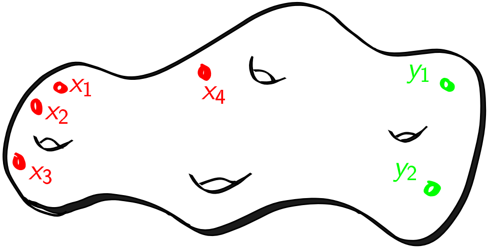
Higher-genus categorified Gromov–Witten invariants and modular quantisation

David Kern
Joint WIP with Hugo Pourcelot

16th June 2025

Motivation

GW invariants: AG formulation of topological quantum string partition functions



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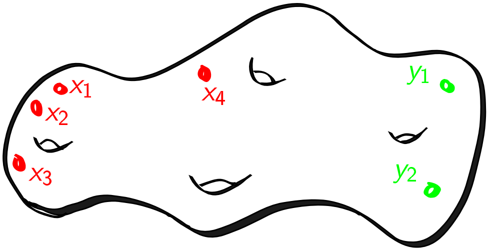

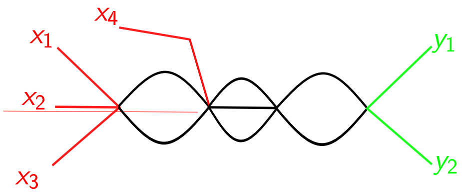

Intuition: Perform quantisation on target by probing it with higher-genus strings

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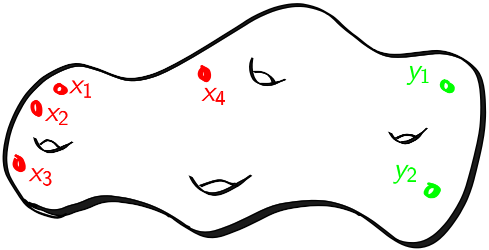

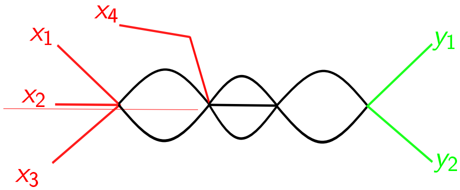

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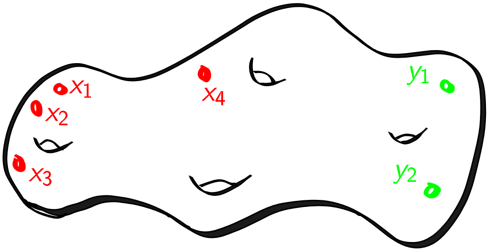

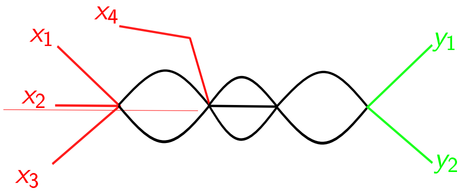

Intuition: Perform $\underbrace{\text{quantisation}}_{\text{Of what?}}$ on target by probing it with higher-genus strings

$\rightsquigarrow$ View Feynman graphs as encoding "quantum" operadic structures

Cohomological Field Theories

A **CohFT** on $(V, q\colon V \xrightarrow{\simeq} V^{\vee})$ is a family of maps $\mu_{g,n}\colon (A_{\bullet}\overline{\mathscr{M}}_{g,n+1}) \otimes V^{\otimes n} \to V$ with compatibilities for the morphisms:

- $\overline{\mathscr{M}}_{g,n+2} \xrightarrow{\mathrm{contract}_{i,j}} \overline{\mathscr{M}}_{g+1,n}$,

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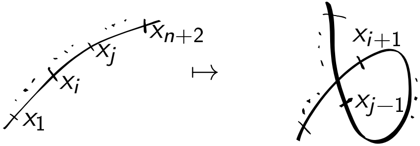

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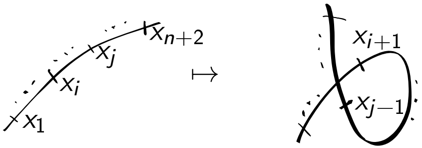

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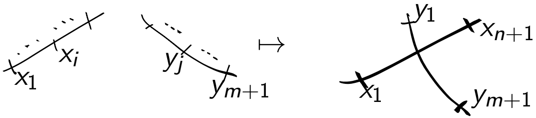

