Derived infinitesimal foliations, after Toën–Vezzosi[TV23]

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KTH/SU Algebraic geometry arχiv Seminar
3rd and 10th November 2023

I will describe the constructions and results of the paper [TV23] (`arXiv:2305.13010`) by Toën–Vezzosi defining derived infinitesimal foliations, a variant of their earlier derived foliations in positive characteristic, that unlike the latter are always formally integrable, and relate to infinitesimal cohomology rather than de Rham or crystalline cohomology. The construction will take us through the world of graded circles in order to capture the algebraic structures of mixed differentials, and use ideas of non-connective derived geometry so as to compare the different shifts between the two kinds of foliations.

1 Flashback to 2020: (infinitesimal) derived foliations in characteristic 0

Reminder 1.1 (Classical foliations). A regular foliation on a manifold M is a decomposition of M into a union of disjoint leaves (with special charts such that the leaves appear locally as a level set of a function) of constant dimension. It induces a distribution $E \hookrightarrow TM$, with E_x consisting of the vectors tangent to the leaf at x.

By the Frobenius integrability theorem, such a distribution comes from a foliation if and only if it is involutive, meaning that it is stable under the Lie bracket of tangent vectors. Thus a regular foliation is equivalent to a Lie algebra $(E, [-,-])$ with an anchor map $(E, [-,-]) \to (TM, [-,-])$, so a Lie algebroid, such that the anchor map is injective. Under dualisation, the involutivity condition means that the quotient $T^\vee M \twoheadrightarrow E^\vee$ is characterised by its kernel being a differential ideal in $\mathrm{dR}(M) = (\wedge^\bullet(T_M^\vee), d_{\mathrm{dR}})$.

The structure of differential ideal in $\mathrm{dR}\, M$ is the one that will be convenient to translate to the derived setting. Note that in this context, the (-1)-truncatedness condition that $E \to TM$ be an injection has no reason to exist, so we need only capture the algebraic structure of differential ideals, also called that of a mixed graded algebra.

We note also, for the integrability, that the notion of leaves will not make sense in the algebraic world. To make sense of integrability, we will use the following reformulation:

passing to the set of components of a foliation $\mathcal{F}$ defines a projection to its leaf space $M \twoheadrightarrow \mathrm{Leaf}(\mathcal{F})$, whose kernel defines a differentiable groupoid. In the algebraic setting, we will get a formal groupoid corresponding to the projection to a formal leaf space.

Definition 1.2 (Mixed graded objects)**.** *The ∞-category of* ***graded derived modules*** *is* $\mathfrak{dMod}_{\Bbbk}^{\mathrm{gr}} := \mathfrak{Fun}(\mathbb{Z}^{\mathrm{discr}}, \mathfrak{dMod}_{\Bbbk})$*. For* $E = \bigoplus_{i\in\mathbb{Z}} E(i)$ *a graded module and* $w \in \mathbb{Z}$*, we write* $E(\!(w)\!) = \bigoplus_{i\in\mathbb{Z}} E(i+w)$ *its weight-shifting by* w*.*

A ***mixed graded structure*** *on* E *is a map* $\varepsilon\colon E \to E(\!(1)\!)[-1]$ (i.e. *ε is a map of weight $+1$ and degree -1) with* $\varepsilon^2 = 0$*.*

We denote $\varepsilon\text{-}\mathfrak{dMod}_{\Bbbk}^{\mathrm{gr}}$ *the ∞-category of mixed graded derived modules.*

