Formal deformation theory

Reading seminar on DT theory

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Section 1: Functors of Artin rings

Formal neighbourhoods

Goal of deformation theory : Study infinitesimal neighbourhoods of points in moduli stacks.

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Formal neighbourhood of a closed affine subscheme

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Neighbourhood of a point

If I is prime so $Z = \{x\}$ is (the closure of) a point, then $\widehat{A}_I \simeq \widehat{(\mathcal{O}_{X,x})_{\mathfrak{m}_x}}$ where $\mathfrak{m}_x = I \cdot \mathcal{O}_{X,x}$.

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The formal neighbourhood $\widehat{X}_Z$ is the topologically locally ringed space

$$\operatorname{Spf}(\widehat{A}_I) \coloneqq \left(|\operatorname{Spec}(\widehat{A}_I/I)|_{\mathrm{Zar}} = |\operatorname{Spec}(A/I)|_{\mathrm{Zar}}, \varprojlim_n \mathcal{O}_{\operatorname{Spec}(A/I^n)} \right).$$

Examples

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- More generally, the completion of $\Bbbk[x_1, \ldots, x_n]$ along the maximal ideal $(x_1, \cdots, x_n)$ is $\Bbbk[\![x_1, \ldots, x_n]\!]$.

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- The formal neighbourhood of $(0,0)$ in $\Bbbk[x,y]/(y^2 - x^2 - x^3)$ is $\Bbbk[\![x,y]\!]/((y-\xi)(y+\xi))$: in $\Bbbk[\![x,y]\!]$, the polynomial $x^2(1+x)$ acquires a square-root $\xi = x\sqrt{1+x}$ (by formal Taylor expansion).

Contents – Section 1: Functors of Artin rings

Artinian algebras

A ring is **Artinian** if it satisfies the descending chain condition for ideals. A **local Artin $\Bbbk$-algebra** is a local ring $(A, \mathfrak{m}_A)$ with A Artinian and $A/\mathfrak{m}_A \simeq \Bbbk$.

Structure theory of Artin rings

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- A a $\Bbbk$-algebra. The square-zero extension $A \oplus A$ is a (non-local) Artin ring. It is isomorphic to $A[\varepsilon]/(\varepsilon^2) \simeq A \otimes_{\Bbbk} \Bbbk[\varepsilon]/(\varepsilon^2)$.

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Write $\mathfrak{Art}_{\Bbbk}$ the category of local Artin $\Bbbk$-algebras and local homomorphisms.

Interpretation: ${\mathfrak{Art}_{\Bbbk}}^{\mathrm{op}}$ is the category of (affine) "fat points".

Complete local rings

A pair (R, I) is **complete** if $R \to \widehat{R}_I$ is an isomorphism.

- If I is nilpotent, then (R, I) is complete.
- If I is finitely generated, $(\widehat{R}_I, \widehat{I}_I = I \cdot \widehat{R}_I)$ is complete. For example, if R is nœtherian.

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Every complete local $\Bbbk$-algebra R is a pro-object in $\mathfrak{Art}_{\Bbbk}$.

Indeed, R is a projective limit $\varprojlim_n R/\mathfrak{m}_R^n$ where each $R/\mathfrak{m}_R^n$ is local nœtherian with nilpotent maximal ideal, so Artinian.

Surjections of Artin rings

A surjection of local Artin $\varphi \colon B \twoheadrightarrow A$ is a **small extension** if $\mathfrak{m}_B \cdot \ker(\varphi) = 0$.
It is **principal** if $\ker(\varphi)$ is a principal ideal of B.

Remark: $\ker(\varphi)$ has a canonical structure of A-module (as $\mathfrak{m}_B \supset I$).

$0 \to \mathbb{k} = (\varepsilon^n) \rightarrowtail \mathbb{k}[\varepsilon]/(\varepsilon^{n+1}) \twoheadrightarrow \mathbb{k}[\varepsilon]/(\varepsilon^n) \to 0$ is a principal small extension.

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Proposition

Every surjection of local Artin rings factors as a composite of small extensions.

For any $\Bbbk$-algebra R and an R-module I, we let $\mathrm{Ex}_\Bbbk(R, I)$ denote the set of isomorphism classes of square-zero extensions of R by I.

- $\mathrm{Ex}_\Bbbk(R, I)$ has a structure of R-module, with $r \cdot [\widetilde{R}] \coloneqq [\alpha_{r,*}\widetilde{R}]$ where $\alpha_r\colon I \to I, i \mapsto r \cdot i$ and, for any $a\colon I \to J$, $a_*\widetilde{R} = \widetilde{R} \amalg_I J$.

Contents – Section 1: Functors of Artin rings

Pre-deformation functors

Fact: The functor of points of a formal scheme is determined by its values on $\Bbbk$-algebras with reduced part $\Bbbk$, *i.e.* it can be recovered from its restriction to $\mathfrak{Art}_{\Bbbk}$.

- A **pre-deformation functor** is a precosheaf $\mathscr{F}$ on $\mathfrak{Art}_{\Bbbk}$, that is a (covariant) functor $\mathfrak{Art}_{\Bbbk} \to \mathfrak{Set}$, such that $\mathscr{F}(\Bbbk) \simeq *$.