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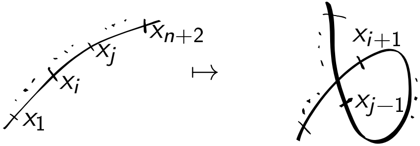

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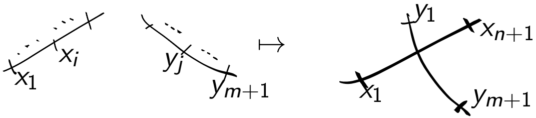

! In genus 0 $\text{glue}_{i,j}$ composition law for an operad $\overline{\mathcal{M}}_0$ and compatibilities $= \overline{\mathcal{M}}_0$-algebra

Gromov–Witten CohFT

Case of $V = A_\bullet(\overline{\mathcal{I}}_\mu X)$ for X a DM stack with rigidified cyclotomic inertia

$$\overline{\mathcal{I}}_\mu X = \coprod_{r \geqslant 1} \mathscr{M}\!\mathit{or}^{\mathsf{repr}}(\mathcal{B}\,\mu_r, X) / \mathcal{B}\,\mu_r$$

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- Moduli of stable maps give

$$\begin{tikzcd}[column sep=small]
 & \coprod_\beta \overline{\mathscr{M}}_{g,n+1}(X,\beta) \arrow[dl, "{(\mathrm{stab},\mathrm{ev}_1,\ldots,\mathrm{ev}_n)}"'] \arrow[dr, "\mathrm{ev}_{n+1}"] & \\
\overline{\mathscr{M}}_{g,n+1} \times (\overline{\mathscr{I}}_\mu X)^n & & \overline{\mathscr{I}}_\mu X
\end{tikzcd}$$

Get the $A_\bullet\overline{\mathscr{M}}$-algebra by push-pull along these spans

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Mann–Robalo: In genus 0, the spans are already a (lax) $\overline{\mathcal{M}}_0$-algebra structure

Contents – Section 1: Some operadic structures

Modular operads

A **modular ∞-operad** $\mathscr{P}$ in a cartesian ∞-category $\mathfrak{T}$ (*e.g.* $\mathfrak{T} = \mathfrak{Sh}(\mathfrak{U})$) has

- an involutive object $(\operatorname{col}\mathscr{P}, (-)^\vee)$ of colours
- objects of multimorphisms $\mathscr{P}(C_1, \ldots, C_n)$ with cyclic $\mathfrak{S}_n$-action

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- composition operations for $D_j = C_i^\vee$

$$\begin{aligned}\mathscr{P}(C_1, \ldots, C_i, \ldots, C_m) \times \mathscr{P}(D_1, \ldots, D_j, \ldots, D_n)\\ \to \mathscr{P}(C_1, \cdots, C_{i-1}, D_1, \ldots, D_{j-1}, D_{j+1}, \ldots, D_n, C_{i+1}, \ldots, C_m)\end{aligned}$$

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- contraction operations for $C_j = C_i^\vee$

$$\mathscr{P}(C_1, \dots, C_i, \dots, C_j, \dots, C_n) \to \mathscr{P}(C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_{j-1}, C_{j+1}, \dots, C_n)$$

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Optional: genus grading

Example: Costello's moduli of unstable curves

Fix $(X, \mathcal{L})$ target polarised stack (*i.e.* $X^{\mathcal{L}\text{-stb}}$ quasi-projective with $\mathcal{L}|_{X^{\mathcal{L}\text{-stb}}}$ ample)

$\mathfrak{M}^{X,\mathcal{L}}_{g,n,(r_1,\dots,r_n)}$ moduli stack of prestable twisted curves

- with ith marking gerbe of type $\mathcal{B}\,\mu_{r_i}$
- with decorations $\beta_j \in \mathrm{Eff}(X, \mathcal{L})$ on each component C_j, st. $\beta_j \neq 0$ if C_j unstable

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Composition along marked gerbes(Abramovich–Graber–Vistoli)

$$\mathfrak{M}_{g,n+1,(r_1,\dots,r_n,s)} \underset{\mathcal{B}^2\,\mu_s}{\times} \mathfrak{M}_{h,p+1,(\bar{s},t_1,\dots,t_p)} \to \mathfrak{M}_{g+h,n+p,(r_1,\dots,r_n,t_1,\dots,t_p)}$$

