Complexes on the Hodge–Tate divisor

David KERN

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Reminder 0.1. Last week, Jon-Magnus defined:

- the Hodge–Tate divisor $\mathcal{WC}art^{\mathrm{HT}} = \mathcal{WC}art \times_{[\widehat{\mathbb{A}}^1/\mathbb{G}_m]} \mathcal{B}\,\mathbb{G}_m$ whose R-points are the Cartier–Witt divisors $I \to W(R)$ such that $I \to I(R) \twoheadrightarrow R$ is the zero map
- for every prism (A, I) a cartesian square

$$\begin{tikzcd} \mathrm{Spf}(A/I) \arrow[r, "\rho_A^{\mathrm{HT}}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{WC}art^{\mathrm{HT}} \arrow[d] \\ \mathrm{Spf}(A) \arrow[r, "\rho_A"] & \mathcal{WC}art \end{tikzcd} \tag{1}$$

- a point $\eta\colon \mathrm{Spf}(\mathbb{Z}_p) \to \mathcal{WC}art^{\mathrm{HT}}$ given by $V(1)$ where V denotes the Verschiebung operator, and which is a faithfully flat surjection
- an equivalence $\mathcal{WC}art^{\mathrm{HT}} \simeq \mathcal{B}_{\mathrm{Spf}\,\mathbb{Z}_p}(\mathbb{G}_m^\sharp)_{\mathrm{Spf}\,\mathbb{Z}_p} = \mathrm{Spf}(\mathbb{Z}_p) \times \mathcal{B}\,\mathbb{G}_m^\sharp$ where $\mathbb{G}_m^\sharp$ is the divided power envelope of $\mathbb{G}_m$ along the ideal $(t-1)$
- an invertible sheaf $\mathcal{I} \to \mathcal{O}_{\mathcal{WC}art}$ given by $(I \to W(R)) \mapsto I \otimes_{W(R)} R$, and called the Hodge–Tate ideal sheaf.

1 Sen operator for complexes on the Hodge–Tate divisor

Notation 1.1. Recall that we denote $\mathcal{D}(\mathcal{WC}art^{\mathrm{HT}})$ the derived ∞-category of quasicoherent sheaves/complexes on $\mathcal{WC}art^{\mathrm{HT}}$, defined formally as $\mathcal{D}(\mathcal{WC}art^{\mathrm{HT}}) = \lim\limits_{\mathrm{Spec}(R) \to \mathcal{WC}art^{\mathrm{HT}}} \mathcal{D}(R)$.

Remark 1.2. We have $\mathcal{D}(\mathcal{WC}art^{\mathrm{HT}}) \xrightarrow{\simeq} \lim_{(A,I)\in \mathfrak{Prism}} \widehat{\mathcal{D}}(A/I)$, where $\widehat{\mathcal{D}}$ denotes the p-complete derived ∞-category, and the functors $\mathcal{D}(\mathcal{WC}art^{\mathrm{HT}}) \to \widehat{\mathcal{D}}(A/I)$ are $(\rho_A^{\mathrm{HT}})^*$.

Example 1.3. For any $n \in \mathbb{Z}$, we have an equivalence $\mathcal{O}_{\mathcal{WC}art^{\mathrm{HT}}}\{n\} \simeq \mathcal{I}^n|_{\mathcal{WC}art^{\mathrm{HT}}}$.

Our goal is to understand $\mathcal{D}(\mathcal{WC}art^{\mathrm{HT}})$, which is equivalently $\widehat{\mathcal{D}}(\mathcal{B}_{\mathrm{Spf}\,\mathbb{Z}_p}(\mathbb{G}_m^\sharp)_{\mathrm{Spf}\,\mathbb{Z}_p})$.

Fairy tale 1.4. We will consider fairy-land to be working integrally.

Note that for any $\mathcal{E} \in \mathcal{D}(\mathcal{B}\,\mathbb{G}_m)$, the endomorphism Θ induces an automorphism $\mathrm{id} + \varepsilon\Theta$ of $\mathcal{E} \otimes_{\mathbb{Z}} \mathbb{Z}[\varepsilon]/(\varepsilon^2)$ which reduces to the identity modulo ε, and it can be recovered from this.

