The Nygaard filtration

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We have seen the conjugate filtration on the Hodge–Tate complex $\overline{\mathbbl{\Delta}}_R$, which is extended by sifted colimits from the Postnikov filtration on the Hodge–Tate complexes of polynomial algebras. Noting that (up to Breuil–Kisin twist) the Hodge–Tate complex is recovered as the associated graded of the filtration $\mathbbl{\Delta}_R^{[\bullet]}$ on the prismatic complex, we will realise the conjugate filtration as the associated graded of a so-called Nygaard filtration on the prismatic complex, or rather first on its Frobenius twist $\Phi^*\mathbbl{\Delta}_R$.

Recall (Theorem 3.6.7) that for any $\mathcal{E} \in \mathcal{D}(\mathcal{WC}art^{\mathrm{HT}})$, there is a cartesian square

$$\begin{tikzcd}
\mathbb{R}\Gamma(\mathcal{WC}art, \mathcal{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{R}\Gamma(\operatorname{Spf}\mathbb{Z}_p, \rho_{\mathrm{dR}}^*\mathcal{E}) \arrow[d] \\
\mathbb{R}\Gamma(\mathcal{WC}art, F^*\mathcal{E}) \arrow[r] & \mathbb{R}\Gamma(\mathcal{WC}art^{\mathrm{HT}}, (F^*\mathcal{E})|_{\mathcal{WC}art^{\mathrm{HT}}})
\end{tikzcd} \tag{1}$$

in $\widehat{\mathcal{D}}(\mathbb{Z}_p)$.

Thus we only need to construct the untwisted Nygaard filtration on $\mathbb{R}\Gamma(\operatorname{Spf}\mathbb{Z}_p, \rho_{\mathrm{dR}}^*\mathcal{H}_{\mathbbl{\Delta}}(R)) = \mathbbl{\Delta}_{\overline{R}/\mathbb{Z}_p}$. To achieve this, we will identify the latter (if $\overline{R}$ is smooth over $\mathbb{F}_p$) with $R\Gamma_{\mathrm{crys}}(\overline{R}/\mathbb{Z}_p)$, or rather (if R is p-torsion-free) with $\widehat{\mathrm{dR}}_R$, the p-complete derived de Rham complex of R, where the filtration will be given by the Hodge filtration.

In general, just as prismatic cohomology can be define as the (derived) global sections of a quasi-coherent complex $\mathcal{H}_{\mathbbl{\Delta}}$, crystalline cohomology (relative to $\mathbb{Z}_p$) can also be computed as the global sections (on the scheme of interest itself, not its Cartier–Witt stack) of the de Rham–Witt complex $\mathscr{W}\Omega_R^\bullet$, first constructed by Illusie.

This complex was endowed by Nygaard with a filtration by subcomplexes $\mathscr{N}^m\mathscr{W}\Omega_R^\bullet$ such that

$$\mathscr{N}^m\mathscr{W}\Omega_R^\bullet / p\mathscr{N}^{m-1}\mathscr{W}\Omega_R^\bullet \simeq \Omega_R^{\geq m}. \tag{2}$$

We will thus likewise endow $F^*\mathbbl{\Delta}_R$ with a Nygaard filtration satisfying a similar behaviour.

1 The relative Nygaard filtration

For this section, fix a bounded prism (A, I).

Construction 1.1. A quasisyntomic $\overline{A}$-algebra R is said to be **large** if R/pR is generated by the images of $\overline{A}$ and $(R/pR)^\flat$, *i.e.* if it admits a surjection from a $\overline{A}\langle X_j^{1/p^\infty}\rangle_{j\in J}$ for some set J. If R is large, then $\Delta_{R/A}$ is a (p, I)-completely flat A-algebra, so in particular $\Phi^*\Delta_{R/A}$ is still a classical algebra concentrated in degree 0.