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Stack of colours $\mathcal{B}^2\mu \coloneqq \coprod_{r\geqslant 1} \mathcal{B}^2\mu_r$, involution taking opposite band

Example: endomorphisms modular operads

$\mathcal{V}^{\otimes}$ symmetric monoidal ∞-category (internal to $\mathfrak{T}$ & tensored over $\mathfrak{T}^{\times}$)

$\{X_\alpha\}_{\alpha \in A}$ family of self-dual objects

$\mathscr{E} := \mathscr{E}nd(\{X_\alpha\})$ modular operad with

- Colors the X_α

A $\mathscr{P}$-algebra in $\mathcal{V}^{\otimes}$ (with carriers the X_α) is a morphism $\mathscr{P} \to \mathscr{E}nd(\{X_\alpha\})$

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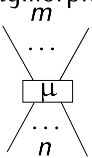
Rk: Can require the X_α only dualisable, by making a choice of $(\star)$

Properads (Vallette)

A **properad** $\mathscr{P}$ in $\mathfrak{E}$ has

- an object $\operatorname{col}\mathscr{P}$ of colours
- objects of polymorphisms $\mathscr{P}(C_1, \ldots, C_m; D_1, \ldots, D_n)$ with $\mathfrak{S}_m \times \mathfrak{S}_n$-action

  depicted as 

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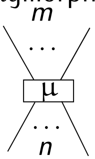
  depicted as 

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Ex.: *Any* symm. mon. ∞-category $\mathfrak{V}^{\otimes}$ gives an ∞-properad $\mathscr{V}$ with
$\mathscr{V}(C_1, \ldots, C_m; D_1, \ldots, D_n) = \mathfrak{V}(C_1 \otimes \cdots \otimes C_m, D_1 \otimes \cdots \otimes D_n)$

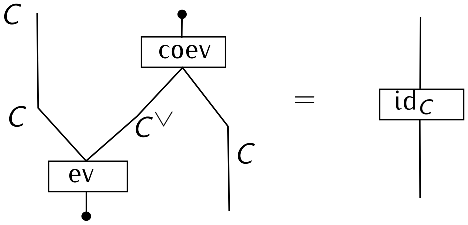
Modular operads as properads

$\mathscr{P}$ an ∞-properad

C is dualisable if $\exists C^\vee$ with

$$\mathrm{ev} \in \mathscr{P}(C, C^\vee;\,) \text{ and } \mathrm{coev} \in \mathscr{P}(;\, C^\vee, C)$$

satisfying triangle identities



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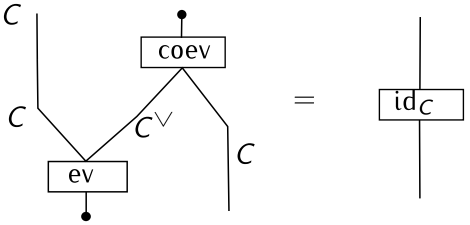

$\implies$ Cyclic action $\mathscr{P}(C_1, \ldots, C_m, C; D_1, \ldots, D_n) \xrightarrow{\simeq} \mathscr{P}(C_1, \ldots, C_m; D_1, \ldots, D_n, C^{\vee})$

Theorem (Barkan–Steinebrunner)

Functor $\mathfrak{ModOprd} \hookrightarrow \mathfrak{Proprd}$ with essential image the fully dualisable ∞-properads

Extensions

Unitality

Colour $C \in \mathscr{P}$ is inwardly (resp. outwardly) unital if $\mathscr{P}(;C)$ (resp. $\mathscr{P}(C;)$) is contractible

If C dualisable, then C inwardly unital iff $C^\vee$ outwardly unital

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Space of directed unital colours

$$\mathrm{DirUnit}(\mathscr{P}) = \coprod_{(C,\varepsilon)\in \mathrm{Col}\,\mathscr{P}\times(\mathrm{in},\mathrm{out})} \mathtt{isUnital}(C,\varepsilon)\Big/\Big((C,\varepsilon)\simeq(C^\vee,\overline{\varepsilon})\Big)$$

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Object of extensions of $\mu \in \mathscr{P}(C_1,\ldots,C_n;D_1,\ldots,D_n)$

$$\operatorname{Ext}(\mu) = \operatorname*{colim}_{P\in\operatorname{DirUnit}(\mathscr{P})} \operatorname{Ext}\big(\mu;(P,\varepsilon)\big)$$