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- A pre-deformation functor **admits a differential calculus** if it preserves pullbacks along $\Bbbk[\varepsilon]/(\varepsilon^2) \twoheadrightarrow \Bbbk$

$$\begin{tikzcd} A' \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Bbbk[\varepsilon]/(\varepsilon^2) \arrow[d, two heads] \\ A \arrow[r] & \Bbbk \end{tikzcd}$$

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\qquad
\begin{tikzcd} \mathscr{F}(A') \arrow[dr, dashed, "\exists ! \Upsilon"] \arrow[drr, bend left, "\mathscr{F}(\mathrm{pr}_2)"] \arrow[ddr, bend right, "\mathscr{F}(\mathrm{pr}_1)"'] & & \\ & \mathscr{F}(A) \underset{*}{\times} \mathscr{F}(\Bbbk[\varepsilon]/(\varepsilon^2)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{F}(\Bbbk[\varepsilon]/(\varepsilon^2)) \arrow[d] \\ & \mathscr{F}(A) \arrow[r] & \mathscr{F}(\Bbbk) = * \end{tikzcd}$$

the dashed map Υ on the right is an isomorphism.

Examples of deformation functors I

Pro-representable functors

(Co)Representable functors

Any local Artin $\Bbbk$-algebra A corepresents the functor $\hbar^A \colon B \mapsto \hom_{\mathfrak{Art}_\Bbbk}(A, B)$. There is a unique $\Bbbk \to A$, and representables preserve all limits.

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Suppose $R \in \widehat{\mathfrak{Art}_\Bbbk}$. The representable $\widehat{\hbar^R}$ is left-exact so $\widehat{\hbar^R}(S) = \varprojlim_n \widehat{\hbar^R}(S/\mathfrak{m}_S^n)$.

- A pre-deformation functor $\mathscr{F}$ can be extended to $\widehat{\mathscr{F}} \colon \widehat{\mathfrak{Art}_\Bbbk} \longrightarrow \mathfrak{Set}$ by

$$\widehat{\mathscr{F}} \colon \widehat{\mathfrak{Art}_\Bbbk} \ni S \simeq \varprojlim_n S/\mathfrak{m}_S^n \quad \mapsto \quad \varprojlim_n \mathscr{F}(S/\mathfrak{m}_S^n) =: \{\text{formal elements over } S\}.$$

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(Co)Representable functors

Any local Artin $\Bbbk$-algebra A corepresents the functor $\mathscr{h}^A\colon B \mapsto \hom_{\mathfrak{Art}_\Bbbk}(A, B)$. There is a unique $\Bbbk \longrightarrow A$, and representables preserve all limits.

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  $\mathscr{F}$ is **pro-representable** if $\widehat{\mathscr{F}}$ is representable by $R \in \widehat{\mathfrak{Art}_\Bbbk}$.
- For such $R \in \widehat{\mathfrak{Art}_\Bbbk}$, we denote $\mathscr{h}^R\colon A \mapsto \hom(R, A)$ the restriction of $\widehat{\mathscr{h}^R}$.
- By Yoneda, morphisms $\mathscr{h}^R \longrightarrow \mathscr{F}$ are in bijection with formal elements of $\mathscr{F}$ over R.

Examples of deformation functors II

Formal neighbourhood of a point

The functor of points of a $\mathbb{k}$-scheme X is $h_X \colon \mathfrak{Aff}_{\mathbb{k}}{}^{\mathrm{op}} = \mathfrak{Alg}_{\mathbb{k}} \to \mathfrak{Ens}, A \mapsto \hom(\operatorname{Spec} A, X)$.
However $\mathfrak{Art}_{\mathbb{k}}$ is not a full subcategory of $\mathfrak{Alg}_{\mathbb{k}}$.

Lemma

Let R be a local ring. There is a bijection between morphisms $f \colon \operatorname{Spec} R \to X$ mapping the unique closed point to $x \in X$ and *local* homomorphisms $f^\sharp \mathscr{O}_{X,x} \to R$.

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Let X be a $\Bbbk$-scheme and $x\colon \operatorname{Spec} \Bbbk \to X$ a point. Its formal neighbourhood is

$$A \mapsto \left\{ f \in \hom(\operatorname{Spec} A, X) \mid p_A^* f \coloneqq f|_{\operatorname{Spec} \Bbbk} = x \right\}$$

where $p_A^\sharp\colon A \twoheadrightarrow A/\mathfrak{m}_A = \Bbbk$ so $f|_{\operatorname{Spec} \Bbbk}\colon \operatorname{Spec} k \xrightarrow{p_A} \operatorname{Spec} A \xrightarrow{f} X$.

Section 2: Tangent spaces to deformation problems

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The tangent space

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Proposition

If $\mathscr{F}$ admits a differential calculus, $T_{\mathscr{F}}$ is a $\mathbb{k}$-vector space.

Construction.

$\mathbb{D}$ is an $\mathbb{k}$-vector space object in $\mathfrak{Art}_{\mathbb{k},/\mathbb{k}}$ with

abelian group structure from $\mu\colon \mathbb{D} \times_{\mathbb{k}} \mathbb{D} \to \mathbb{D}, (a + b\varepsilon, a + b'\varepsilon) \mapsto a + (b + b')\varepsilon$,

scalar multiplication from $\rho_\lambda\colon \mathbb{D} \to \mathbb{D}, a + b\varepsilon \mapsto a + \lambda b\varepsilon$ for $\lambda \in \mathbb{k}$.