Construction 1.5 (Sen operator). Let R be a $\mathbb{Z}[\varepsilon]/(\varepsilon^2)$-algebra. We let $[\varepsilon] \in W(R)$ denote the Teichmüller lift of ε. It is killed by the Frobenius morphism of $W(R)$, so multiplication by $(1 + [\varepsilon])$ acts as the identity on the Verschiebung ideal $VW(R)$.

For any Cartier–Witt divisor $(I \xrightarrow{\alpha} R)$ in the Hodge–Tate divisor, that is such that α factors through $VW(R) \subseteq W(R)$, multiplication by $(1 + [\varepsilon])$ then defines an automorphism of (I, α). Lettting R and (I, α) vary, this determines an automorphism γ of the projection $\pi\colon \mathcal{W}\mathcal{C}art^{\mathrm{HT}} \times \operatorname{Spec}(\mathbb{Z}[\varepsilon]/(\varepsilon^2)) \to \mathcal{W}\mathcal{C}art^{\mathrm{HT}}$. In particular, for any $\mathcal{E} \in \mathcal{D}(\mathcal{W}\mathcal{C}art^{\mathrm{HT}})$, it induces an automorphism of $\pi^*\mathcal{E} = \mathcal{E} \otimes_{\mathbb{Z}} \mathbb{Z}[\varepsilon]/(\varepsilon^2)$ which reduces to the identity modulo ε.

We denote $\Theta_{\mathcal{E}}$ the corresponding endomorphism of $\mathcal{E}$, and call it the **Sen operator** on $\mathcal{E}$.

For any $\mathcal{E} \in \mathcal{D}(\mathcal{W}\mathcal{C}art^{\mathrm{HT}})$, we will write $\mathcal{E}_\eta \coloneqq \eta^*\mathcal{E} \in \mathcal{D}(\operatorname{Spf} \mathbb{Z}_p) \simeq \widehat{\mathcal{D}}(\mathbb{Z}_p)$, and we will still write $\Theta_{\mathcal{E}}\colon \mathcal{E}_\eta \to \mathcal{E}_\eta$ for the restriction of $\Theta_{\mathcal{E}}$.

Example 1.6. Take $\mathcal{E} = \eta_*\mathcal{O}_{\operatorname{Spf}\mathbb{Z}_p}$, so that we have $\eta^*\mathcal{E} \simeq \widehat{\mathcal{O}_{\mathbb{G}_m^\sharp}}$ (where $\widehat{\mathcal{O}}$ again denotes the p-completion).

The automorphism γ of $\pi\colon \mathcal{W}\mathcal{C}art^{\mathrm{HT}} \times \operatorname{Spec}(\mathbb{Z}[\varepsilon]/(\varepsilon^2)) \to \mathcal{W}\mathcal{C}art^{\mathrm{HT}}$ used to construct the Sen operator restricts to an automorphism of $\operatorname{Spf}(\mathbb{Z}_p[\varepsilon]/(\varepsilon^2)) \to \operatorname{Spf}\mathbb{Z}_p \xrightarrow{\eta} \mathcal{W}\mathcal{C}art^{\mathrm{HT}}$. Recall that we have $\operatorname{Aut}(\eta) \simeq \mathbb{G}_m^\sharp$, and this automorphism becomes $1 + \varepsilon \in \mathbb{G}_m^\sharp(\mathbb{Z}_p[\varepsilon]/(\varepsilon^2))$.

One can then see that for any $f(t) \in \widehat{\mathcal{O}_{\mathbb{G}_m^\sharp}}$ we have $f((1+\varepsilon)t) = f(t) + \varepsilon\Theta_{\mathcal{E}}(f(t))$, so that $\Theta_{\mathcal{E}}$ maps $f(t) \mapsto t\frac{\partial}{\partial t}f(t)$.

Example 1.7. For any Cartier–Witt divisor $(I \xrightarrow{\alpha} R)$, the restriction $W(R) \twoheadrightarrow R$ maps $1+[\varepsilon]$ to $1 + \varepsilon$, so the Sen operator on $\mathcal{I}(I, \alpha) = I \otimes_{W(R)} R$ is the identity.