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Extensions and prestable curves

$\forall r \geqslant 1$, have $\mathfrak{M}(;r) = \mathcal{B}^2\,\mu_r$

Consequence: Only colour 1 is unital, and $\operatorname{Ext}(C) = \mathfrak{M}^{X,\mathcal{L}}_{g,n+1,(r_1,\dots,r_n,1)} \underset{\mathfrak{M}^{X,\mathcal{L}}_{g,n,(r_1,\dots,r_n)}}{\times} \{C\}$

Lemma (Costello)

$\mathfrak{M}^{X,\mathcal{L}}_{g,n+1,(r_1,\dots,r_n,1)} \to \mathfrak{M}^{X,\mathcal{L}}_{g,n,(r_1,\dots,r_n)}$ is the universal curve $\mathfrak{C}^{X,\mathcal{L}}_{g,n,(r_1,\dots,r_n)} \to \mathfrak{M}^{X,\mathcal{L}}_{g,n,(r_1,\dots,r_n)}$

$$\implies \operatorname{Ext}(C) \simeq \mathfrak{C}^{X,\mathcal{L}}_{g,n} \underset{\mathfrak{M}^{X,\mathcal{L}}_{g,n}}{\times} \{C\} = C$$

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Recover moduli of $\mathcal{L}$-stable maps to X

$\mathcal{M}or(C,X) \overset{\text{opn}}{\supset} \overline{\mathscr{M}}^{\mathcal{L}}_{g,n}(X,\beta)_C$

Extensions and prestable curves

$\forall r \geqslant 1$, have $\mathfrak{M}(;r) = \mathcal{B}^2\,\mu_r$

Consequence: Only colour 1 is unital, and $\operatorname{Ext}(C) = \mathfrak{M}^{X,\mathcal{L}}_{g,n+1,(r_1,\dots,r_n,1)} \underset{\mathfrak{M}^{X,\mathcal{L}}_{g,n,(r_1,\dots,r_n)}}{\times} \{C\}$

Lemma (Costello)

$\mathfrak{M}^{X,\mathcal{L}}_{g,n+1,(r_1,\dots,r_n,1)} \to \mathfrak{M}^{X,\mathcal{L}}_{g,n,(r_1,\dots,r_n)}$ is the universal curve $\mathfrak{C}^{X,\mathcal{L}}_{g,n,(r_1,\dots,r_n)} \to \mathfrak{M}^{X,\mathcal{L}}_{g,n,(r_1,\dots,r_n)}$

$$\implies \operatorname{Ext}(C) \simeq \mathfrak{C}^{X,\mathcal{L}}_{g,n} \underset{\mathfrak{M}^{X,\mathcal{L}}_{g,n}}{\times} \{C\} = C$$

Recover moduli of $\mathcal{L}$-stable maps to X

$\mathcal{M}or(C,X) \overset{\text{opn}}{\supset} \overline{\mathscr{M}}^{\mathcal{L}}_{g,n}(X,\beta)_C$ so $\mathcal{M}or(\mathfrak{C}^{X,\mathcal{L}}_{g,n},X) \overset{\text{opn}}{\supset} \overline{\mathscr{M}}^{\mathcal{L}}_{g,n}(X,\beta)$

The brane action

The **universe** of an ∞-topos $\mathfrak{T}$ is the internal category $\mathfrak{T}_{/-}$

Theorem (Toën, Mann–Robalo in genus 0, K.–Pourcelot)

Let $\mathscr{P}$ be an $\mathfrak{T}$-properad. There is a lax morphism of 2-properads $\mathscr{P} \to \mathscr{Cospan}(\mathfrak{T}_{/-})^{\amalg}$ mapping $\mu \in \mathscr{P}(C_1, \ldots, C_m; D_1, \ldots, D_n)$ to

$$\begin{tikzcd}
 & \mathrm{Ext}(\mu) & \\
\coprod_{i=1}^{m} \mathrm{Ext}(\mathrm{id}_{C_i}) \arrow[ur, "\mu \circ_i -"] & & \coprod_{j=1}^{m} \mathrm{Ext}(\mathrm{id}_{D_j}) \arrow[ul, "- \circ_j \mu"']
\end{tikzcd}$$

