Moduli of polarised K3 surfaces

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1 The moduli problem

1.1 Definitions

We work over a fixed base scheme $S = \operatorname{Spec} \mathbb{Z}$.

Definition 1. *Let* X *be a K3 surface. A* ***polarisation*** *of* X *is a choice of* $\mathcal{L} \in \operatorname{Pic}(X)$ *which is ample and primitive (i.e. indivisible). The* ***degree*** *of the polarisation is the even integer* $\mathcal{L}^2$.

Remark 2. If $\mathcal{L}$ is a polarisation of degree $2d$, then any smooth curve in $|\mathcal{L}|$ is of genus $g = d + 1$.

Recall that for any S-morphism $f \colon X \to T$, the relative Picard sheaf $\mathscr{Pic}_{X/T}$ is the (for our purposes étale) sheafification of the presheaf $(T' \to T) \mapsto \operatorname{Pic}(X \times_T T') / \operatorname{Pic}(T')$; explicitly its value at $(T' \to T)$ is $\Gamma(T', \mathbb{R}^1(f \times_T T')_* \mathbb{G}_m)$.

Definition 3. *Let* T *be an S-scheme. A* ***family of polarised K3 surfaces of degree*** $2d$ *parameterised by* T *is a smooth proper morphism* $f \colon X \to T$ *endowed with* $\mathcal{L} \in \mathscr{Pic}_{X/T}(T)$ *such that the fibre* $(X_{\overline{t}}, \mathcal{L}_{\overline{t}})$ *at a geometric point* $\overline{t} \colon \operatorname{Spec} \overline{\Omega} \to T$ *is a polarised K3 surface of degree* $2d$.

If $(f \colon X \to T, \mathcal{L})$ *and* $(f' \colon X' \to T, \mathcal{L}')$ *are two families of polarised K3 surfaces of degree* $2d$, *an isomorphism of families is given by a* T*-isomorphism* $\psi \colon X \xrightarrow{\simeq} X'$ *such that there is a line bundle* $\mathcal{L}_0 \in \operatorname{Pic}(T)$ *with* $\psi^* \mathcal{L}' \simeq \mathcal{L} \otimes f^* \mathcal{L}_0$.

For any family $(f\colon X \to T, \mathcal{L})$ and any morphism $(\tau\colon T' \to T)$, we have a base-changed family of polarised K3 surfaces $(f \times_T T'\colon X \times_T T' \to T', (X \times_T \tau)^*\mathcal{L})$ over T', making this definition (pseudo-)functorial in $T \in \mathfrak{Sch}_{/S}$.

We thus have, for every $T \in \mathfrak{Sch}_{/S}$, a category $\mathfrak{K}_d(T)$ of T-families of polarised K3 surfaces of degree 2d, and for every $f\colon T' \to T$ a base-change functor $f^*\colon \mathfrak{K}_d(T) \to \mathfrak{K}_d(T')$.

1.2 A reformulation of the moduli problem

As we require all K3 surfaces to be projective, they can be seen as certain subschemes of a $\mathbb{P}^N_\Omega$; furthermore the datum of the embedding into $\mathbb{P}^N$ is equivalent to the choice of a polarisation. Hence we can view the moduli problem of polarised K3 surfaces (of some fixed degree) as a sub-moduli problem of subschemes of a certain projective space (of dimension depending on the degree).

Definition 4. *Let $P \to S$ be a flat projective S-scheme, with the canonical polarisation $\mathcal{O}_{P/S}(1)$ and let $\mathcal{F}$ be a coherent $\mathcal{O}_P$-module. The* ***Hilbert polynomial*** *of $\mathcal{F}$ is $h^{\mathcal{F}} \in \mathbb{Q}[t]$ determined by the fact that $h^{\mathcal{F}}(i) = \chi(P; \mathcal{F}(i))$ for any $i \in \mathbb{Z}$.*

If Z is a closed subscheme of P, we define its Hilbert polynomial to be that of the corresponding ideal $\mathcal{I}_{Z \hookrightarrow P} \subset \mathcal{O}_P$.

Example 5. If $(X, \mathcal{L})$ is a polarised K3 surface of degree 2d with $\mathcal{L}$ very ample, then by the Riemann–Roch theorem for surfaces $h^{(X,\mathcal{L})}(t) = h_d(t) := dt^2 + 2$ (recall $\chi(X, \mathcal{O}_X) = 2$ so $\chi(X, \mathcal{L}) = \frac{\mathcal{L}^2}{2} + 2$).