For R a large $\overline{A}$-algebra, we define $\mathrm{Fil}^n_{\mathrm{Ng}}\,\Phi^*\Delta_{R/A}$ as the ideal

$$\mathrm{Fil}^n_{\mathrm{Ng}}\,\Phi^*\Delta_{R/A} = \{x \in \Phi^*\Delta_{R/A} \mid \varphi(x) \in I^n\Delta_{R/A}\}. \tag{3}$$

Remember that $R \mapsto \Delta_{R/A}$ satisfies p-quasisyntomic descent. We have more generally (Lemma 5.1.7, proved in a similar way using the Hodge–Tate comparison) that for any $M \in \widehat{\mathcal{D}}(A)$ with cohomology bounded below, the functor $R \mapsto M\widehat{\otimes}^{\mathbb{L}}_A\Delta_{R/A}$ satisfies p-quasisyntomic descent. In particular, taking M to be the A-algebra $A \xrightarrow{\Phi} A$, we have quasisyntomic descent for $\Phi^*\Delta_{-/A}$.

Now the large quasisyntomic $\overline{A}$-algebras form a basis for the p-quasisyntomic topology on $\mathfrak{CAlg}^{\mathrm{QSyn}}_{\overline{A}}$, so a functor having descent implies that it is the right Kan extension of its restriction to large algebras. We thus define $\mathrm{Fil}^n_{\mathrm{Ng}}\,\Phi^*\Delta_{-/A}$ as the right Kan extension of $R \mapsto \mathrm{Fil}^n_{\mathrm{Ng}}\,\Phi^*\Delta_{R/A}$ along the inclusion of large algebras into all quasisyntomic $\overline{A}$-algebras, and obtain a canonical isomorphism $\mathrm{Fil}^0_{\mathrm{Ng}}\,\Phi^*\Delta_{R/A} \simeq \Phi^*\Delta_{R/A}$.

Note that, by construction, it comes equipped with a functorial "filtered Frobenius" $\mathrm{Fil}^\bullet\,\varphi\colon \mathrm{Fil}^\bullet_{\mathrm{Ng}}\,\Phi^*\Delta_{-/A} \to \Delta^{[\bullet]}_{-/A} = I^{\times\bullet}\Delta_{-/A}$.

Proposition 1.2 (Bhatt–Scholze, Theorem 15.2)**.** *The image of* $\mathrm{gr}^n\,\varphi\colon\ \mathrm{gr}^n_{\mathrm{Ng}}\,\Phi^*\Delta_{R/A} \rightarrowtail \overline{\Delta}_{R/A}\{n\}$ *is* $\mathrm{Fil}^{\mathrm{conj}}_n\,\overline{\Delta}_{R/A}\{n\}$.

Proof. This follows from the cohomology computation

$$H^m(\mathrm{gr}^n_{\mathrm{Ng}}\,\Phi^*\Delta_{R/A}) \xrightarrow{\simeq} \begin{cases} H^m_{\overline{\Delta}}(R/A)\{n\} \simeq \widehat{\Omega}^m_{R/\overline{A}}\{n-m\} & \text{if } m \le n \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

It is established by using descent to reduce to the case of quasiregular semiperfectoid rings over a perfect prism, where it is a long computation. □

We now want to extend this functor to all animated commutative algebras by deriving it.

Scholium 1.3 (proof of Proposition 5.1.1)**.** *The functor* $R \mapsto \mathrm{Fil}^n_{\mathrm{Ng}}\,\Phi^*\Delta_{R/A}$ *is a left Kan extension of its restriction to the category* $\mathfrak{Poly}_{\overline{A}}$ *of finitely generated polynomial* $\overline{A}$*-algebras.*

Proof. The case $n = 0$ (and $n < 0$ is the same) is clear since $R \mapsto \mathrm{Fil}^0_{\mathrm{Ng}}\,\Phi^*\Delta_{R/A} \simeq \Phi^*\Delta_{R/A}$ commutes with sifted limits. We then proceed by induction, using the identification $\mathrm{gr}^n_{\mathrm{Ng}}\,\Phi^*\Delta_{R/A} \simeq \mathrm{Fil}^{\mathrm{conj}}_n\,\overline{\Delta}_{R/A}\{n\}$, the assignment of the latter commuting with sifted colimits. □

Proposition 1.4 (Proposition 5.3.8). *Let* R *be a regular Noetherian* $\mathbb{F}_p$*-algebra. Then the crystalline comparison equivalence* $\gamma_{\Delta}^{\mathrm{crys}}\colon \Delta_R \simeq \mathbb{R}\Gamma_{\mathrm{crys}}(R/\mathbb{Z}_p)$ *can be promoted to an equivalence* $\mathrm{Fil}^{\bullet}_{\mathrm{Ng}}\,\Delta_{R/\mathbb{Z}_p} \simeq \mathscr{N}^{\bullet}\mathscr{W}\Omega_R$ *in* $\widehat{\mathcal{DF}}(\mathbb{Z}_p)$.