Sketch of construction

Sequence of reductions

- Lax morphisms of 2-properads $\mathscr{P} \to \mathscr{Cospan}(\mathfrak{T}_{/\_})$ ($\mathscr{Cospan}(\mathfrak{T}_{/\_})$ monoidal)
  $\leftrightarrow$ monoidal lax 2-functors $\mathscr{Env}(\mathscr{P}) \to \mathscr{Cospan}(\mathfrak{T}_{/\_})$
- Lax functors $\mathfrak{Env}(\mathscr{P}) \to \mathscr{Cospan}(\mathfrak{T}_{/\_})$ are functors $\mathfrak{Tw}(\mathfrak{Env}(\mathscr{P})) \to \mathfrak{T}_{/\_}$
- $\mathfrak{T}_{/\_}$ is the discrete fibrations classifier in $\mathfrak{Cat}(\mathfrak{T})$

$\rightsquigarrow$ Discrete fibration over $\mathfrak{Tw}(\mathfrak{Env}(\mathscr{P}))$

Dendroidal encoding of operads

A **level forest** of height n is a functor $T_\bullet \colon [n] \to \mathfrak{Fin}_*^{\mathrm{act}} \simeq \mathfrak{Fin}$
Level **tree** if $T_n \simeq \langle 1 \rangle$

Elementaries

Height 0: the root $\big([0], \langle 1 \rangle\big)$
"walking colour"

Height 1: k-ary corolla $\left([1], \begin{array}{c} \langle k \rangle \\ \downarrow \\ \langle 1 \rangle \end{array}\right)$

Denoted $\star_{k,1}$

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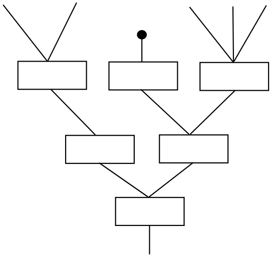
Composite of height [3]

$$\begin{array}{l}\langle 5 \rangle \\ \downarrow \{1,2\},\{\},\{3,4,5\} \\ \langle 3 \rangle \\ \downarrow \{1\},\{2,3\} \\ \langle 2 \rangle \\ \downarrow \{1,2\} \\ \langle 1 \rangle\end{array}$$



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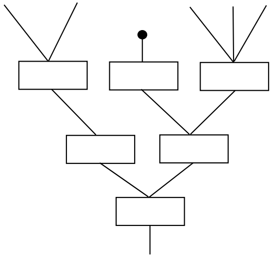

Segal condition: preserve decomposition as gluing of elementaries

Properadic operations as level graphs

A **level graph** of height n is a functor $[n] \to \mathfrak{Cospan}(\mathfrak{Fin})$

$$\begin{tikzcd} 3 \arrow[d, "\{1,2\},\{3\}"] \\ 2 \\ 5 \arrow[u, "\{1,2\},\{3,4,5\}"'] \arrow[d, "\{1\},\{2,3\},\{4,5\}"] \\ 3 \\ 4 \arrow[u, "\{1,2,3\},\{\},\{4\}"'] \end{tikzcd}$$

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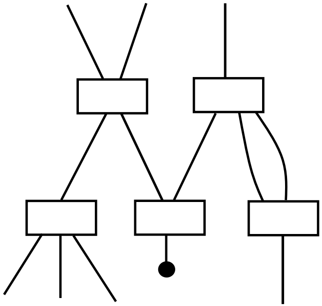

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3 \\
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4
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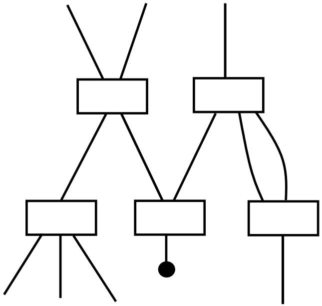

Corollas

$\star_{k,\ell} = \big([1], \langle k\rangle \to \langle 1\rangle \leftarrow \langle \ell\rangle\big)$

Monoidal envelopes for ∞-properads

Theorem (Barkan–Steinebrunner)