Proposition 6 ([Huy5, Theorem 2.7], Saint-Donat)**.** *Let $\mathcal{L}$ be an ample line bundle on a K3 surface. Then $\mathcal{L}^3$ is very ample.*

It follows that any polarised K3 $(X, \mathcal{L})$ of degree 2d can be embedded in $\mathbb{P}^N$ for $N = h_d(3 \times 1) - 1 = 9d + 1$ with $\mathcal{O}_{\mathbb{P}^N}(1)|_X = \mathcal{L}^3$ and with Hilbert polynomial $h_d(3t)$.

Definition 7. *Let $(\varpi\colon P \to S, \mathcal{O}_{P/S}(1))$ be a flat projective S-scheme and let $h \in \mathbb{Q}[t]$. The Hilbert functor $\mathscr{Hilb}^h_{P/S}$ is defined by mapping $T \to S$ to the set of closed subschemes Z of $P \times_S T$ of Hilbert polynomials h such that $Z \to T$ is of finite presentation, flat and proper.*

Construction 8 (Grothendieck). For $r > 0$, we may define a morphism

$$\begin{aligned} \alpha_r\colon \mathscr{Hilb}^h_{P/S} &\to \mathscr{Grass}\big(h^{\mathscr{O}_P}(r) - h(r), \varpi_*\mathscr{O}_P(r)\big), \\ (\mathcal{I}_Z \rightarrowtail \mathscr{O}_P \twoheadrightarrow \mathscr{Q}) &\mapsto \ker\big(\varpi_*\mathscr{O}_P(r) \twoheadrightarrow \varpi_*\mathscr{Q}(r)\big). \end{aligned} \tag{1}$$

Theorem 9 (Grothendieck)**.** *For sufficiently big $r \gg 0$, α_r is an embedding of the Hilbert sheaf as a closed subscheme of a Grassmannian.*

Note that by definition, there is a universal closed subscheme $\mathscr{Z} \subset \mathscr{Hilb}^h_{P/S} \times_S P$.

We let $h = h_d(3\cdot)$.

Lemma 10 ([Huy5, Proposition 2.1]). *There exists a (locally closed, and in fact open) subscheme* $\mathscr{H} \subset \mathscr{H}ilb^{h}_{\mathbb{P}^N_S}$ *characterised by the following universal property:*

An S*-morphism* $T \to \mathscr{H}ilb^{h}_{\mathbb{P}^N_S}$ *factors through* $\mathscr{H}$ *if and only if* $f\colon \mathscr{X}_T = \mathscr{X} \times_{\mathscr{H}ilb^{h}_{\mathbb{P}^N_S}} T \to T$ *satisfies the conditions*

1. f *is a smooth family all of whose fibres are K3 surfaces;*

2. *writing* $\varpi\colon \mathscr{X}_T \to \mathbb{P}^N_S$ *for the canonical projection, there exist some* $\mathcal{L} \in \mathrm{Pic}(\mathscr{X}_T)$ *and* $\mathcal{L}_0 \in \mathrm{Pic}(T)$ *such that* $\varpi^*\mathcal{O}(1) \simeq \mathcal{L}^3 \otimes \mathcal{L}_0$*;*

3. *the line bundle* $\mathcal{L}$ *of item 2 can be taken to be primitive in all geometric fibres;*

4. *for all fibres* $\mathscr{X}_t$ *of* f*, restriction induces an isomorphism* $\Gamma(\mathbb{P}^N_{\kappa(s)}, \mathcal{O}(1)) \xrightarrow{\simeq} \Gamma(\mathscr{X}_t, \mathcal{L}_t^3)$*.*

Proposition 11. *The* $\mathcal{L}$ *in item 2 is uniquely determined (modulo line bundles coming from* T*).*

Corollary 12. *We can define for any* S*-scheme* $T \to S$ *a functor* $\mathscr{H}(T) \to \mathfrak{K}_d(T)$ *by mapping* $Z \subset \mathbb{P}^n_T$ *to* $(Z \to T, \mathcal{L})$*, and this is natural, i.e. compatible with the base-change functors* $\mathfrak{K}_d(T) \to \mathfrak{K}_d(T')$*.*

Finally note that the natural $\mathbb{P}GL_{N+1}$-action on $\mathbb{P}^N$ induces one on $\mathscr{H}ilb^{h}_{\mathbb{P}^N_S}$, and that the conditions characterising $\mathscr{H}$ are invariant under this action. Hence the functors defined extend to functors on the action groupoids $\Theta_T\colon [\mathscr{H}(T)/\mathbb{P}GL_{N+1}(T)] \to \mathfrak{K}_d(T)$ (which can be interpreted more geometrically as an equivariant map between the objects), that is every orbit of the action is indeed mapped to an isomorphism class.