For any $\mathcal{E}, \mathcal{F} \in \mathcal{D}(\mathcal{W}\mathcal{C}art^{\mathrm{HT}})$, we have an equivalence $\Theta_{\mathcal{E}\otimes\mathcal{F}} \simeq (\Theta_{\mathcal{E}} \otimes \mathrm{id}_{\mathcal{F}}) + (\mathrm{id}_{\mathcal{E}} \otimes \Theta_{\mathcal{F}})$. It follows that $\Theta_{\mathcal{I}|^n_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}}}}$ acts as multiplication by n on $\mathcal{I}|^n_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}}} \simeq \mathcal{O}_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}}}\{n\}$.

Note that since the Sen operator is zero on $\mathcal{O}_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}}}$, the unit map $\upsilon\colon \mathcal{O}_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}}} \to \eta_*\eta^*\mathcal{O}_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}}} = \eta_*\mathcal{O}_{\operatorname{Spf}\mathbb{Z}_p}$ factors through

$$\left(\eta_*\mathcal{O}_{\operatorname{Spf}\mathbb{Z}_p}\right)^{\Theta=0} \coloneqq \operatorname{fib}\left(\Theta_{\eta_*\mathcal{O}_{\operatorname{Spf}\mathbb{Z}_p}}\right). \tag{2}$$

Lemma 1.8 (Proposition 3.5.10). *The unit map $\mathcal{O}_{\mathcal{W}\mathcal{C}art^{HT}} \to \left(\eta_*\mathcal{O}_{\operatorname{Spf}\mathbb{Z}_p}\right)^{\Theta=0}$ is an equivalence.*

Proof. Since $\eta\colon \operatorname{Spf}\mathbb{Z}_p \to \mathcal{W}\mathcal{C}art^{\mathrm{HT}}$ is faithfully flat, it is enough to show that $\eta^*\mathcal{O}_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}}} \to \eta^*\left(\eta_*\mathcal{O}_{\operatorname{Spf}\mathbb{Z}_p}\right)^{\Theta=0}$ is an equivalence. We have computed that the sequence whose exactness is under question is

$$\mathbb{Z}_p \to \widehat{\mathcal{O}_{\mathbb{G}_m^\sharp}} \xrightarrow{t\frac{\partial}{\partial t}} \widehat{\mathcal{O}_{\mathbb{G}_m^\sharp}}. \tag{3}$$

To compute it, we identify $\widehat{\mathcal{O}_{\mathbb{G}_m^\sharp}}$ with the formal series $f(t) = \sum_{n\in\mathbb{N}} c_n \frac{(t-1)^n}{n!}$ such that $(c_n)_{n\in\mathbb{N}}$ converges to 0 in $\mathbb{Z}_p$. The differential operator $t\frac{\partial}{\partial t}$ then takes the form

$$\sum_{n\in\mathbb{N}} c_n \frac{(t-1)^n}{n!} \mapsto \sum_{n\in\mathbb{N}} (c_{n+1} + nc_n) \frac{(t-1)^n}{n!} \tag{4}$$

from which we easily see that it is surjective and its kernel consists of those series $\sum_{n\in\mathbb{N}} c_n \frac{(t-1)^n}{n!}$ whose coefficients $c_n = 0$ for $n > 0$, so the unit copy of $\mathbb{Z}_p$. ☐

Corollary 1.9 (Proposition 3.5.11). *For any $\mathcal{E} \in \mathcal{D}(\mathcal{W}\mathcal{C}art^{HT})$, there is a fibre sequence*

$$\mathbb{R}\Gamma(\mathcal{W}\mathcal{C}art^{HT}, \mathcal{E}) \to \mathcal{E}_\eta \to \mathcal{E}_\eta. \tag{5}$$

Proof. Tensor the fibre sequence $\mathcal{O}_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}}} \to \eta_*\mathcal{O}_{\mathrm{Spf}\,\mathbb{Z}_p} \xrightarrow{\Theta} \eta_*\mathcal{O}_{\mathrm{Spf}\,\mathbb{Z}_p}$ by $\mathcal{E}$ and use the projection formula $\mathcal{E} \otimes \eta_*\mathcal{O}_{\mathrm{Spf}\,\mathbb{Z}_p} \simeq \eta_*(\eta^*\mathcal{E} \otimes \mathcal{O}_{\mathrm{Spf}\,\mathbb{Z}_p})$, so that the desired formula is obtained by taking global sections. ☐