2 Hodge comparison

Lemma 2.1 (Proposition 5.2.5). *Let* R *be an animated commutative* $\overline{A}$*-algebra. There is an equivalence*

$$\overline{\gamma}_R\colon \overline{A} \overset{\mathbb{L}}{\underset{A}{\otimes}} \Phi^*\Delta_{R/A} \xrightarrow{\simeq} \widehat{\mathrm{dR}}_{R/\overline{A}} \tag{5}$$

in $\widehat{\mathcal{D}}(\overline{A})$.

Corollary 2.2 (Corollary 5.2.8). *For every animated commutative* $\overline{A}$*-algebra* R*, there is a fibre sequence*

$$I \overset{\mathbb{L}}{\underset{A}{\otimes}} \mathrm{Fil}^{\bullet-1}_{\mathrm{Ng}}\,\Phi^*\Delta_{R/A} \xrightarrow{\nu} \mathrm{Fil}^{\bullet}_{\mathrm{Ng}}\,\Phi^*\Delta_{R/A} \xrightarrow{\mathrm{Fil}^{\bullet}(\gamma_R)} \mathrm{Fil}^{\bullet}_{\mathrm{Hod}}\,\widehat{\mathrm{dR}}_{R/\overline{A}} \tag{6}$$

Proof. The map ν comes from multiplicativity of the filtration, and the fact that $\mathrm{Fil}^m_{\mathrm{Ng}}\,F^*\Delta_{\overline{A}/A} \simeq I^m$ (by convention, A if $m \leq 0$). We want to show that the map $\mathrm{Fil}^{\bullet}(\overline{\gamma}_R)\colon \mathrm{cofib}(\nu) \to \mathrm{Fil}^{\bullet}_{\mathrm{Hod}}\,\widehat{\mathrm{dR}}_{R/\overline{A}}$ (which is defined because the composition $\mathrm{Fil}^{\bullet}(\gamma_R) \circ \nu$ vanishes in the polynomial case due to Beilinson t-structure considerations) is an equivalence in each weight. We will proceed by induction on the weight m: for $m = 0$ (or $m \leq 0$) this is the equivalence $\overline{\gamma}_R$ from the previous lemma, and for the induction step we use that the associated graded corresponds to the Hodge–Tate comparison $\mathrm{gr}^{\mathrm{conj}}_m\,\overline{\Delta}\{m\} \simeq \mathbb{L}\widehat{\Omega}^m_{R/\overline{A}}[m]$ composed with the relative Frobenius

$$\varphi\colon\ \mathrm{gr}^m_{\mathrm{Ng}}\,\Phi^*\Delta_{R/A}/I\,\mathrm{gr}^{m-1}_{\mathrm{Ng}}\,\Phi^*\Delta_{R/A} \simeq \mathrm{gr}^{\mathrm{conj}}_m\,\overline{\Delta}_{R/A}\{n\}. \tag{7}$$

$\square$

Construction 2.3. The globalised version of the de Rham comparison is an equivalence

$$F^*\mathcal{H}_{\Delta}(R)|_{\mathcal{WC}art^{\mathrm{HT}}} \simeq \widehat{\mathrm{dR}}_R \otimes \mathcal{O}_{\mathcal{WC}art^{\mathrm{HT}}}. \tag{8}$$