Then, as $\mathscr{F}$ preserves the relevant fibre products, define

$+\colon \mathscr{F}(\mathbb{D}) \times \mathscr{F}(\mathbb{D}) \xrightarrow{\Upsilon^{-1}} \mathscr{F}(\mathbb{D} \times_{\mathbb{k}} \mathbb{D}) \xrightarrow{\mathscr{F}(\mu)} \mathscr{F}(\mathbb{D})$ and so on. $\square$

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If $\varphi\colon \mathscr{F} \to \mathscr{G}$ is a transformation, $d\varphi \coloneqq \varphi_{\mathbb{D}}\colon T_{\mathscr{F}} \to T_{\mathscr{G}}$ is called its **differential**.

Example 0: Pro-representable functors

If $\mathscr{F} = h^R$ is pro-representable, $T_{h^R} = T_{R/\Bbbk,\mathfrak{m}_R} \simeq T_{R/\Bbbk}$ where $\mathfrak{m}_R$ is the unique closed point of $\operatorname{Spec} R$ (so the tangent space is the tangent module).

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Proof.

We use the characterisation of $T_{R/\Bbbk,\mathfrak{m}_R}$ as the $\Bbbk$-dual of $\Omega^1_{R/\Bbbk,\mathfrak{m}_R} = \Omega^1_{R/\Bbbk} \otimes_R \Bbbk$, so

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Recall that $\Bbbk$-linear derivations from R to an R-module M are in bijection with maps of R-augmented $\Bbbk$-algebras into the square-zero extension $R \oplus M$:

$$T_{R/\Bbbk} \simeq \hom_{\Bbbk,/R}(R, R \oplus \Bbbk) \simeq \hom_{\Bbbk,/\Bbbk}(R, \Bbbk \oplus \Bbbk) = \hom_{\Bbbk,/\Bbbk}(R, \mathbb{D}).$$

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Finally, a local morphism $R \to \mathbb{D}$ is exactly a morphism of $\Bbbk$-augmented $\Bbbk$-algebras. $\square$

Extending tangent modules

Recall that the tangent *module* to a $\Bbbk$-algebra R is $T_{R/\Bbbk} = \hom_R(\Omega_{R/\Bbbk}, R) = \mathrm{Der}_\Bbbk(R, R)$.

- For any morphism $S \to R$, there is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Der}_S(R, I) \arrow[r] & \mathrm{Der}_\Bbbk(R, I) \arrow[r] & \mathrm{Der}_\Bbbk(S, I) \otimes_S R \arrow[dll, rounded corners] \\
& \mathrm{Ex}_S(R, I) \arrow[r] & \mathrm{Ex}_\Bbbk(R, I) \arrow[r] & \mathrm{Ex}_\Bbbk(S, I) \otimes_S R,
\end{tikzcd}$$

  hence one also writes $\mathrm{Ex}_\Bbbk(R, R) =: \mathrm{H}^1(T^\bullet_{R/\Bbbk})$ $(= \mathrm{Ext}^1_R(\mathbb{L}\Omega^{1,\bullet}_{R/\Bbbk}, R))$.

Extending tangent modules

Recall that the tangent *module* to a $\Bbbk$-algebra R is $T_{R/\Bbbk} = \hom_R(\Omega_{R/\Bbbk}, R) = \mathrm{Der}_\Bbbk(R, R)$.

- For any morphism $S \to R$, there is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Der}_S(R, I) \arrow[r] & \mathrm{Der}_\Bbbk(R, I) \arrow[r] & \mathrm{Der}_\Bbbk(S, I) \otimes_S R \arrow[dll, rounded corners] \\ & \mathrm{Ex}_S(R, I) \arrow[r] & \mathrm{Ex}_\Bbbk(R, I) \arrow[r] & \mathrm{Ex}_\Bbbk(S, I) \otimes_S R, \end{tikzcd}$$

  hence one also writes $\mathrm{Ex}_\Bbbk(R, R) =: \mathrm{H}^1(T^\bullet_{R/\Bbbk})$ $(= \mathrm{Ext}^1_R(\mathbb{L}\Omega^{1,\bullet}_{R/\Bbbk}, R))$.

- As a quasicoherent $\mathscr{O}_{\mathrm{Spec}\, R}$-module, $\mathrm{Ex}_\Bbbk(R, R)$ is supported on the singular locus of $\mathrm{Spec}\, R \to \mathrm{Spec}\, \Bbbk$.

- If R is reduced, then $\mathrm{Ex}_\Bbbk(R, I) \simeq \mathrm{Ext}^1_R(\Omega^1_{R/\Bbbk}, I)$.

Action of the tangent space

Lemma

For any principal extension $(t) \rightarrowtail \widetilde{A} \overset{p}{\twoheadrightarrow} A$, $T_{\mathscr{F}}$ acts on the fibres of $\mathscr{F}(\widetilde{A}) \xrightarrow{\mathscr{F}p} \mathscr{F}(A)$.

Proof.

- $\mathbb{D}$ acts on $\widetilde{A}$ by $\mathbb{D} \times_{\Bbbk} \widetilde{A} \xrightarrow{\text{act}} \widetilde{A}, (\alpha + \beta\varepsilon, a) \mapsto a + \beta t$ where t generates (t).