Proposition 1.10 (Corollary 3.5.14). *A quasicoherent complex $\mathcal{E} \in \mathcal{D}(\mathcal{W}\mathcal{C}art^{HT})$ is equivalent to a Breuil–Kisin twist $\mathcal{O}_{\mathcal{W}\mathcal{C}art^{HT}}\{n\}$, for $n \in \mathbb{Z}$, if and only if*

1. *the fibre $\mathcal{E}_\eta$ is equivalent to $\mathbb{Z}_p$,*
2. *the Sen operator $\Theta_\mathcal{E} \colon \mathcal{E}_\eta \to \mathcal{E}_\eta$ is given by multiplication by n.*

Lemma 1.11 (Proposition 3.5.15). *The ∞-category $\mathcal{D}(\mathcal{W}\mathcal{C}art^{HT})$ is generated by $\mathcal{O}_{\mathcal{W}\mathcal{C}art^{HT}}$ under colimits, shifts, and positive Breuil–Kisin twists.*

Proof. By corollary 1.9, any $\mathcal{E} \in \mathcal{D}(\mathcal{W}\mathcal{C}art^{\mathrm{HT}})$ fits in a cofibre sequence $\eta_*\mathcal{E}_\eta[-1] \xrightarrow{\Theta} \eta_*\mathcal{E}_\eta[-1] \to \mathcal{E}$, so is a colimit of shifts of $\eta_*\mathcal{E}_\eta$. Thus we only need to show that the full subcategory $\mathfrak{K}$ generated under shifts and colimits by the $\mathcal{O}_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}}}\{n\}$, $n \geq 0$, contains the objects of the form $\mathcal{F} = \eta_*\mathcal{M}$ for any $\mathcal{M} \in \mathcal{D}(\mathrm{Spf}\,\mathbb{Z}_p)$. Since $\mathcal{D}(\mathrm{Spf}\,\mathbb{Z}_p)$ is generated under shifts and colimits by $\mathcal{O}_{\mathrm{Spf}\,\mathbb{Z}_p}$, it will suffice to consider the case $\mathcal{M} = \mathcal{O}_{\mathrm{Spf}\,\mathbb{Z}_p}$.

Viewing $\mathcal{D}(\mathcal{W}\mathcal{C}art^{\mathrm{HT}})$ as the ∞-category of $\mathcal{O}_{\mathbb{G}_m^\sharp}$-comodules in $\widehat{\mathcal{D}}(\mathbb{Z}_p)$, the object $\mathcal{F} = \eta_*\mathcal{O}_{\mathrm{Spf}\,\mathbb{Z}_p}$ corresponds to the p-complete regular corepresentation of $\mathcal{O}_{\mathbb{G}_m^\sharp}$, denoted $\widehat{\mathcal{O}_{\mathbb{G}_m^\sharp}}$.

Letting $V_{\leq n}$ denote the sub-$\mathbb{Z}_p$-module of $\widehat{\mathcal{O}_{\mathbb{G}_m^\sharp}}$ generated by the $\frac{(t-1)^m}{m!}$ for $m \leq n$, one can check that the coaction $c \colon \widehat{\mathcal{O}_{\mathbb{G}_m^\sharp}} \to \mathcal{O}_{\mathbb{G}_m^\sharp} \widehat{\otimes} \widehat{\mathcal{O}_{\mathbb{G}_m^\sharp}}$ leaves $V_{\leq n}$ stable so that it becomes a $\mathcal{O}_{\mathbb{G}_m^\sharp}$-comodule: indeed this coaction is determined by $c(t) = t \otimes t$, and so one can see that

$$c\left(\frac{(t-1)^n}{n!}\right) \in t^n \otimes \frac{(t-1)^n}{n!} + \mathcal{O}_{\mathbb{G}_m^\sharp} \widehat{\otimes} V_{\leq n-1}.$$

This means that $V_{\leq n}$ corresponds to an object $\mathcal{F}_{\leq n} \in \mathcal{D}(\mathcal{W}\mathcal{C}art^{\mathrm{HT}})$. Furthermore, the Sen operator acts as multiplication by n on the quotient $(\mathcal{F}_{\leq n}/\mathcal{F}_{\leq n-1})_\eta$, so by the recognition principle the latter is isomorphic to $\mathcal{O}_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}}}\{n\}$. We can thus prove by induction on n that $\mathcal{F}_{\leq n} \in \mathfrak{K}$ for all $n \geq 0$, since we have a cofibre sequence $\mathcal{F}_{\leq n-1} \to \mathcal{F}_{\leq n} \to \mathcal{O}_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}}}\{n\}$. Finally, $\mathcal{F} = \mathrm{colim}_n\, \mathcal{F}_{\leq n}$ is in $\mathfrak{K}$. ☐

Despite working on the Hodge–Tate divisor, it is also possible to leverage this result to obtain a description of the derived ∞-category of the entire Cartier–Witt stack (which we will not need).