Proposition 13. *The functor* Θ_T *is an embedding.*

Proof. Let $f, f'\colon Z, Z' \to T$ be closed subschemes of $\mathbb{P}^N_T$ satisfying the conditions defining $\mathscr{H}(T)$, and let $\psi\colon Z \to Z'$ be an isomorphism of polarised K3 surfaces, so there is an $\mathcal{L}_0 \in \mathrm{Pic}(T)$ so that $\psi^*(i^*_{Z'}\mathcal{O}(1)) \simeq i^*_Z\mathcal{O}(1) \otimes f^*\mathcal{L}_0$.

Note that $f_*(i^*_Z\mathcal{O}(1))^{\otimes 3}$ is locally free of rank $N+1$, and in fact the embedding i_Z gives a trivialisation of it on T. As this is also true with Z', we have a series of isomorphisms

$$\mathcal{O}_T^{N+1} \simeq f'_*(i^*_{Z'}\mathcal{O}(1)^{\otimes 3}) \simeq f_*(i^*_Z\mathcal{O}(1))^{\otimes 3} \otimes \mathcal{L}_0^{\otimes 3} \simeq \mathcal{O}_T^{N+1} \otimes \mathcal{L}_0^{\otimes 3} \qquad (2)$$

giving an element of $\mathbb{P}GL_{N+1}(T)$, which corresponds to an automorphism of $\mathbb{P}^N_T$ exchanging Z and Z'. What is more, this automorphism is the only one that can induce ψ, so Θ is fully faithful. □

2 Geometricity of the moduli problem

Proposition 14. *The functor* Θ_T *becomes essentially surjective after passing to an étale cover of* T*.*

Proof. Suppose given a family of polarised K3 surfaces $(f\colon X \to T, \mathcal{L})$. Once again $f_*(\mathcal{L}^{\otimes 3})$ is locally trivial of rank $N+1$, and we may restrict it to a cover $\coprod_i T_i \to T$ to assume that it is free. Here however we must work with an étale cover rather than a Zariski open cover as we used the étale relative Picard stacks so $\mathcal{L}$ itself need only be defined after passing to an étale cover. Then the adjunction counit $f^*f_*(\mathcal{L}^{\otimes 3}|_{T_i}) = f^*\mathcal{O}_{T_i}^{N+1} = \mathcal{O}_X^{N+1} \to \mathcal{L}^{\otimes 3}|_{T_i}$ defines a closed embedding $X \hookrightarrow \mathbb{P}^{N+1}_{\coprod_i T_i}$. □

We see finally that the comparison between the moduli problem for polarised K3 surfaces and the action groupoid of $\mathscr{H}$ under $\mathbb{P}GL_{N+1}$ cannot be performed fibrewise, but both must be considered as global geometric objects: stacks, or homotopy sheaves of groupoids.

Remark 15 (Motivation for stacks)*.* The topos of étale sheaves on S is cocomplete, so it does not lack quotients: we may always define X/G as the coequaliser of $G \times X \rightrightarrows X$. The issue is that this quotient might be useless; for example if G acts trivially on X then the coequaliser is $X \xrightarrow{\mathrm{id}_X} X$ which remembers nothing about the presence of an action. More precisely, in a topos all epimorphisms and all equivalence relations are effective; this means that an epi $\varpi\colon E \to B$ can be recovered as the quotient of its kernel pair $E \times_B E \rightrightarrows E$ (and vice-versa, any equivalence relation is the kernel pair of its quotient). But in the case of a trivial action, the morphism $G \times X \to X \times X$ is very far from being monic, so there is no hope of it defining an internal equivalence relation.

We need to refine the construction of the quotient by checking farther into these diagrams: we could consider $G \times G \times X \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} G \times X \rightrightarrows X$ where the first stage at least remembers the (non-trivial) multiplication of G and thus captures some of the information of the group structure. This corresponds to looking at 2-equivalence relations in a 2-category, and thus to obtain the right quotients we need to complete with respect to the 2-colimits.

Just as the cocompletion of a small category $\mathfrak{C}$ is its category of presheaves, its 2-cocompletion is the 2-category of pseudofunctors $\mathfrak{C}^{\mathrm{op}} \to \mathfrak{Grpd}$, which we may call preprestacks. To ensure that the moduli problems are sufficiently local, we should as in the case of sheaves restrict to those pseudofunctors which satisfy descent, called stacks (*i.e.* for any cover $T' = \coprod_i T_i \to T$ the functor

$$\mathcal{F}(T) \ni s \mapsto \Big(s|_{T'} = (s|_{T_i})_i, (s|_{T_i}|_{T_{ij}} \xrightarrow{\simeq} s|_{T_j}|_{T_{ij}})_{i,j} \Big) \tag{3}$$

associating to each section its canonical descent datum induces an equivalence with the category of descent data along $T' \to T$). An equivalent, more explicit condition for a preprestack to be a stack is that its morphisms presheaves be actual sheaves (*i.e.* morphisms glue "on the nose", the only way they can) and that its objects glue up to isomorphisms. When only the morphisms glue (equivalently, when the functors in Equation 3 are just fully faithful), we speak of a prestack.