Recalling the de Rham point $\rho_{\mathrm{dR}}\colon \mathrm{Spf}\,\mathbb{Z}_p \to \mathcal{WC}art$ (and writing $\mathcal{E}_{\mathrm{dR}} \coloneqq \mathbb{R}\Gamma(\mathrm{Spf}\,\mathbb{Z}_p, \rho^*_{\mathrm{dR}}\mathcal{E})$, called the **de Rham specialisation**, for any complex $\mathcal{E}$ on $\mathcal{WC}art$) and the commutative diagram

$$\begin{tikzcd} \mathcal{WC}art^{\mathrm{HT}} \arrow[r, hook] \arrow[d] & \mathcal{WC}art \arrow[d, "F"] \\ \mathrm{Spf}\,\mathbb{Z}_p \arrow[r, "\rho_{\mathrm{dR}}"'] & \mathcal{WC}art \end{tikzcd} \tag{9}$$

this translates to an equivalence

$$\mathcal{H}_{\Delta}(R)_{\mathrm{dR}} \otimes \mathcal{O}_{\mathcal{WC}art^{\mathrm{HT}}} \simeq \widehat{\mathrm{dR}}_R \otimes \mathcal{O}_{\mathcal{WC}art^{\mathrm{HT}}}. \tag{10}$$

More generally, the fibre sequence globalises to

$$\mathcal{I}\,\mathrm{Fil}^{\bullet-1}_{\mathrm{Ng}}\,F^*\mathcal{H}_{\Delta}(R) \to \mathrm{Fil}^{\bullet}_{\mathrm{Ng}}\,F^*\mathcal{H}_{\Delta}(R) \to \iota_*\big(\mathrm{Fil}^{\bullet}_{\mathrm{Hod}}\,\widehat{\mathrm{dR}}_R \otimes \mathcal{O}_{\mathcal{WC}art^{\mathrm{HT}}}\big) \tag{11}$$

where $\iota\colon \mathcal{WC}art^{\mathrm{HT}} \hookrightarrow \mathcal{WC}art$ denotes the inclusion.

Also setting, for any n, the nth twist

$$\mathrm{Fil}^{\bullet}_{\mathrm{Ng}}\,F^*(\mathcal{H}_{\Delta}(R)\{n\}) \simeq \mathrm{Fil}^{\bullet}_{\mathrm{Ng}}\,F^*\mathcal{H}_{\Delta}(R) \otimes F^*(\mathcal{O}_{\mathcal{WC}art}\{n\}), \tag{12}$$

and using the facts that $F^*(\mathcal{O}_{\mathcal{WC}art}\{n\}) \simeq \mathcal{I}^{-n}\mathcal{O}_{\mathcal{WC}art}\{n\}$ and $\mathcal{I}^{-n}|_{\mathcal{WC}art^{\mathrm{HT}}} \simeq \mathcal{O}_{\mathcal{WC}art^{\mathrm{HT}}}\{-n\}$, we obtain for any n the fibre sequence

$$\mathcal{I}\,\mathrm{Fil}^{\bullet-1}_{\mathrm{Ng}}\,F^*(\mathcal{H}_{\Delta}(R)\{n\}) \to \mathrm{Fil}^{\bullet}_{\mathrm{Ng}}\,F^*(\mathcal{H}_{\Delta}(R)\{n\}) \to \iota_*\big(\mathrm{Fil}^{\bullet}_{\mathrm{Hod}}\,\widehat{\mathrm{dR}}_R \otimes \mathcal{O}_{\mathcal{WC}art^{\mathrm{HT}}}\big) \tag{13}$$

whose last arrow gives on global sections a comparison map

$$\begin{aligned}\mathbb{R}\Gamma(\mathcal{WC}art, \mathrm{Fil}^{\bullet}_{\mathrm{Ng}}\,F^*(\mathcal{H}_{\Delta}(R)\{n\})) &\to \mathbb{R}\Gamma(\mathcal{WC}art^{\mathrm{HT}}, \mathrm{Fil}^{\bullet}_{\mathrm{Hod}}\,\widehat{\mathrm{dR}}_R \otimes \mathcal{O}_{\mathcal{WC}art^{\mathrm{HT}}}) \\ &\simeq \mathrm{Fil}^{\bullet}_{\mathrm{Hod}}\,\widehat{\mathrm{dR}}_R \widehat{\otimes}^{\mathbb{L}} \mathbb{R}\Gamma(\mathcal{WC}art^{\mathrm{HT}}, \mathcal{O}_{\mathcal{WC}art^{\mathrm{HT}}})\end{aligned} \tag{14}$$