Corollary 1.12 (Corollary 3.5.16)**.** *The ∞-category $\mathcal{D}(\mathcal{W}\mathcal{C}art)$ is generated under colimits and shifts by the invertible sheaves $\mathcal{I}^n$ for $n \in \mathbb{Z}$.*

Proof. Omitted for time. $\square$

We can finally arrive at the main result.

Theorem 1.13 (Theorem 3.5.8)**.** *The functor*

$$\begin{aligned} \mathcal{D}(\mathcal{W}\mathcal{C}art^{HT}) &\to \mathcal{D}(\mathbb{Z}[\Theta]) \\ \mathcal{E} &\mapsto (\mathcal{E}_\eta, \Theta_\mathcal{E}) \end{aligned} \tag{6}$$

is fully faithful, with essential image the objects $M \in \mathcal{D}(\mathbb{Z}[\Theta])$ satisfying:

1. *The complex M is p-complete.*
2. *The action of $\Theta^p - \Theta$ on the cohomology $H^\bullet(\mathbb{F}_p \otimes^{\mathbb{L}} M)$ is locally nilpotent.*

Proof. We start by establishing full faithfulness, *i.e.* that for any $\mathcal{E}, \mathcal{F} \in \mathcal{D}(\mathcal{W}\mathcal{C}art^{\mathrm{HT}})$ we have $\hom(\mathcal{E}, \mathcal{F}) \simeq \hom\big((\mathcal{E}_\eta, \Theta_\mathcal{E}), (\mathcal{F}_\eta, \Theta_\mathcal{F})\big)$. Thanks to lemma 1.11, we may assume that $\mathcal{E}$ is $\mathcal{O}_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}}}\{n\}$ for some n, and even that this n is 0 up to replacing $\mathcal{F}$ by $\mathcal{F}\{-n\}$.

We then have, on the left hand side of the desired equivalence, that $\hom(\mathcal{E}, \mathcal{F}) = \mathbb{R}\Gamma(\mathcal{W}\mathcal{C}art^{\mathrm{HT}}, \mathcal{F})$. By corollary 1.9 this is equivalent to $\mathrm{fib}(\mathcal{F}_\eta \to \mathcal{F}_\eta)$.

On the other hand, since the Sen operator on $\mathcal{O}_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}},\eta}$ acts by zero, $\hom\big((\mathcal{O}_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}},\eta}, 0), (\mathcal{F}_\eta, \Theta_\mathcal{F})\big)$ also forces the Sen operator on $\mathcal{F}_\eta$ to become zero, or in other words takes the fibre of $\Theta_\mathcal{F}$. Thus both sides are indeed equivalent.

Let us now sketch the necessity and sufficience of the conditions identifying the essential image of the functor. For necessity, the first condition is clear, and for the second we can again reduce to the case of $\mathcal{E} = \mathcal{O}_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}}}\{n\}$ on which the Sen operator acts by multiplication by n, so that it follows from $n \cong n^p \mod p\mathbb{Z}$. Now to establish sufficiency, it is enough to show that the full subcategory of $\mathcal{D}(\mathbb{Z}[\Theta])$ spanned by the objects satisfying the conditions is generated (under colimits and shifts) by the $\mathcal{O}_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}}}\{n\}_\eta$ for $n \in \mathbb{Z}$.