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- The action preserves the fibres:

$$\begin{tikzcd}
T_{\mathscr{F}} \times \mathscr{F}(\widetilde{A}) \arrow[r, "\mathscr{F}(\text{act},\text{pr}_2)\circ\Upsilon^{-1}"'] \arrow[rrr, bend left, "\text{pr}_2"] \arrow[rrd, bend right, "\mathscr{F}(\text{act})\circ\Upsilon^{-1}"'] & \mathscr{F}(\widetilde{A} \times_A \widetilde{A}) \arrow[r, "!(\mathscr{F}\text{pr}_1, \mathscr{F}\text{pr}_2)"'] & \times \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{F}(\tilde{A}) \arrow[d, "\mathscr{F}(p)"] \\
 & & \mathscr{F}(\widetilde{A}) \arrow[r, "\mathscr{F}(p)"'] & \mathscr{F}(A)
\end{tikzcd}$$

$\square$

Action for representables

Proposition

If $\mathscr{F} = \hbar^R$ is pro-representable, the action of $T_{\hbar^R}$ is transitive and free on non-empty fibres.

Proof.

Let $\varpi\colon \widetilde{A} \twoheadrightarrow A$ be a principal extension with kernel $I \simeq \Bbbk$ and $\varphi \in \hbar^R(A) = \hom(R, A)$. Fix $\widetilde{\varphi}\colon R \to \widetilde{A}$ in the fibre.

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For any other $\widetilde{\psi}$ such that $\varpi \circ \widetilde{\psi} = \varphi$, one checks that $\widetilde{\varphi} - \widetilde{\psi}$ is a derivation $R \longrightarrow I$.

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In any category with products, an internal group action such that $G \times X \xrightarrow{(\mathrm{act}, \mathrm{pr}_2)} X \times X$ is an isomorphism is called a **torsor**.

$\Rightarrow$ Every choice of "base-point" $x \colon * \to X$ trivialises the torsor by $G \times * \simeq X \times *$.

Torsors in algebraic geometry

Let $\mathscr{G}$ be a sheaf of groups and $\mathscr{X}$ a $\mathscr{G}$-sheaf. If $\mathscr{X}(U) = \emptyset$, then $(\mathscr{G} \times \mathscr{X})(U) = \emptyset$ and the torsor condition is trivially satisfied over U.

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Let $\coprod_\alpha U_\alpha \twoheadrightarrow X$ be an open cover, and $(\gamma_{\alpha\beta})$ a Čech cocycle. For any open $V \subset X$, define

$$\mathscr{S}(V) \coloneqq \{(g_\alpha \in \mathscr{G}(V \times_X U_\alpha)) \mid \gamma_{\alpha\beta} g_\beta = g_\alpha\}$$

with the obvious restriction maps and $\mathscr{G}$-action. Any coboundary gives a trivial torsor.

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Contents – Section 2: Tangent spaces to deformation problems

Flatness over the dual numbers

Given a $\Bbbk$-scheme X and a $\Bbbk$-algebra A, we write $X_A = X \otimes_{\Bbbk} A = X \times_{\operatorname{Spec} \Bbbk} \operatorname{Spec} A$.

Differential modules

An $\mathcal{O}_{X_{\mathbb{D}}}$-module consists of:

- a sheaf on $|X|_{\mathrm{Zar}}$
- with a structure of module over $\mathcal{O}_X \otimes_{\Bbbk} \mathbb{D} = \mathcal{O}_X[\varepsilon]/(\varepsilon^2)$

Modules over $\mathcal{O}_X[\varepsilon]/(\varepsilon^2)$ correspond to **differential $\mathcal{O}_X$-modules**: pairs $\mathscr{F} = (\mathscr{F}_0, \psi)$ where $\mathscr{F}_0$ is an $\mathcal{O}_X$-module and $\psi \colon \mathscr{F}_0 \to \mathscr{F}_0$ such that $\psi \circ \psi = 0$.

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Flatness

Suppose $A \twoheadrightarrow \Bbbk$ is a principal extension. A quasicoherent $\mathcal{O}_{X_A}$-module $\mathscr{F}$ is flat over A iff $\mathscr{F} \otimes \mathcal{O}_X = \mathscr{F}|_X$ is flat over $\Bbbk$ and $\mathscr{F} \otimes \mathfrak{m}_A \mathcal{O}_{X_A} \xrightarrow{\simeq} \mathfrak{m}_A \mathscr{F}$.

Remark: When $A = \mathbb{D} = \Bbbk[\varepsilon]/(\varepsilon^2)$, then $\varepsilon \mathcal{O}_{X_{\mathbb{D}}} \simeq \mathcal{O}_X$, so $\mathscr{F}_0$ splits (over $\Bbbk$, not $\mathcal{O}_X$).

Example 1: Deformations of a scheme

- Let X_0 be a finite type $\Bbbk$-scheme. A deformation of X_0 over $A \in \mathfrak{Art}_{\Bbbk}$ is $X \to \operatorname{Spec}(A)$ flat and surjective with an isomorphism $\vartheta\colon X \otimes_A \Bbbk \xrightarrow{\simeq} X_0$.
A morphism $(X, \vartheta) \to (X', \vartheta')$ is $f\colon X \to X'$ such that $\vartheta' \circ (f \otimes_A \Bbbk) = \vartheta$.

- The functor of deformations of X_0 is $\mathscr{D}\!\mathit{ef}_{X_0}\colon A \mapsto \{A\text{-deformations of } X_0\}/\simeq$.
Functoriality is by taking pullbacks of families: if $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$, then $\big(\mathscr{D}\!\mathit{ef}_{X_0}(f^\sharp)\big)(X) = f^*X \coloneqq X \otimes_A B$.