Lemma 16. *The inclusion 2-functor of stacks into preprestacks admits a left adjoint "stackification" (or associated stack) 2-functor. If $\mathcal{F}$ is a prestack, we may explicitly describe the sections of its associated stack $\mathcal{F}^+$ in the following way: an object of $\mathcal{F}^+(T)$ is given by a covering*

$T' = \coprod_i T_i \to T$, *a section* $s = (s_i)$ *of* $\mathcal{F}(T')$, *and a descent datum for* s *along* $T' \to T$, *that is isomorphisms* $s_i|_{T_i \times_T T_j} \xrightarrow{\simeq} s_j|_{T_i \times_T T_j}$ *satisfying the cocycle condition.*

Example 17. Let $X_1 \rightrightarrows X_0$ be an internal groupoid in schemes (or algebraic spaces). Then $T \mapsto (X_1(T) \rightrightarrows X_0(T))$ is not a stack, though it is a prestack, and its associated stack is the 2-quotient.

Example 18. Consider the case $(X_1 \rightrightarrows X_0) = (G \times X \rightrightarrows X)$, where G is a group scheme acting on a scheme X. We write $[X/G]$ the stack quotient; the category $[X/G](T)$ is the category of G-torsors on T equipped with a G-equivariant morphism to X.

In particular, the classifying stack $\mathscr{B}G = [*/G]$ parametrises G-torsors.

We have introduced stacks to remedy the problems which might prevent a quotient sheaf from being a scheme. Our goal is indeed to treat them as geometric object, and apply to them the tools of algebraic geometry, so it is necessary to have a notion of "algebraicity" (or more generally, geometricity), for a stack.

Definition 19. *A stack* $\mathcal{F}$ *is an* ***Artin stack*** *if its diagonal* $\Delta_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ *is representable by algebraic spaces and it admits a smooth atlas* $U \to \mathcal{F}$, *that is an epimorphism of stacks which is representable and smooth, with* U *a coproduct of affine schemes.*

We say $\mathcal{F}$ *is a* ***Deligne–Mumford stack*** *if it can be covered by an étale (rather than just smooth) atlas.*

Here a morphism of stacks $p\colon \mathcal{F} \to \mathcal{G}$ *is said to be* ***representable*** *by algebraic spaces if, for any morphism* $U \to \mathcal{G}$ *from an affine scheme* $U = \mathrm{Spec}(\mathcal{O}_U(U))$, *the fibre* $\mathcal{F} \times_{\mathcal{G}} U$ *is an algebraic space.*

Construction 20. If $\mathcal{F}$ is an algebraic stack, a choice of atlas $U = \coprod_i U_i \to \mathcal{F}$ gives rise to its kernel pair $U \times_{\mathcal{F}} U = \coprod_{i,j} U_i \cap U_j \rightrightarrows U$ (which we see as an equivalence relation specifying how the intersections are glued together) which is a groupoid in algebraic spaces, both of whose structure projections are smooth (or étale if the atlas is). Conversely, the quotient of such a smooth groupoid is an algebraic stack, and this is an equivalence of 2-categories.

In particular, a quotient by an appropriate group action is an Artin stack.

Remark 21.

- Representability of the diagonal is actually implied by the conditions on the atlas.
- A Zariski open immersion is in particular étale, so schemes are DM stacks. Algebraic spaces are also DM stacks (as follows directly from their definition).
- An Artin stack is DM if and only if its diagonal is furthermore unramified.

If $x\colon \mathrm{Spec}\,\Omega \to \mathcal{X}$ is a point of a stack, its **isotropy group** is the group Ω-scheme $G_x = \mathscr{I}\!som(x, x)$. This final remark leads to the philosophy that a DM stack is an Artin stack with finite isotropy groups, which can be made precise:

Proposition 22 ([Stk, tags 0DSM and 0DSN])**.** *Any Artin stack* $\mathcal{X}$ *contains a largest open Deligne–Mumford stack* $\mathscr{X}_{\mathrm{DM}}$, *called its DM locus. A point* $x\colon \mathrm{Spec}\,\Omega \to \mathscr{X}$ *is in the DM locus if and only if its isotropy group* G_x *is unramified over* Ω.