There is a fully faithful functor

$$\mathscr{E}\!nv\colon \mathfrak{Propr\mathfrak{d}} \to \mathfrak{CMon}((\infty,1)\text{-}\mathfrak{Cat})_{/\mathfrak{CoSpan}(\mathfrak{Fin})} \qquad \text{(where } \mathfrak{CoSpan}(\mathfrak{Fin}) \simeq \mathscr{E}\!nv(*)\text{)}$$

and for any ∞-properad $\mathscr{P}$

$$\mathrm{N}_n \mathscr{E}\!nv(\mathscr{P}) \simeq \operatorname*{colim}_{T \text{ of height } n} \mathscr{P}(T)$$

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- Objects: $\mathtt{N}_0 \mathscr{E}\mathit{nv}(\mathscr{P}) \simeq \operatorname*{colim}_{k \in \mathfrak{Fin}} \mathscr{P}([0], \langle k \rangle)$

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- Morphisms: $\mathrm{N}_1\mathscr{E}nv(\mathscr{P}) \simeq \operatorname*{colim}_{\langle k\rangle\to\langle p\rangle\leftarrow\langle \ell\rangle} \prod_{i\in\langle p\rangle} \mathscr{P}(\star_{k_i,\ell_i})$

Monoidal envelopes for ∞-properads

Theorem (Barkan–Steinebrunner)

There is a fully faithful functor

$$\mathscr{Env}\colon \mathfrak{Propd} \to \mathfrak{CMon}((\infty,1)\text{-}\mathfrak{Cat})_{/\mathfrak{CoSpan}(\mathfrak{Fin})} \qquad \text{(where } \mathfrak{CoSpan}(\mathfrak{Fin}) \simeq \mathscr{Env}(*)\text{)}$$

and for any ∞-properad $\mathscr{P}$

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Upshot: Recover extensions by going to $\langle k+1\rangle$ or $\langle \ell+1\rangle$ in the colim

Back to Gromov–Witten theory

$$
\begin{tikzcd}
\mathfrak{M}^{X,\mathcal{L}} \arrow[r] \arrow[d, "\mathrm{stab}"'] & \mathscr{C}\!\mathit{ospan}(\mathfrak{dSt}_{/-}) \arrow[r, "\mathscr{M}\!\mathit{or}^{\mathrm{repr}}(-,X)"] & \mathscr{S}\!\mathit{pan}(\mathfrak{dSt}_{/-}) \\
\overline{\mathscr{M}} & &
\end{tikzcd}
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\overline{\mathscr{M}} \arrow[urr, dashed, bend right, "\mathrm{LKE}"'] & &
\end{tikzcd}
$$

On the unique colour of $\overline{\mathscr{M}}$

$$
\begin{aligned}
\mathrm{LKE}(\cdot) &\simeq \operatorname*{colim}_{\coprod_r \mathcal{B}^2 \mu_r} \mathscr{M}\!\mathit{or}^{\mathrm{repr}}(\mathrm{Ext}(\mathrm{id}_r), X) \simeq \coprod_r \operatorname*{colim}_{\mathcal{B}(\mathcal{B}\,\mu_r)} \mathscr{M}\!\mathit{or}^{\mathrm{repr}}(\Omega\,\mathcal{B}^2\,\mu_r, X) \\
&\simeq \coprod_r \mathscr{M}\!\mathit{or}^{\mathrm{r}}(\mathcal{B}\,\mu_r, X)/\,\mathcal{B}\,\mu_r
\end{aligned}
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&\simeq \coprod_r \mathscr{M}\!\mathit{or}^{\mathrm{r}}(\mathcal{B}\,\mu_r, X) / \mathcal{B}\,\mu_r
\end{aligned}
$$

$\Longrightarrow$ Lax $\overline{\mathscr{M}}$-action on $\overline{\mathscr{I}}_{\mu} X$ relayed by $\overline{\mathscr{M}}^{\mathcal{L}}_{g,n}(X) \subset \mathscr{M}\!\mathit{or}^{\mathrm{repr}}\big(\mathrm{Ext}(\mathfrak{C}^{X,\mathcal{L}}_{g,n}), X\big)$

Geometry of the laxity

Laxity for a map of 2-properads $\mathscr{P} \to \mathscr{E}$ is

$$\begin{tikzcd} \mathscr{P}(n,1) \times \mathscr{P}(1,p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \mathscr{E}(n,1) \times \mathscr{E}(1,p) \arrow[d] \\ \mathscr{P}(n,p) \arrow[r] & \mathscr{E}(n,p) \end{tikzcd}$$