This means showing that every nonzero M in this subcategory admits a nonn-trivial map from some $\mathcal{O}_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}}}\{n\}_\eta[m]$. In fact, from knowing that the action of the Sen operator on $\mathcal{O}_{\mathcal{W}\mathcal{C}art^{\mathrm{HT}}}\{n\}_\eta$ is multiplication by n, we can write the latter object as $\mathbb{Z}_p[\Theta]/(\Theta - n)[m]$. Now we have assumed that some cohomology group $H^m(\mathbb{F}_p \otimes^{\mathbb{L}} M)$ contains a nonzero element μ annihilated by some power of $\Theta^p - \Theta$, and so an element μ' annihilated by $\Theta^p - \Theta$. But on $\mathbb{F}_p$ we have the decomposition $\Theta^p - \Theta = \prod_{0 \le n < p}(\Theta - n)$, so we deduce the existence of an element μ'' annihilated by some $\Theta - n$, and thus defining a map from $\mathbb{Z}_p[\Theta]/(\Theta - n)[m]$. $\square$

2 Understanding the structure of the Hodge–Tate divisor

2.1 The Frobenius operator of the Cartier–Witt stack

Construction 2.1.1 (The Frobenius endomorphism). Let $(I \xrightarrow{\alpha} R) \in \mathcal{WC}art(R) \subset \mathcal{C}art(W(R))$. Then $F(I, \alpha)$ is still a Cartier–Witt divisor, where F denotes the Frobenius of $W(R)$. Thus F determines an endomorphism of the stack $\mathcal{WC}art$, also denoted F and called the Frobenius of $\mathcal{WC}art$.

Example 2.1.2. For any prism (A, I), the Frobenius $A \to A$ coming from the δ-structure on A fits in a commutative diagram

$$\begin{tikzcd} \operatorname{Spf}(A) \arrow[r, "\rho_A"] \arrow[d, "F"'] & \mathcal{WC}art \arrow[d, "F"] \\ \operatorname{Spf}(A) \arrow[r, "\rho_A"'] & \mathcal{WC}art \end{tikzcd} \tag{7}$$

Example 2.1.3. For every $n \in \mathbb{Z}$, we have an isomorphism of invertible sheaves $F^*\mathcal{O}_{\mathcal{WC}art}\{n\} \simeq \mathscr{I}^{\otimes -n}\mathcal{O}_{\mathcal{WC}art}\{n\}$ where $F^*\colon \mathcal{D}(\mathcal{WC}art) \to \mathcal{D}(\mathcal{WC}art)$ is pullback along the Frobenius.

Definition 2.1.4 (The de Rham point). *Let (A, I) be the crystalline prism $(\mathbb{Z}_p, (p))$. The induced morphism $\rho_A\colon \operatorname{Spf}\mathbb{Z}_p \to \mathcal{WC}art$ is denoted ρ_{dR} and called the* ***de Rham point*** *of $\mathcal{WC}art$.*

Proposition 2.1.5 (Proposition 3.6.6). *There is a commutative square of stacks*

$$\begin{tikzcd} \mathcal{WC}art^{HT} \arrow[r, hook] \arrow[d] & \mathcal{WC}art \arrow[d, "F"] \\ \operatorname{Spf}\mathbb{Z}_p \arrow[r, "\rho_{dR}"'] & \mathcal{WC}art \end{tikzcd} \tag{8}$$

Theorem 2.1.6 (Theorem 3.6.7). *For any $\mathcal{E} \in \mathcal{D}(\mathcal{WC}art^{HT})$, there is a cartesian square*

$$\begin{tikzcd} \mathbb{R}\Gamma(\mathcal{WC}art^{HT}, (F^*\mathcal{E})|_{\mathcal{WC}art^{HT}}) & \mathbb{R}\Gamma(\mathcal{WC}art, F^*\mathcal{E}) \arrow[l] \\ \mathbb{R}\Gamma(\operatorname{Spf}\mathbb{Z}_p, \rho_{dR}^*\mathcal{E}) \arrow[u] & \mathbb{R}\Gamma(\mathcal{WC}art, \mathcal{E}) \arrow[l] \arrow[u] \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \tag{9}$$

in $\widehat{\mathcal{D}}(\mathbb{Z}_p)$.