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  A morphism $(X, \vartheta) \to (X', \vartheta')$ is $f \colon X \to X'$ such that $\vartheta' \circ (f \otimes_A \Bbbk) = \vartheta$.
- An A-deformation $X \to \operatorname{Spec} A$ is **locally trivial** if there is a cover $\coprod_\alpha U_\alpha \twoheadrightarrow X_0$ such that $X|_{U_\alpha} \simeq U_\alpha \otimes_{\Bbbk} A$.

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- Subfunctor $\mathscr{Def}^{\mathrm{triv}}_{X_0}$ of locally trivial deformations.

Deformations of smooth affines

Lemma

Every deformation of a smooth affine $\Bbbk$-scheme $X_0 = \operatorname{Spec} R_0$ is trivial.

Proof.

Let $X \in \mathscr{D}\!\mathit{ef}_{X_0}(\mathbb{D})$. Note first that X is affine, $X = \operatorname{Spec} R$, and smooth over $\mathbb{D}$ (by flatness).

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$$\begin{tikzcd} R_0 & R \arrow[l] \arrow[dl, dashed] \\ R_0[\varepsilon]/(\varepsilon^2) \arrow[u, two heads] & \Bbbk \arrow[l] \arrow[u] \end{tikzcd}$$

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As $\Bbbk \to R$ is smooth, the dashed lift exists. It has an inverse from the bilinear $R_0 \times \mathbb{D} \to R$. Triviality over $A \in \mathfrak{Art}_{\Bbbk}$ is by induction on $\dim_{\Bbbk}(A)$: decompose in successive extensions. $\square$

Tangent to locally trivial deformations of a scheme

Theorem

There is an isomorphism KS: $T_{\mathscr{D}ef^{\mathrm{triv}}_{X_0}} \xrightarrow{\simeq} \check{\mathrm{H}}^1(X_0, \mathscr{T}_{X_0})$.

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- Over $U_{\alpha\beta} = U_\alpha \times_X U_\beta$, the transition functions $\vartheta_\beta \circ \vartheta_\alpha^{-1}\colon U_{\alpha\beta} \otimes_{\Bbbk} \mathbb{D} \to U_{\alpha\beta} \otimes_{\Bbbk} \mathbb{D}$ are automorphisms of $U_{\alpha\beta} \otimes_{\Bbbk} \mathbb{D}$ fixing $U_{\alpha\beta}$. Such automorphisms correspond to elements of $\mathrm{Der}_{\Bbbk}(\mathscr{O}_{U_{\alpha\beta}}, \mathscr{O}_{U_{\alpha\beta}}) \simeq \Gamma(U_{\alpha\beta}, \mathscr{T}_{X_0})$.

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  Indeed, in an affine chart $U_{\alpha\beta} = \mathrm{Spec}\, B$, an automorphism $B \otimes_{\Bbbk} \mathbb{D} \xrightarrow{\simeq} B \otimes_{\Bbbk} \mathbb{D}$ reducing to id_B modulo ε is of the form $b \mapsto b + \delta(b)\varepsilon$. By before δ must be a $\mathbb{D}$-derivation $B \otimes_{\Bbbk} \mathbb{D} \to B$, and $\mathrm{Der}_{\mathbb{D}}(B \otimes_{\Bbbk} \mathbb{D}, B) \simeq \mathrm{Der}_{\Bbbk}(B, B)$.

Tangent to deformations of a scheme

Proposition

If X_0 is of finite type over $\Bbbk$, then $T_{\mathscr{D}ef_{X_0}} \simeq \mathrm{Ex}_{\Bbbk}(\mathscr{O}_{X_0}, \mathscr{O}_{X_0})$. In particular, if X_0 is reduced, $T_{\mathscr{D}ef_{X_0}} \simeq \mathrm{Ext}^1_{\mathscr{O}_{X_0}}(\Omega^1_{X_0}, \mathscr{O}_X)$.

Proof.

- Let X be a first-order deformation. Since $\mathscr{O}_X$ is flat over $\Bbbk$ it splits as a $\Bbbk$-linear extension $0 \to \varepsilon\mathscr{O}_X \rightarrowtail \mathscr{O}_X \twoheadrightarrow \mathscr{O}_{X_0} \to 0$, with $\varepsilon\mathscr{O}_X = \mathscr{O}_{X_0}$. Hence we get a $\Bbbk$-linear self-extension of $\mathscr{O}_{X_0}$.

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- Given an extension $0 \to \mathscr{O}_{X_0} \overset{i}{\rightarrowtail} \mathscr{A} \overset{p}{\twoheadrightarrow} \mathscr{O}_{X_0} \to 0$, one can endow $\mathscr{A}$ with a $\mathbb{D} \otimes_{\Bbbk} \mathscr{O}_{X_0}$-algebra structure by having $i \circ p$ act as ε.

□

Example of the projective spaces

The Euler sequence $0 \to \mathscr{O}_{\mathbb{P}^n_{\Bbbk}} \to \mathscr{O}_{\mathbb{P}^n_{\Bbbk}}(1)^{\oplus n+1} \to \mathscr{T}_{\mathbb{P}^n_{\Bbbk}/\Bbbk} \to 0$ implies that $\mathrm{H}^1(\mathbb{P}^n_{\Bbbk}, \mathscr{T}_{\mathbb{P}^n_{\Bbbk}/\Bbbk}) = 0$ for $n \geq 1$, so $\mathbb{P}^n_{\Bbbk}$ is rigid.