Definition 2.4 (The absolute Nygaard filtration)**.** *We define the Nygaard-filtered absolute prismatic complex of* R*, along with its filtered de Rham comparison map, as fitting in the cartesian square*

$$\begin{tikzcd} \mathrm{Fil}^{\bullet}_{\mathrm{Ng}}\,\Delta_R\{n\} \arrow[r, "\mathrm{Fil}^{\bullet}(\gamma^{\mathrm{dR}}_{\Delta}\{n\})"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fil}^{\bullet}_{\mathrm{Hod}}\,\widehat{\mathrm{dR}}_R \arrow[d] \\ \mathbb{R}\Gamma(\mathcal{WC}art, \mathrm{Fil}^{\bullet}_{\mathrm{Ng}}\,F^*\mathcal{H}_{\Delta}(R)\{n\}) \arrow[r] & \mathrm{Fil}^{\bullet}_{\mathrm{Hod}}\,\widehat{\mathrm{dR}}_R \widehat{\otimes}^{\mathbb{L}} \mathbb{R}\Gamma(\mathcal{WC}art^{\mathrm{HT}}, \mathcal{O}_{\mathcal{WC}art^{\mathrm{HT}}}) \end{tikzcd} \tag{15}$$

We now want to ensure that this definition refines the prismatic complex and the de Rham comparison. As announced above, we will apply the cartesian square of eq. (14) to the sheaf $\mathcal{H}_{\Delta}(R)$.

Lemma 2.5 (Proposition 5.4.8)**.** *For any animated commutative ring* R*, there is an equivalence*

$$\alpha_R\colon \mathcal{H}_{\Delta}(R)_{\mathrm{dR}} \simeq \widehat{\mathrm{dR}}_R, \tag{16}$$

which after tensoring with $\mathcal{O}_{\mathcal{WC}art^{\mathrm{HT}}}$ *recovers the equivalence* $\mathcal{H}_{\Delta}(R)_{\mathrm{dR}} \otimes \mathcal{O}_{\mathcal{WC}art^{\mathrm{HT}}} \simeq \widehat{\mathrm{dR}}_R \otimes \mathcal{O}_{\mathcal{WC}art^{\mathrm{HT}}}$.

Before continuing, we can use this result to obtain a crystalline specialisation.

Theorem 2.6 (Theorem 5.4.2)**.** *Let* R *be an animated commutative ring and denote* $\overline{R} = R \otimes^{\mathbb{L}} \mathbb{F}_p$*. There is a canonical equivalence* $\Delta_{\overline{R}} \simeq \widehat{\mathrm{dR}}_R$ *in* $\mathfrak{CAlg}(\widehat{\mathcal{D}}(\mathbb{Z}_p))$.

Remark 2.7. As a result, if $\overline{R}$ is quasisyntomic, we have an equivalence $\mathbb{R}\Gamma_{\mathrm{crys}}(\overline{R}/\mathbb{Z}_p) \simeq \widehat{\mathrm{dR}}_R$.

Going back to the Cartier–Witt stack, we define the de Rham comparison map as

$$\gamma_{\Delta}^{\mathrm{dR}}\colon \Delta_R = \mathbb{R}\Gamma(\mathcal{WC}art, \mathcal{H}_{\Delta}(R)) \to \mathcal{H}_{\Delta}(R)_{\mathrm{dR}} \xrightarrow{\alpha_R} \widehat{\mathrm{dR}}_R \tag{17}$$

and also obtain:

Proposition 2.8 (Corollary 5.4.14). *For any animated commutative ring* R *and every integer* n*, the de Rham comparison map fits into the cartesian square*

$$\begin{tikzcd}
\Delta_R\{n\} \arrow[r, "\gamma_{\Delta}^{\mathrm{dR}}\{n\}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widehat{\mathrm{dR}}_R \arrow[d] \\
\mathbb{R}\Gamma(\mathcal{WC}art, F^*(\mathcal{H}_{\Delta}(R)\{n\})) \arrow[r] & \widehat{\mathrm{dR}}_R \widehat{\otimes}^{\mathbb{L}} \mathbb{R}\Gamma(\mathcal{WC}art^{\mathrm{HT}}, \mathcal{O}_{\mathcal{WC}art^{\mathrm{HT}}}).
\end{tikzcd} \tag{18}$$

As a consequence, we do find that the weight-0 part of $\mathrm{Fil}^{\bullet}(\gamma_{\Delta}^{\mathrm{dR}}\{n\})$ identifies with $\gamma_{\Delta}^{\mathrm{dR}}\{n\}\colon \Delta_R\{n\} \to \widehat{\mathrm{dR}}_R$.