For $\overline{\mathscr{M}} \to \mathscr{Span}(\mathfrak{dSt}_{/\_})$ get

$$\coprod_{\gamma+\delta=\beta} \overline{\mathscr{M}}_{g,n}(X,\gamma) \underset{\overline{\mathscr{I}}_\mu X}{\times} \overline{\mathscr{M}}_{h,p}(X,\delta) \to \overline{\mathscr{M}}_{g+h,n+p}(X,\beta) \underset{\overline{\mathscr{M}}_{g+h,n+p}}{\times} (\overline{\mathscr{M}}_{g,n} \times \overline{\mathscr{M}}_{h,p})$$

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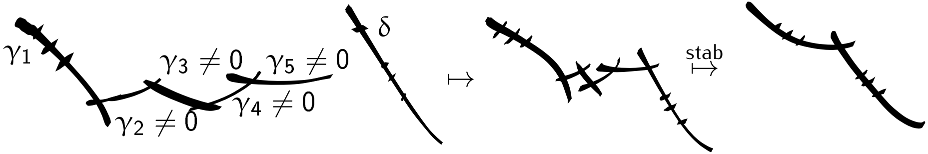

Categorical GW actions

Let $\mathfrak{D}\colon \mathfrak{Span}(\mathfrak{dSt}) \to (\infty,1)\text{-}\mathfrak{Cat}$ be a 6-functors formalism (*e.g.* $\mathfrak{D} = \mathfrak{QCoh}$)

$\rightsquigarrow$ $\mathfrak{dSt}$-2-category $\mathfrak{D}\text{-}\mathscr{M}\!\mathit{od}\colon T \mapsto \mathfrak{D}(T)\text{-}\mathfrak{Mod}$

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By composing: Lax morphism $\overline{\mathscr{M}} \to \mathscr{Span}(\mathfrak{dSt}_{/\_})^{\times} \to \mathfrak{D}\text{-}\mathscr{Mod}^{\otimes}$

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Linearisation

Assume all $\mathfrak{D}(T)$ compactly generated and $\mathfrak{D}(\overline{\mathscr{M}}_{g,n})$ smooth and proper

E.g. for $\mathfrak{D} = \mathfrak{QCoh}$, if all stacks are perfect

Then get a lax $\mathfrak{D}(\overline{\mathscr{M}})$-algebra structure

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- For $\mathfrak{D} = \mathfrak{QCoh}$, specialises to $\mathfrak{Coh}^{\mathrm{b}}$ and $\mathfrak{Perf}$
- Decategorifies to $\mathrm{K}(\mathfrak{D})$

Contents - Section 3: Quantising categories of quasicoherent sheaves

Riva's category of push-pull spans

Data: Functor $\mathcal{D}^{\otimes}\colon \mathfrak{dSt}^{\mathrm{op}} \to \mathfrak{SMCat}$ st each $f^*\colon \mathcal{D}Y \to \mathcal{D}X$ has a right adjoint f_*

Construction (Riva, *cf.* also Liu–Zheng, Aoki)

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- composition of 2-cells $\alpha\colon Z_1 \Rightarrow Z_2$ and $\beta\colon Z_2 \Rightarrow Z_3$ is

$$(\mathrm{pr}_{1,3})_*\left(\mathrm{pr}_{1,2}^*\,\alpha \otimes \mathrm{pr}_{2,3}^*\,\beta\right) \qquad \text{where} \qquad \mathrm{pr}_{i,j}\colon Z_1 \underset{X\times Y}{\times} Z_2 \underset{X\times Y}{\times} Z_3 \to Z_i \underset{X\times Y}{\times} Z_j$$

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Rk: Similar to small quantum product

$$\gamma_1 \bullet \gamma_2 = \sum_{\beta} Q^{\beta}\,(\mathrm{ev}_3)_*\big(\,ev_1^*\gamma_1 \smile \mathrm{ev}_2^*\,\gamma_2 \frown \big[\overline{\mathscr{M}}_{0,3}(X,\beta)\big]^{\mathrm{vir}}\big) \text{ on } A^{\bullet}X \otimes \mathrm{Nov}$$

Categories over an operad (May, Trimble)

$\mathfrak{O}$ monochromatic planar 1-operad, $\mathfrak{V}^{\otimes}$ monoidal category tensored