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Computation for the affine line

Consider a first-order deformation of $\mathbb{P}^1_{\Bbbk}$ trivialised by the standard covering $\mathbb{P}^1_{\Bbbk} \simeq U_0 \amalg_{U_{01}} U_1$, where $U_0 = \mathbb{A}^1_{\Bbbk} = \operatorname{Spec} \Bbbk[x]$, $U_1 = \operatorname{Spec} \Bbbk[x^{-1}]$ and $U_{01} = \operatorname{Spec} \Bbbk[x, x^{-1}]$. The transition function is an automorphism of $\Bbbk[x, x^{-1}] \otimes_{\Bbbk} \mathbb{D}$ reducing to the identity, so a $\Bbbk$-derivation of $\Bbbk[x, x^{-1}]$.

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- A derivation $\delta \colon \Bbbk[x, x^{-1}] \to \Bbbk[x, x^{-1}]$ satisfies $0 = \delta(1) = \delta(xx^{-1}) = x\delta(x^{-1}) + x^{-1}\delta(x)$ so $\delta(x^{-1}) = -x^{-2}\delta(x)$. It is determined by its action on x, which must be $\Bbbk$-linear.

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Computation for the affine line

Consider a first-order deformation of $\mathbb{P}^1_{\Bbbk}$ trivialised by the standard covering $\mathbb{P}^1_{\Bbbk} \simeq U_0 \amalg_{U_{01}} U_1$, where $U_0 = \mathbb{A}^1_{\Bbbk} = \operatorname{Spec} \Bbbk[x]$, $U_1 = \operatorname{Spec} \Bbbk[x^{-1}]$ and $U_{01} = \operatorname{Spec} \Bbbk[x, x^{-1}]$. The transition function is an automorphism of $\Bbbk[x, x^{-1}] \otimes_{\Bbbk} \mathbb{D}$ reducing to the identity, so a $\Bbbk$-derivation of $\Bbbk[x, x^{-1}]$.

- A derivation $\delta \colon \Bbbk[x, x^{-1}] \to \Bbbk[x, x^{-1}]$ satisfies $0 = \delta(1) = \delta(xx^{-1}) = x\delta(x^{-1}) + x^{-1}\delta(x)$ so $\delta(x^{-1}) = -x^{-2}\delta(x)$. It is determined by its action on x, which must be $\Bbbk$-linear.
- Let $\delta_0 \colon \Bbbk[x] \to \Bbbk[x]$ be the derivation defined by $x \mapsto \delta(x)$ and $\delta_1 \colon \Bbbk[x^{-1}] \to \Bbbk[x^{-1}]$ defined by $\delta_1(x^{-1}) = -\delta(x^{-1})$. Then $\delta_0 \otimes_{\Bbbk} \Bbbk[x^{-1}] - \delta_1 \otimes_{\Bbbk} \Bbbk[x]$ gives back δ.

Hence the Čech cocycle defined by δ is a coboundary, and defines a trivial deformation.

Example 2: Deformations of a quasicoherent sheaf

Let $\mathscr{F}_0$ be a quasicoherent sheaf on X_0, flat over $\Bbbk$.

- An A-**deformation** of $\mathscr{F}_0$ is a quasicoherent sheaf $\mathscr{F}$ on the trivial deformation $X_A \coloneqq X_0 \times_{\operatorname{Spec} \Bbbk} \operatorname{Spec} A$, flat over A, with an isomorphism $\mathscr{F}|_{X_0} = \mathscr{F} \otimes_A \Bbbk \xrightarrow{\simeq} \mathscr{F}_0$.
- The functor of deformations of $\mathscr{F}_0$ is $\mathscr{D}\!ef_{\mathscr{F}_0} \colon A \mapsto \{A\text{-deformations of } \mathscr{F}_0\}/\simeq$.
  If $f \colon \operatorname{Spec} B \to \operatorname{Spec} A$ and $\mathscr{F} \in \mathscr{D}\!ef_{\mathscr{F}_0}(A)$, then $(\mathscr{D}\!ef_{\mathscr{F}_0} f)(\mathscr{F}) \coloneqq \mathscr{F} \otimes_A B$, that is $(\mathscr{D}\!ef_{\mathscr{F}_0} f)(\mathscr{F}) = p_f^* \mathscr{F}$ where $p_f \colon X_B \to X_A$.

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Restrictions and inductions of sheaves

The principal extension $0 \to \Bbbk \hookrightarrow \mathbb{D} \twoheadrightarrow \Bbbk \to 0$ induce maps $p \colon X_{\mathbb{D}} \to X_0$ and $i \colon X_0 \to X_{\mathbb{D}}$.

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- For $\mathscr{F}$ an $\mathscr{O}_{X_{\mathbb{D}}}$-module: $i^*\mathscr{F} = \mathscr{F}|_{X_0} = \mathscr{F}/(\varepsilon)$, while $p_*\mathscr{F}$ is $\mathscr{F}$ seen as an $\mathscr{O}_X$-module.

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- For $\mathscr{F}_0$ an $\mathscr{O}_{X_0}$-module: $i_* \mathscr{F}_0$ is $\mathscr{F}_0$ seen as an $\mathscr{O}_X[\varepsilon]/(\varepsilon^2)$-module with trivial ε-action, while $p^* \mathscr{F}_0 = \mathscr{F}_0 \otimes \Bbbk[\varepsilon]/(\varepsilon^2)$. The differential $\mathscr{O}_{X_0}$-module is $(\mathscr{F}_0 \oplus \mathscr{F}_0, \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right))$.