3 The associated graded of the Nygaard filtration

Construction 3.1. Recall that the relative Nygaard filtration comes equiped with a relative Frobenius morphism $\mathrm{Fil}^{\bullet}\,\varphi\colon\ \mathrm{Fil}^{\bullet}_{\mathrm{Ng}}\,\Phi^*\Delta_{R\otimes^{\mathbb{L}}A/A} \to \Delta^{[\bullet]}_{R\otimes^{\mathbb{L}}A/A}$ for any bounded prism (A, I) and any animated commutative ring R. By functoriality in (A, I), this determines a morphism

$$\mathrm{Fil}^{\bullet}\,\phi\colon\ \mathrm{Fil}^{\bullet}_{\mathrm{Ng}}\,F^*\mathcal{H}_{\Delta}(R) \to \mathcal{I}^{\times\bullet}\mathcal{H}_{\Delta}(R) \tag{19}$$

also called the **relative Frobenius**.

Now tensoring (in weight n) with the isomorphism $F^*(\mathcal{O}_{\mathcal{WC}art}\{n\}) \simeq \mathcal{I}^{-n}\{n\}$ and passing to global sections we get a map

$$\varphi\{n\}\colon\ \mathrm{Fil}^{n}_{\mathrm{Ng}}\,\Delta_R\{n\} \to \mathbb{R}\Gamma(\mathcal{WC}art, \mathrm{Fil}^{n}_{\mathrm{Ng}}\,F^*(\mathcal{H}_{\Delta}(R)\{n\})) \to \Delta_R\{n\}. \tag{20}$$

Note that when taking $n = 0$, this produces a Frobenius map $\Delta_R \to \Delta_R$.

Remark 3.2. The relative Frobenius map ϕ induces the map of associated graded

$$\mathrm{gr}^{m}\,\phi\colon\ \mathrm{gr}^{m}_{\mathrm{Ng}}\,F^*\mathcal{H}_{\Delta}(R) \to \iota_*\mathcal{H}_{\overline{\Delta}}(R)\{m\} \tag{21}$$

and factors through an isomorphism $\mathrm{gr}^{m}_{\mathrm{Ng}}\,F^*\mathcal{H}_{\Delta}(R) \simeq \iota_*\,\mathrm{Fil}^{\mathrm{conj}}_{m}\,\mathcal{H}_{\overline{\Delta}}(R)\{m\}$.

From this (and the previous discussion) we see that the associated graded of the Nygaard filtration fits in the cartesian square

$$\begin{tikzcd}
\mathrm{gr}^{m}_{\mathrm{Ng}}\,\Delta_R\{n\} \arrow[r, "\gamma_{\Delta}^{\mathrm{Hod}}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{L}\widehat{\Omega}^{m}_R[-m] \arrow[d] \\
\mathbb{R}\Gamma(\mathcal{WC}art^{\mathrm{HT}}, \mathrm{Fil}^{\mathrm{conj}}_{m}\,\mathcal{H}_{\overline{\Delta}}(R)\{m\}) \arrow[r] & \mathbb{R}\Gamma(\mathcal{WC}art^{\mathrm{HT}}, \mathrm{gr}^{\mathrm{conj}}_{m}\,\mathcal{H}_{\overline{\Delta}}(R)\{m\})
\end{tikzcd} \tag{22}$$

The associated graded then also fits in a fibre sequence with the diffracted Hodge complex:

$$\mathrm{gr}^m_{\mathrm{Ng}} \Delta_R\{n\} \to \mathrm{Fil}^{\mathrm{conj}}_m \widehat{\Omega}^{D}_R \xrightarrow{\Theta+m} \mathrm{Fil}^{\mathrm{conj}}_{m-1} \widehat{\Omega}^{D}_R \tag{23}$$

Proposition 3.3 (Proposition 5.5.12). *Let R be an animated commutative ring and let n be an integer. Then:*

