathfrak{Ring})) \\ (A, M) &\mapsto (A \to \mathbb{L}\operatorname{Sym}^\bullet_A M) \end{aligned} \tag{9}$$

is left-adjoint to $(A \to B) \mapsto (A, B^\circ)$.

If we wish to obtain exterior and divided power algebras, we need to consider the appropriate shifts in every position. For this, it is useful to remember that $\mathbb{L}\operatorname{Sym}^\bullet_A M$ is a $\mathbb{Z}$-graded animated ring. Indeed, we wish to obtain animated ring structures on the graded modules

$$\mathbb{L}\wedge^\bullet_A M = \Sigma^{-\bullet}\mathbb{L}\operatorname{Sym}^\bullet(\Sigma M) \qquad \text{and} \qquad \mathbb{L}\Gamma^\bullet_A M = \Sigma^{-2\bullet}\mathbb{L}\operatorname{Sym}^\bullet(\Sigma^2 M), \tag{10}$$

that is on the **shearings** of $\mathbb{L}\operatorname{Sym}^\bullet_A M$. In fact, we will focus on the even shearing as the odd ones are less understood.

In fact, to simplify matters, we will use the fact that the grading is actually an $\mathbb{N}$-grading (since shearing $\mathbb{Z}$-gradings requires the use of non-connective derived rings, and is poorly behaved).

Lemma 4.1.4 ([Lur15, Proposition 3.4.5]). *There is a $\mathbb{Z}$-graded $\mathscr{E}_2$-ring spectrum $\mathbb{S}[\beta]$, whose underlying graded $\mathscr{E}_1$-ring spectrum is freely generated by $\Sigma^{-2}\mathbb{S}(1)$,* i.e. *by a generator in degree 2 and weight 1.*

More specifically, multiplication by β^n provides an equivalence $\Sigma^{-2n}\mathbb{S} \to \mathbb{S}[\beta]_n$.

However, this $\mathscr{E}_2$-structure is provably not $\mathscr{E}_3$; indeed there is an explicit topological extension to it extending to an $\mathscr{E}_3$-structure.

Corollary 4.1.5 ([ABM22, Proposition 3.1]). *There is an $\mathscr{E}_2$-monoidal equivalence $\mathfrak{Sp}^{gr} \to \mathfrak{Sp}^{gr}$ given by $(M_i) \mapsto (\Sigma^{2i}M_i)$.*

Remark 4.1.6. By [Rak20, Proposition 3.3.4], the shearing functor admits an $\mathscr{E}_\infty$-monoidal structure over $\mathbb{Z}$.

4.2 Spectral skew-fields and chromatic heights

Lemma 4.2.1. *Let A be an $\mathscr{E}_1$-ring spectrum. The following are equivalent:*

- *$\pi_\bullet A$ is a graded skew-field, that is every nonzero homogeneous element is invertible,*
- *every classical graded $\pi_\bullet A$-module is free,*
- *every A-module (spectrum) is free.*

Definition 4.2.2. *An $\mathscr{E}_1$-ring spectrum A satisfying the equivalent conditions of the lemma is said to be a* ***spectral skew-field****.*

If A is further endowed with a structure of $\mathscr{E}_\infty$-ring spectrum, we say it is a ***spectral field****.*

Definition 4.2.3. *Two spectral skew-fields A and B* ***have the same characteristic*** *if $A \otimes B \neq 0$.*

Proposition 4.2.4 ([Lur24]). *Let A be a spectral skew-field. Then:*

- *If the skew-field $\pi_0 A$ has characteristic zero, then A has the same characteristic as $\mathbb{Q}$.*
- *If the skew-field $\pi_0 A$ has characteristic $p > 0$ and $A^\bullet(\mathcal{B}\,\mathbb{Z}/(p)) = \pi_\bullet \operatorname{hom}(\Sigma^\infty_+ \mathcal{B}\,\mathbb{Z}/(p), A)$ has infinite rank over $\pi_\bullet A$, then A has the same characteristic as $\mathbb{F}_p$.*
- *Otherwise, if $\pi_0 A$ has characteristic $p > 0$, then $A^\bullet(\mathcal{B}\,\mathbb{Z}/(p))$ has rank p^n over $\pi_\bullet A$ for some $n \in \mathbb{N}$.*

Theorem 4.2.5 (Morava (*cf.* [JW75]), [Lur10, Lecture 24 Proposition 9, Lecture 25 Corollary 9]). *For any $(p, n) \in \mathbb{P} \times \overline{\mathbb{N}}$, there exists a spectral skew field of π_0-characteristic p and height n: the p-local Morava K-theory of chromatic height n, denoted $K_{(p)}(n)$.*

In particular, for any spectral skew-field A, there is (p, n) such that A carries a structure of $K_{(p)}(n)$-module.

Remark 4.2.6. For $n > 0$, we have $\pi_\bullet K_{(p)}(n) \simeq \mathbb{F}_p[v_{(p),n}, v_{(p),n}^{-1}]$ where $v_{(p),n}$ has degree $2(p^n - 1)$.

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