Tangent to deformations of a sheaf

Theorem

If $\mathscr{F}_0$ is locally free, $T_{\mathscr{D}ef_{\mathscr{F}_0}} \simeq \check{\mathrm{H}}^1(X_0, \mathscr{H}om(\mathscr{F}_0, \mathscr{F}_0))$.

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Tangent to deformations of a quasicoherent sheaf

Proposition

For any $\mathscr{F}_0$ quasicoherent, $T_{\mathscr{Def}_{\mathscr{F}_0}} \simeq \mathrm{Ext}^1_{\mathscr{O}_{X_0}}(\mathscr{F}_0, \mathscr{F}_0)$.

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- Given an $\mathscr{O}_{X_0}$-extension of $\mathscr{F}_0$ by $\mathscr{F}_0$, there is again a unique $\mathscr{O}_{X_{\mathbb{D}}}$-module structure restricting to $\mathscr{F}_0$: for $0 \to \mathscr{F}_0 \xrightarrow{i} \mathscr{E} \xrightarrow{p} \mathscr{F}_0 \to 0$ the nilpotent endomorphism is $i \circ p \colon \mathscr{F} \to \mathscr{F}$.

Section 3: Extending deformations

Contents – Section 3: Extending deformations

Formal smoothness

A morphism of schemes $\varphi\colon X \to Y$ is formally smooth when it has the right lifting property against square-zero immersions, *i.e.* if for any square-zero immersion $\iota\colon \operatorname{Spec}(A/I) \hookrightarrow \operatorname{Spec} A$

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$$\hom(\operatorname{Spec} A, X) \to \hom(\operatorname{Spec} A/I, X) \underset{\hom(\operatorname{Spec} A/I, Y)}{\times} \hom(\operatorname{Spec} A, Y)$$
$$\psi \mapsto (\psi \circ \iota, \varphi \circ \psi)$$

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is surjective.

Formally smooth morphism

A morphism of pre-deformation functors $\varphi\colon \mathscr{F} \to \mathscr{G}$ is **formally smooth** if for every small surjection $A \twoheadrightarrow B$, the map $\mathscr{F}(A) \to \mathscr{F}(B) \times_{\mathscr{G}(B)} \mathscr{G}(A)$ is surjective.

$\mathscr{F}$ is formally smooth if $\mathscr{F} \to *$ is so, *i.e.* $\mathscr{F}(A) \twoheadrightarrow \mathscr{F}(B)$ for every $A \twoheadrightarrow B$.

Versal families

A pre-deformation functor $\mathscr{F}$ is pro-representable iff there is an isomorphism $\mathscr{h}_R \to \mathscr{F}$, which by Yoneda corresponds to a formal family $\widehat{\xi} \in \widehat{\mathscr{F}}(R)$. Such a family is called **universal**.

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Atlases for deformation problems

A **semi-universal family** for $\mathscr{F}$ is a formal element $\widehat{\xi} \in \widehat{\mathscr{F}}(R)$ such that the corresponding $\hbar^R \to \mathscr{F}$ is smooth and its differential is an isomorphism $T_{R/\Bbbk} \xrightarrow{\simeq} T_{\mathscr{F}}$.

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Remark: If $\varphi \colon \mathscr{E} \to \mathscr{F}$ is smooth, it is surjective (thus so is its differential).

Proposition

If $(R, \widehat{\xi})$ and $(S, \widehat{\psi})$ are two semi-universal families, there is an isomorphism $(R, \widehat{\xi}) \simeq (S, \widehat{\psi})$ inducing a uniquely determined $T_{R/\Bbbk} \simeq T_{S/\Bbbk}$.
If $(R, \widehat{\xi})$ and $(S, \widehat{\psi})$ are universal, the isomorphism between them is unique.

Schlessinger's criterion

Definition (deformation functor)

A pre-deformation functor $\mathscr{F}$ is a **deformation functor** if

1. for every small surjection $p\colon \widetilde{A} \twoheadrightarrow A$ and every $f\colon B \longrightarrow A$, the map $\Upsilon_{p,f}\colon \mathscr{F}(\widetilde{A} \times_A B) \longrightarrow \mathscr{F}(\widetilde{A}) \times_{\mathscr{F}(A)} \mathscr{F}(B)$ is surjective
2. $\mathscr{F}$ admits a differential calculus, *i.e.* $\Upsilon_{p,f}$ is bijective when $p\colon \mathbb{D} \longrightarrow \mathbb{k}$.

$\mathscr{F}$ is **homogeneous** if the maps $\Upsilon_{p,f}$ are always bijective.

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Theorem (Schlessinger)

- A pre-deformation functor $\mathscr{F}$ admits a semi-universal formal family if and only if it is a deformation functor with finite-dimensional tangent space.
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Remark: For every $\Bbbk \hookrightarrow \widetilde{A} \twoheadrightarrow A$, $T_{\mathscr{F}}$ acts transitively on the fibres of $\mathscr{F}(\widetilde{A}) \longrightarrow \mathscr{F}(A)$.