Finally, our fibrewise comparison can now be interpreted as follows:

Theorem 23. *The 2-natural transformation of prestacks Θ made up of the Θ_T exhibits $\mathfrak{K}_d$ as the stackification of the action groupoids of $\mathscr{H}$ by $\mathbb{P}GL_{N+1}$, that is Θ corresponds to an equivalence of stacks $[\mathscr{H}/\mathbb{P}GL_{N+1}] \simeq \mathfrak{K}_d$.*

Corollary 24. *The moduli stack $\mathfrak{K}_d$ of polarised K3 surfaces of degree $2d$ is an Artin stack.*

In fact, more is true: even though $\mathbb{P}GL_{N+1}$ is far from being a finite group, $\mathfrak{K}_d$ is a DM stack.

Proposition 25 ([Huy5, Proposition 4.10])**.** *The diagonal of $\mathfrak{K}_d$ is unramified.*

Proof. Recall that a morphism is unramified if and only if all its geometric fibres are reduced and discrete. Let $x\colon \operatorname{Spec}\overline{\Omega} \to \mathfrak{K}_d \times \mathfrak{K}_d$ be a geometric point; its fibre $\mathfrak{K}_{d,x} := \mathfrak{K}_d \times_{\mathfrak{K}_d \times \mathfrak{K}_d} \operatorname{Spec}\overline{\Omega}$ is the algebraic space mapping an $\overline{\Omega}$-scheme T to the sets of triples of T-families $(X_0, \mathcal{L}_0), (X, \mathcal{L}), (X', \mathcal{L}')$ equipped with isomorphisms $(X_0, \mathcal{L}_0) \xrightarrow{\simeq} (X, \mathcal{L})$ and $(X_0, \mathcal{L}_0) \xrightarrow{\simeq} (X', \mathcal{L}')$ (equivalently, we can, and will, forget $(X_0, \mathcal{L}_0)$ and just consider the single composed isomorphism). We will now see, following [Huy5, Proposition 3.3], that for $T = \operatorname{Spec} k$, for any pair of polarised K3 surfaces $(X, \mathcal{L}), (X', \mathcal{L}')$ over k, the set of isomorphisms $(X, \mathcal{L}) \xrightarrow{\simeq} (X', \mathcal{L}')$ is a finite set of reduced points.

Let $f\colon X \xrightarrow{\simeq} X'$ be such an isomorphism (compatible with the polarisations). We identify it with its graph $\Gamma_f \subset X \times X'$, whose Hilbert polynomial with respect to $\mathcal{L} \boxtimes \mathcal{L}'$ is $h_d(2t)$, so that it corresponds uniquely to a k-point of $\mathscr{H}\!ilb^{h_d(2t)}_{X\times X'}$. Now in a Hilbert scheme, the tangent space at a point $[Z]$ corresponding to a closed subscheme Z with ideal $\mathcal{I}_Z$ is $\hom_k(\mathcal{I}_Z, \mathcal{O}_Z)$, which is furthermore identified with $\Gamma(Z, \mathcal{N}_Z)$ if the embedding of Z is smooth (with normal bundle $\mathcal{N}_Z$).

In our case, from the identification $(\mathrm{id}_X, f)\colon X \xrightarrow{\simeq} \Gamma_f \subset X \times X'$ and $\mathcal{T}_{X\times X'}|_{\Gamma_f} \simeq \mathcal{T}_X \oplus f^*\mathcal{T}_{X'}$ which splits the exact sequence $0 \to \mathcal{T}_{\Gamma_f} \to \mathcal{T}_{X\times X'}|_{\Gamma_f} \to \mathcal{N}_{\Gamma_f/X\times X'} \to 0$, we find that $\Gamma(\Gamma_f, \mathcal{N}_{\Gamma_f/X\times X'}) \simeq \Gamma(X, f^*\mathcal{T}_{X'}) \simeq \Gamma(X, \mathcal{T}_X)$ which is trivial as X is a K3 surface. This means that f defines a reduced isolated point of $\mathscr{H}\!ilb^{h_d(2t)}_{X\times X'}$, and as a projective scheme can only have a finite number of irreducible components it follows that the set of such isomorphisms is finite. □

Being a Deligne–Mumford stack has several very pleasant implications on the structure of $\mathfrak{K}_d$.

Definition 26. *A* ***coarse moduli space*** *for an algebraic stack $\mathscr{X}$ is an algebraic space X with a morphism $q\colon \mathscr{X} \to X$ which is initial among morphisms to algebraic spaces and induces an isomorphism $\pi_0(\mathscr{X}(\bar{k})) \xrightarrow{\simeq} X(\bar{k})$.*

Note that we also have $q_*\mathcal{O}_{\mathscr{X}} = \mathcal{O}_X$, and in fact this property characterises q.