- *For every integer $m \geq 0$, the map $\Delta_R\{n\}/\mathrm{Fil}^m_{\mathrm{Ng}} \Delta_R\{n\} \to \widehat{\mathrm{dR}}_R/\mathrm{Fil}^m_{\mathrm{Hod}} \widehat{\mathrm{dR}}_R$ induced by $\mathrm{Fil}^m(\gamma^{\mathrm{dR}}_{\Delta}\{n\})$ is an isogeny: that is, its fiber is annihilated by p^k for $k \gg 0$.*
- *For $0 \leq m \leq p$, the induced map $\Delta_R\{n\}/\mathrm{Fil}^m_{\mathrm{Ng}} \Delta_R\{n\} \to \widehat{\mathrm{dR}}_R/\mathrm{Fil}^m_{\mathrm{Hod}} \widehat{\mathrm{dR}}_R$ is an equivalence.*

Proof. We prove the first assertion, by induction on m. It is then enough to show that

$$\mathrm{gr}^m(\gamma^{\mathrm{dR}}_{\Delta}\{n\})\colon\ \mathrm{gr}^m_{\mathrm{Ng}} \Delta_R\{n\} \to \mathrm{gr}^m_{\mathrm{Hod}} \widehat{\mathrm{dR}}_R \simeq \mathbb{L}\widehat{\Omega}^m_R[-m] \tag{24}$$

is an isogeny. We identified the fibre of this map with $\left(\mathrm{Fil}^{\mathrm{conj}}_{m-1} \widehat{\Omega}^{D}_R\right)^{\Theta=-m}$.

We now prove, by induction on i, that $\left(\mathrm{Fil}^{\mathrm{conj}}_{i} \widehat{\Omega}^{D}_R\right)^{\Theta=-m}$ is annihilated by a power of p for any $0 \leq i < m$. For the induction step we only need to show that this is the case for $\left(\mathrm{gr}^{\mathrm{conj}}_{i} \widehat{\Omega}^{D}_R\right)^{\Theta=-m}$, which follows from the fact that $\Theta + m$ acts by multiplication by $m - i$ on $\mathrm{gr}^{\mathrm{conj}}_{i} \widehat{\Omega}^{D}_R$.

For the second assertion, we use that $m - i$ is a unit in $\mathbb{Z}_p$ for $0 \leq i < m < p$ and follow the same path. □

Proposition 3.4 (Corollary 5.5.14, Remark 5.5.15). *For every integers m and n, there is an isomorphism $\mathrm{gr}^m_{\mathrm{Ng}} \Delta_R \xrightarrow{\simeq} \mathrm{gr}^m_{\mathrm{Ng}} \Delta_R\{n\}$.*

Proof. For any n, the element $1 \in H^0(\mathrm{gr}^0_{\mathrm{Hod}} \widehat{\mathrm{dR}}_R)$ admits a unique lift $e^n \in H^0(\mathrm{gr}^0_{\mathrm{Ng}} \Delta_R\{n\})$ through $\mathrm{gr}^0(\gamma^{\mathrm{dR}}_{\Delta}\{n\})$. The isomorphism is given by multiplication by this e^n with inverse from e^{-n}. □

Proposition 3.5 (Corollary 5.5.18). *For any animated commutative ring R and any $n < p$, there is an equivalence $\mathrm{gr}^n_{\mathrm{Ng}} \Delta_R \simeq \mathbb{L}\widehat{\Omega}^n_R[-n]$.*

Proposition 3.6 (Proposition 5.5.20). *For any integers m and n, the functor $\mathrm{gr}^m_{\mathrm{Ng}} \Delta_{-}\{n\}$ satisfies p-completely faithfully flat descent.*

Proof. Because the diffracted Hodge complex does. □

Corollary 3.7 (Corollary 5.5.22, Corollary 5.5.21). *For any integers m and n, the functor $\mathrm{Fil}^m_{\mathrm{Ng}} \Delta_{-}\{n\}$ satisfies étale descent, and p-quasisyntomic descent for its restriction to $\mathfrak{CAlg}^{\mathrm{QSyn}}$.*

Proof. By induction, using the associated graded in the induction step. □