Corollary 2.1.7 (Corollary 3.6.9). *The functor $\mathcal{D}(\mathcal{WC}art) \xrightarrow{(F^*, \rho_{dR}^*)} \mathcal{D}(\mathcal{WC}art) \times_{\mathcal{D}(\mathcal{WC}art^{HT})} \mathcal{D}(\operatorname{Spf}\mathbb{Z}_p)$ is fully faithful.*

Example 2.1.8 (Lemma 3.6.16). For any $n \in \mathbb{Z}$, the complex $\mathbb{R}\Gamma\big(F^{-1}\mathcal{WC}art^{\mathrm{HT}}, F^*(\mathcal{O}_{\mathcal{WC}art^{\mathrm{HT}}}\{n\})\big)$ is equivalent to either $\mathbb{Z}_p \oplus \mathbb{Z}_p[-1]$ or $\mathbb{Z}/(p^k)[-1]$ for some $k > 0$.

2.2 Covering by the q-de Rham prism

Construction 2.2.1 (The q-de Rham point). Consider the q-de Rham prism $(\mathbb{Z}_p[\![q-1]\!], ([p]_q))$ and its induced morphism $\rho_{qdR}\colon \operatorname{Spf}(\mathbb{Z}_p[\![q-1]\!]) \to \mathcal{W}\mathcal{C}art$ which is a faithfully flat cover because the q-de Rham prism is transversal.

The group $\mathbb{Z}_p^\times$ acts on $\operatorname{Spf}(\mathbb{Z}_p[\![q-1]\!])$ by $(u, q) \mapsto q^u$, and ρ_{qdR} is $\mathbb{Z}_p^\times$-equivariant

Proposition 2.2.2 (Proposition 3.8.2)**.** *This map determines a morphism of stacks* $[\rho_{qdR}]\colon \big[\operatorname{Spf}(\mathbb{Z}_p[\![q-1]\!])/\mathbb{Z}_p^\times\big] \to \mathcal{W}\mathcal{C}art$*, where* $\mathbb{Z}_p^\times$ *is seen as a pro-finite étale group scheme.*

Remark 2.2.3. The composite $\operatorname{Spf}\mathbb{Z}_p \xrightarrow{q\mapsto 1} \big[\operatorname{Spf}(\mathbb{Z}_p[\![q-1]\!])/\mathbb{Z}_p^\times\big] \xrightarrow{[\rho_{qdR}]} \mathcal{W}\mathcal{C}art$ coincides with the de Rham point ρ_{dR}.

As a consequence, we have a commutative square

$$\begin{tikzcd}
{[\operatorname{Spf}(\mathbb{Z}_p)/\mathbb{Z}_p^\times]} \arrow[r] \arrow[d] & \big[\operatorname{Spf}(\mathbb{Z}_p[\![q-1]\!])/\mathbb{Z}_p^\times\big] \arrow[d, "\rho_{qdR}"] \\
\operatorname{Spf}\mathbb{Z}_p \arrow[r, "\rho_{dR}"'] & \mathcal{W}\mathcal{C}art
\end{tikzcd} \tag{10}$$

Theorem 2.2.4 (Theorem 3.8.3)**.** *Suppose* p *is odd. Then the functor*

$$\mathcal{D}(\mathcal{W}\mathcal{C}art) \to \mathcal{D}\big(\big[\operatorname{Spf}(\mathbb{Z}_p[\![q-1]\!])/\mathbb{Z}_p^\times\big]\big) \underset{\mathcal{D}([\operatorname{Spf}(\mathbb{Z}_p)/\mathbb{Z}_p^\times])}{\times} \mathcal{D}(\operatorname{Spf}\mathbb{Z}_p) \tag{11}$$

is fully faithful.

Corollary 2.2.5. *For any* $\mathcal{E} \in \mathcal{D}(\mathcal{W}\mathcal{C}art)$*, the square*

$$\begin{tikzcd}
\mathbb{R}\Gamma\big([\operatorname{Spf}(\mathbb{Z}_p)/\mathbb{Z}_p^\times], \rho_{dR}^*\mathcal{E}\big) & \mathbb{R}\Gamma\big(\big[\operatorname{Spf}(\mathbb{Z}_p[\![q-1]\!])/\mathbb{Z}_p^\times\big], \rho_{qdR}^*\mathcal{E}\big) \arrow[l] \\
\mathbb{R}\Gamma(\operatorname{Spf}\mathbb{Z}_p, \rho_{dR}^*\mathcal{E}) \arrow[u] & \mathbb{R}\Gamma(\mathcal{W}\mathcal{C}art, \mathcal{E}) \arrow[l] \arrow[u]
\end{tikzcd} \tag{12}$$

is cartesian.