Theorem 27 (Keel–Mori, [Ryd12, Theorem 6.12])**.** *Let $\mathscr{X}$ be an Artin stack with finite inertia (i.e. $\mathscr{X} \times_{\mathscr{X}\times\mathscr{X}} \mathscr{X} \to \mathscr{X}$ is finite). Then $\mathscr{X}$ admits a coarse moduli space.*

3 Properties of the moduli stack $\mathfrak{K}_d$

3.1 Local structure and quasi-projectivity

Proposition 28 ([Ris06, § Approach via geometric invariant theory, after Corollary 4.3.10]). *Let* k *be an algebraically closed field of characteristic* 0*. Then the coarse moduli space of* $\mathfrak{K}_d \otimes_{\mathbb{Z}} k$ *is quasi-projective.*

Theorem 29 (Luna's étale slice, [AHR19, Theorem 1.1]). *Let* $\mathscr{X}$ *be a quasi-separated Artin stack and* x *a smooth closed point of* $\mathscr{X}$ *whose isotropy group algebraic space* G_x *is linearly reductive. Then there is an affine pointed* G_x*-scheme* (W, w) *and an étale morphism* $([W/G_x], w) \to (\mathscr{X}, x)$ *inducing an isomorphism of isotropy groups at the selected points.*

Proposition 30 ([AHR19, Theorem 2.9]). *Let* $\mathscr{X}$ *be a nœtherian Artin stack admitting a coarse moduli space* $q\colon \mathscr{X} \to X$ *with affine diagonal, and let* x *be a closed point of* $\mathscr{X}$*. Then there is a cartesian diagram*

$$\begin{array}{ccc} [W/G_x] & \longrightarrow & \mathscr{X} \\ \big\downarrow & \lrcorner & \big\downarrow q \\ W/\!/G_X & \longrightarrow & X \end{array} \tag{4}$$

such that $W/\!/G_x \to X$ *is an étale neighbourhood of* $q(x)$*, where* $W/\!/G_x$ *denotes the good GIT quotient (the coarse moduli space of the stack quotient).*

Hence, if $\mathfrak{K}_d$ is smooth as a Deligne–Mumford stack, it will be étale locally the quotient of a smooth affine scheme by a finite group.

Theorem 31. $\mathfrak{K}_d \otimes_{\mathbb{Z}} \mathbb{Z}[(2d)^{-1}]$ *is a smooth Deligne–Mumford stack of dimension* 19*.*

As $\mathfrak{K}_d$ is of finite type we only have to check formal smoothness. We will do this at the level of points, or rather of their formal neighbourhoods. We thus fix a point $x\colon \operatorname{Spec} k \to \mathfrak{K}_d$, corresponding to a polarised K3 surface X over k. We will study the smoothness of $\widehat{(\mathfrak{K}_d)}_x$, that is of the deformation problem of $(X, \mathcal{L})$.

3.2 Deformation theory of K3 surfaces

Let k be a field. For any complete local nœtherian k-algebra Λ with residue field k, we denote $\mathfrak{Art}_\Lambda$ the category of local artinian Λ-algebras with residue field k, and $\widehat{\mathfrak{Loc}}_\Lambda$ the category of complete local nœtherian Λ-algebras . A **functor** of Artin rings (relative to Λ) is a copresheaf on $\mathfrak{Art}_\Lambda$, a functor $\mathfrak{Art}_\Lambda \to \mathfrak{Sets}$.

Example 32. Let $A \in \widehat{\mathfrak{Loc}}_\Lambda$ with unique maximal ideal $\mathfrak{m}_A$ and residue field $A/\mathfrak{m}_A = k$. Then we define a functor $\hbar_R$ by restricting the covariant Yoneda embedding of R along the inclusion $\mathfrak{Art}_\Lambda \subset \widehat{\mathfrak{Loc}}_\Lambda$. For any $B \in \mathfrak{Art}_\Lambda$ we have

$$\hbar_A(B) = \hom\Big(\varinjlim_n A/\mathfrak{m}_A^{n+1}, B\Big) = \varprojlim_n \hom(A/\mathfrak{m}_A^{n+1}, B), \tag{5}$$

so any such functor is called **pro-representable**.

Remark 33. Any functor of Artin ring $\mathscr{F}$ can be extended to a copresheaf $\widehat{\mathscr{F}}$ on $\widehat{\mathfrak{Loc}_\Lambda}$ by setting $\widehat{\mathscr{F}}(A) = \varprojlim_n \mathscr{F}(A/\mathfrak{m}_A^{n+1})$. We find a tautological correspondence between $\widehat{\mathscr{F}}(A)$ and the set of natural transformations $\mathscr{h}_A \to \mathscr{F}$.