An $\mathfrak{O}$-category $\mathfrak{C}$ has

- a set of objects $\operatorname{obj}\mathfrak{C}$
- $\forall X, Y \in \operatorname{obj}\mathfrak{C}$, object $\mathfrak{C}(X, Y)$ in $\mathfrak{V}$

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Examples

- An $\mathscr{Assoc}$-category is a category (enriched in $\mathfrak{V}$)
- An $\mathscr{A}_\infty$-category is an $\mathscr{A}_\infty$-category

Looking through the macrocosm principle

A “many-objects (planar) operad” is a **virtual double category**, consisting of:

- category of objects and vertical arrows
- horizontal arrows (without composition)
- multicells $\begin{array}{ccccc} \cdot \longrightarrow \cdot & & \cdots & & \cdot \longrightarrow \cdot \\ \Big\downarrow & & \Downarrow & & \Big\downarrow \\ \cdot & & \xrightarrow{\hspace{6em}} & & \cdot \end{array}$ with box stacking composition

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- horizontal arrows (without composition)
- multicells

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with box stacking composition

Base-change $f^* \colon \mathfrak{VDC}_{S'} \to \mathfrak{VDC}_S$ for $f \colon S \to S'$ of vertical cats

- When f is $S \to *$, **codiscrete** virtual double category on an operad

Looking through the macrocosm principle

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Equivalence between $\mathcal{P}$-categories with objects set S and morphisms $S^*\mathcal{P} \to \mathcal{B}\,\mathfrak{V}$

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Symmetric case: Horizontal arrows have same source and target

Categories over an ∞-properad

Plus construction (..., Blom–Loubaton–Ruit)

In dendroidal model, add $\big([0], \langle 0\rangle\big)$ and $\big([1], \langle 0\rangle \to \langle 0\rangle\big)$ as elementaries

Representation $\big([0], \langle 0\rangle\big)$ as $\cdot$ an object and $\big([1], \langle 0\rangle \to \langle 0\rangle\big)$ as $\downarrow$

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Do the same for directed graphs: bivirtual (symmetric) double categories

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$$\star_{k,\ell} \text{ as } \begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] & \cdots \arrow[r] \arrow[d, phantom, "\Downarrow"] & \cdot \arrow[r, "k"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdots \arrow[r] & \cdot \arrow[r, "\ell"'] & \cdot \end{tikzcd}$$

The A-model $\overline{\mathscr{M}}$-category

We have: functor $T \to \mathfrak{D}(T)$ to $\mathfrak{D}(\overline{\mathscr{M}})$-monoidal categories

The expectation

There is an $(\infty, 3)$-category locally over $\mathfrak{D}(\overline{\mathscr{M}})$ with

- 0- and 1-cells as in $\mathfrak{Span}(\mathfrak{dSt})$
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Question: How to compare to deformation quantisation?

Toward the algebraic Fukaya category

Topological recursion for WKB quantisation (Chekhov–Eynard–Orantin)

Summing over topol. rec. graphs produces structure constants for formal series in $\hbar$

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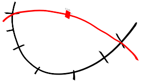
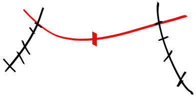
Topological recursion operads (Safnuk)

- Elementaries as for modular operads
- Contraction and gluing add rational bridge





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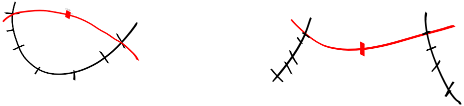

$\Longrightarrow$ Expect comparison functor $\mathfrak{ModOprd} \to \mathfrak{TR}\text{-}\mathfrak{Oprd}$

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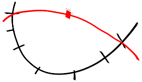

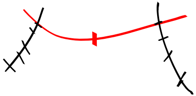

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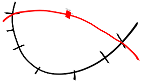

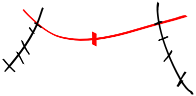

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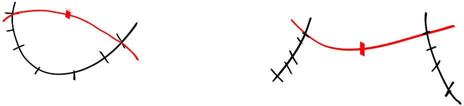

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Conjecture

The composite $\mathfrak{ModOprd}_{\mathbb{C}} \to \mathfrak{Oprd}_{\mathbb{C}[[\hbar]]}$ maps $\overline{\mathscr{M}} \mapsto \mathscr{Assoc}[[\hbar]]$