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Extensions and obstruction calculus

Definition (obstruction spaces)

An **obstruction theory** for a pre-deformation functor $\mathscr{F}$ is a $\Bbbk$-vector space $\mathfrak{v}$ with, for every small extension $(E)\colon 0 \to I \hookrightarrow \widetilde{A} \twoheadrightarrow A \to 0$ a map $o_{(E)}\colon \mathscr{F}(A) \to \mathfrak{v} \otimes_{\Bbbk} I$ which is functorial in morphisms of extensions, and such that when $A = \Bbbk$ (so $\mathscr{F}(A) = *$), $o_{(E)}(*) = 0$.

We get for every $A \in \mathfrak{Art}_{\Bbbk}$, $\xi \in \mathscr{F}(A)$, and every $\Bbbk$-vector space I a $\Bbbk$-linear map $o_{(-)}(\xi)\colon \operatorname{Ex}_{\Bbbk}(A, I) \to \mathfrak{v} \otimes I$.

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Definition (obstruction spaces)

An **obstruction theory** for a pre-deformation functor $\mathscr{F}$ is a $\Bbbk$-vector space $\mathfrak{v}$ with, for every small extension $(E)\colon 0 \to I \hookrightarrow \widetilde{A} \twoheadrightarrow A \to 0$ a map $o_{(E)}\colon \mathscr{F}(A) \to \mathfrak{v} \otimes_{\Bbbk} I$ which is functorial in morphisms of extensions, and such that when $A = \Bbbk$ (so $\mathscr{F}(A) = *$), $o_{(E)}(*) = 0$.

We get for every $A \in \mathfrak{Art}_{\Bbbk}$, $\xi \in \mathscr{F}(A)$, and every $\Bbbk$-vector space I a $\Bbbk$-linear map $o_{(-)}(\xi)\colon \operatorname{Ex}_{\Bbbk}(A, I) \to \mathfrak{v} \otimes I$.

Lemma

If $(\mathfrak{v}, (o_{(E)})_{(E)})$ is an obstruction theory for $\mathscr{F}$, then whenever an element $\xi \in \mathscr{F}(A)$ lifts along a small extension $(E)\colon \widetilde{A} \twoheadrightarrow A$ to $\widetilde{\xi} \in \mathscr{F}(\widetilde{A})$ we have $o_{(E)}(\xi) = 0$.

An obstruction theory is called **complete** if vanishing of the obstruction $o_{(E)}(\xi)$ is equivalent to the existence of a lift along (E).

Universal obstruction theories

If $\mathscr{F}, \mathscr{G}$ are pre-deformation functors endowed with complete obstruction theories $(\upsilon^{\mathscr{F}}, (o_E^{\mathscr{F}}))$ and $(\upsilon^{\mathscr{G}}, (o_E^{\mathscr{G}}))$, an **obstruction map** for $\varphi\colon \mathscr{F} \to \mathscr{G}$ is a linear map $o^{\varphi}\colon \upsilon^{\mathscr{F}} \to \upsilon^{\mathscr{G}}$ such that for every $\xi \in \mathscr{F}(A)$ and every I, the following triangle commutes:

$$\begin{tikzcd} & \mathrm{Ex}_{\Bbbk}(A, I) \arrow[dl, "o^{\mathscr{F}}_{(-)}(\xi)"'] \arrow[dr, "o^{\mathscr{G}}_{(-)}(\varphi_A(\xi))"] & \\ \upsilon^{\mathscr{F}} \arrow[rr, "o^{\varphi}"] & & \upsilon^{\mathscr{G}} \end{tikzcd}.$$

Theorem (Fantechi–Manetti)

Every deformation functor admits an initial obstruction theory, which is furthermore complete.

The universal obstruction theory for $\hbar^R$ is $\mathrm{H}^1(T^{\bullet}_{R/\Bbbk,\mathfrak{m}_R}) \coloneqq \mathrm{Ex}_{\Bbbk}(R, \Bbbk)$.

Obstructions and smoothness

Proposition

Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of deformation functors. If φ has an injective obstruction map and $d\varphi$ is surjective then φ is smooth.

$\Rightarrow$ A deformation functor is smooth if and only if it has 0 as obstruction theory.

For a scheme X_0 of finite type over $\Bbbk$, $\mathrm{H}^2(X_0, \mathscr{T}_{X_0})$ is an obstruction space for $\mathscr{D}\!\mathit{ef}_{X_0}^{\mathrm{triv}}$. Hence if X_0 is smooth and $\mathrm{H}^2(X_0, \mathscr{T}_{X_0}) = 0$, then $\mathscr{D}\!\mathit{ef}_{X_0}$ is smooth of dimension $\dim \mathrm{H}^1(X_0, \mathscr{T}_{X_0})$.

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Dimension from obstructions

- If $\mathscr{F} \to \mathscr{G}$ is smooth, then any obstruction theory for $\mathscr{G}$ gives one for $\mathscr{F}$; in particular any obstruction theory on a deformation functor induces one on a semi-universal family.
- If υ is an obstruction theory for $\mathscr{h}^R$, then $\dim_{\Bbbk}(T_{R/\Bbbk}) \geq \dim_{\mathrm{Krull}}(R) \geq \dim_{\Bbbk}(T_{R/\Bbbk}) - \dim_{\Bbbk}(\upsilon)$.

Other questions in deformation theory

Stacky aspects

- Automorphisms of deformations as obstructions to pro-representability

Formal aspects

- Effectivity of deformations
- Algebraisability of formal families

Cohomological aspects

- Extensions and obstructions from the cotangent complex
- (Derived) Deformation problems and differential graded Lie algebras

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