Recall that a surjection $B \to A$ of local Λ-algebras is called a **small extension** if its kernel is a principal ideal annihilated by $\mathfrak{m}_B$ and that (by [Stk, tag 06GE]) every surjection in $\mathfrak{Art}_\Lambda$ factors as a composition of small extensions.

Definition 34. *A functor of Artin rings $\mathscr{F} : \mathfrak{Art}_\Lambda \to \mathfrak{Sets}$ is a* ***formal moduli problem*** *if*

- $\mathscr{F}(k) = *$ *and*
- *for any small extension $A' \twoheadrightarrow A$ in $\mathfrak{Art}_\Lambda$ and any morphism $B \to A$ the map $\mathscr{F}(A' \times_A B) \to \mathscr{F}(A') \times_{\mathscr{F}(A)} \mathscr{F}(B)$ is bijective.*

Definition 35. *The* ***tangent space*** *to a functor of Artin rings $\mathcal{F}$ is $\mathcal{F}(k[\varepsilon])$ (where ε is nilpotent of order 2, i.e. $k[\varepsilon] = k[\varepsilon]/(\varepsilon^2)$). An element of $\mathcal{F}(k[\varepsilon])$ is also called a* ***first-order deformation*** *of its image in $\mathcal{F}(k)$ (under $k[\varepsilon] \twoheadrightarrow k$).*

Lemma 36 ([Ser06, Corollary 2.2.11])**.** *Let $\mathcal{F}$ be a formal moduli problem prorepresented by R. Then $\dim(R) = \dim_k(\mathcal{F}(k[\varepsilon]))$.*

Theorem 37 (Schlessinger, [Ser06, Theorem 2.3.2])**.** *A functor of Artin rings is prorepresentable if and only if it is a formal moduli problem with finite-dimensional tangent space.*

We will now specialise to the functor of deformations of a given k-scheme.

Example 38. Let X_0 be a k-scheme. A deformation of X_0 over $A \in \mathfrak{Art}_\Lambda$ is an A-scheme X such that $X \otimes_A k \simeq X_0$. The deformation functor $\mathscr{Def}_{X_0}$ of X_0 maps $A \in \mathfrak{Art}_\Lambda$ to the set of isomorphism classes of A-deformations of X_0.

Proposition 39 ([Ser06, Theorem 2.4.1. (ii)])**.** *If X_0 is smooth over k, then $\mathscr{Def}_{X_0}(k[\varepsilon]) \simeq H^1(X_0, \mathcal{T}_{X_0})$.*

The cohomology class corresponding to first-order deformation ξ of X_0 is called its **Kodaira–Spencer class** $\kappa(\xi)$.

Definition 40. *An* ***obstruction space*** *for a functor of Artin rings $\mathcal{F}$ is a k-vector space $o(\mathcal{F})$ such that for every $A \in \mathfrak{Art}_\Lambda$ and any A-deformation $\xi \in \mathcal{F}(A)$ there is a linear map $\mathrm{Ext}^1_\Lambda(A, k) \to o(\mathcal{F})$ whose kernel consists of the isomorphism classes of extensions $\widetilde{A}$ such that ξ is in the image of $\mathcal{F}(\widetilde{A}) \to \mathcal{F}(A)$.*

Lemma 41 ([Ser06, Proposition 2.2.10])**.** *A functor of Artin rings is smooth if and only if it is unobstructed (i.e. its obstruction space is zero).*

Proposition 42 ([Ser06, Proposition 2.4.6])**.** *Let X_0 be a smooth algebraic variety. Then $H^2(X_0, \mathcal{T}_{X_0})$ is an obstruction space for $\mathscr{Def}_{X_0}$.*

Theorem 43 ([Ser06, Corollary 2.6.4])**.** *If X_0 is a projective scheme with $H^0(X_0, \mathcal{T}_{X_0}) = 0$ then $\mathscr{Def}_{X_0}$ is prorepresentable. If furthermore X_0 is smooth and $H^2(X_0, \mathcal{T}_{X_0}) = 0$ then $\mathscr{Def}_{X_0}$ is prorepresentable by a (formally smooth) power series ring.*

Corollary 44. *If* X_0 *satisfies the hypotheses of the theorem, then* $\dim(\mathscr{Def}_{X_0}) = \dim_k H^1(X_0, \mathcal{T}_{X_0})$, *so there is a (non-canonical) isomorphism of* $\mathscr{Def}_{X_0}$ *with the functor prorepresented by* $\widehat{\mathrm{Sym}}(k^{h^1(X_0,\mathcal{T}_{X_0})})$.

In particular, when X_0 is our K3 surface we find (from the Hodge diamond) that its deformation functor is prorepresented by $k[[x_1, \dots, x_{20}]]$.

Remark 45. The prorepresentability of the deformation gives a universal formal deformation of X_0; however it is not algebraisable, corresponding to the fact (mentioned in the first talk) that there are non-algebraic K3 surfaces.

Finally, we must take into account the polarisation in the deformations.

Definition 46. *If* X_0 *has a polarisation* $\mathcal{L}_0$*, for any infinitesimal deformation* $\xi\colon X \to \operatorname{Spec} A$ *of* X*, a deformation of* $\mathcal{L}_0$ *along* ξ *consists of a line bundle* $\mathcal{L}$ *on* X *such that* $\mathcal{L}|_{X_0} = \mathcal{L}_0$*. A deformation of* $(X_0, \mathcal{L}_0)$ *is by definition a deformation of* X_0 *and a deformation of* $\mathcal{L}_0$ *along it. We define a functor of Artin rings* $\mathscr{Def}_{(X_0,\mathcal{L}_0)}$ *by mapping* A *to the set of isomorphism classes of* A*-deformations of* $(X_0, \mathcal{L}_0)$.

Construction 47. Suppose X_0 is a smooth algebraic variety. Then the logarithmic de Rham differential induces a morphism $c\colon H^1(X_0, \mathcal{O}_{X_0}^\times) \to H^1(X_0, \Omega^1_{X_0})$. As $\Omega^1_{X_0}$ is locally free, there is an isomorphism $H^1(X_0, \Omega^1_{X_0}) \simeq \operatorname{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{T}_{X_0}, \mathcal{O}_{X0})$, so that c maps any invertible sheaf $\mathcal{L}_0$ to (the isomorphism class of) an $\mathcal{O}_{X_0}$-extension $0 \to \mathcal{O}_{X_0} \to \mathcal{E}_{\mathcal{L}_0} \to \mathcal{T}_{X_0} \to 0$, called the Atiyah extension of $\mathcal{L}_0$. For any $n \neq 0$, we have $\mathcal{E}_{\mathcal{L}_0^n} \simeq \mathcal{E}_{\mathcal{L}_0}$.

Theorem 48 ([Ser06, Theorem 3.3.11])**.** *Suppose* X_0 *is a smooth algebraic variety.*

- *The tangent space* $\mathscr{Def}_{(X_0,\mathcal{L}_0)}(k[\varepsilon])$ *is* $H^1(X_0, \mathcal{E}_{\mathcal{L}_0})$.
- $H^2(X_0, \mathcal{E}_{\mathcal{L}_0})$ *is an obstruction space for* $\mathscr{Def}_{(X_0,\mathcal{L}_0)}$.
- *Given a deformation* ξ *of* X_0 *over* $k[\varepsilon]$*, there exists a deformation of* $\mathcal{L}_0$ *along* ξ *if and only if* $\langle \kappa(\xi) \smile c(\mathcal{L}) \rangle = 0$ *in* $H^2(X_0, \mathcal{O}_{X_0})$*, where* $\kappa(\xi)$ *is the Kodaira–Spencer class of the first-order deformation* ξ*,* $(\cdot \smile \cdot)\colon H^1(X_0, \mathcal{T}_{X_0}) \times H^1(X_0, \Omega^1_{X_0}) \to H^2(X_0, \mathcal{T}_{X_0} \otimes \Omega^1_{X_0})$ *is the cup-product of classes and* $\langle \cdot \rangle\colon H^2(X_0, \mathcal{T}_{X_0} \otimes \Omega^1_{X_0}) \to H^2(X_0, \mathcal{O}_{X_0})$ *is induced by the duality pairing.*

Corollary 49. *If* $(X_0, \mathcal{L}_0)$ *is a polarised K3 surface, then* $\mathscr{Def}_{(X_0,\mathcal{L}_0)}$ *is smooth of dimension* 19.

Proof. In this case the cup-product $H^1(X_0, \mathcal{T}_{X_0}) \times H^1(X_0, \Omega^1_{X_0}) \to H^2(X_0, \mathcal{O}_{X_0}) \simeq k$ is given by Serre duality and thus surjective, so for any non-trivial $\mathcal{L}_0$ the morphism $\langle \cdot \smile c(\mathcal{L}_0) \rangle$ is surjective. Hence the space of possible deformations of $\mathcal{L}_0$, its kernel, is a codimension 1 subspace of the 20-dimensional $H^1(X_0, \mathcal{T}_{X_0})$. □

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