Remark 1.3 (Tate realisation)*.* To make the construction clearer, let us identify $\mathfrak{dMod}_{\Bbbk}$ with $\mathfrak{Ch}(\mathfrak{Mod}_{\Bbbk})[\mathrm{qis}^{-1}]$. The ∞ functor $\varepsilon\text{-}\mathfrak{dMod}_{\Bbbk}^{\mathrm{gr}} \to \mathfrak{dMod}_{\Bbbk}$ sending E to $\prod_{p\in\mathbb{Z}} E(p)[-2p]$ with the *total* differential upgrades naturally to an ∞-functor $|-|^t\colon \varepsilon\text{-}\mathfrak{dMod}_{\Bbbk}^{\mathrm{gr}} \to \mathfrak{dMod}_{\Bbbk}^{\mathrm{fil}}$ (by setting $F^i|E|^t = \prod_{p\geq -i} E(p)[-2p]$), called the **Tate realisation**. By [TV20b, Propositon 1.3.1], it is fully faithful, and induces an equivalence with the *complete* filtered modules (meaning such that $F^iE \xrightarrow{\simeq} \lim_{j\leq i}(F^iE)/(F^jE)$).

Construction 1.4 (Redshift)*.* There is a self-equivalence $\mathscr{RS}(-)$ of $\mathfrak{dMod}_{\Bbbk}^{\mathrm{gr}}$ sending $E = \bigoplus_i E(i)$ to $\bigoplus_i E(i)[-2i]$. The image of a mixed graded structure $\varepsilon\colon E \to E(\!(1)\!)[-1]$ is a (square-zero) map $\eta = \mathscr{RS}(\varepsilon)\colon \mathscr{RS}(E) \to \mathscr{RS}(E)(\!(1)\!)[1]$.

We denote the ∞-category of such $\eta\text{-}\mathfrak{dMod}_{\Bbbk}^{\mathrm{gr}}$. On this category, the Tate construction takes the (simpler) form $E \mapsto \prod_p E(p)$.

Remark 1.5*.* We have $\varepsilon\text{-}\mathfrak{dMod}_{\Bbbk}^{\mathrm{gr}} \simeq \mathfrak{dMod}_{\Bbbk[\varepsilon]}^{(\mathrm{gr})}$ and $\eta\text{-}\mathfrak{dMod}_{\Bbbk}^{\mathrm{gr}} \simeq \mathfrak{dMod}_{\Bbbk[\eta]}^{(\mathrm{gr})}$ where $\Bbbk[\varepsilon] \simeq \Bbbk \oplus \Bbbk(\!(-1)\!)[1]$ and $\Bbbk[\eta] \simeq \Bbbk \oplus \Bbbk(\!(-1)\!)[-1]$ are the square-zero extensions by a generator ε (resp. η) of weight 1 and cohomological degree -1 (resp. 1).

Definition 1.6 (Derived foliation)**.** *Let X be a derived $\Bbbk$-scheme of finite presentation. A* ***derived foliation*** *$\mathcal{F}$ on X is a sheaf of mixed graded commutative derived algebras* $\mathrm{dR}(\mathcal{F})$ *such that*

- *the (derived) $\mathcal{O}_X$-modules* $\mathrm{dR}(\mathcal{F})(1)[-1] =: \mathbb{L}_{\mathcal{F}}$ *is perfect and connective (of cohomological Tor-amplitude in* $(-\infty, 0]$*),*
- $\mathrm{dR}(\mathcal{F})$ *is quasi-free, that is its underlying graded commutative derived algebra is freely generated by a perfect (derived) $\mathcal{O}_X$-module (which is* $\mathbb{L}_{\mathcal{F}}[1] = \mathrm{dR}(\mathcal{F})(1)$*).*

The ∞-category of derived foliations on X is the *opposite* of the full sub-∞-category of the mixed graded commmutative derived $\mathcal{O}_X$-algebras spanned by the derived foliations.

Example 1.7*.*
- The terminal foliation 1_X has $\mathbb{L}_{1_X} = \mathbb{L}_X$, so that $\mathrm{dR}(1_X)$ is the de Rham algebra of X, with mixed graded structure induced by the de Rham differential.
- The initial foliation 0_X has $\mathbb{L}_{0_X} = 0$, so that $\mathrm{dR}(0_X) = \mathcal{O}_X$.

Definition 1.8 (Infinitesimal derived foliation)**.** *An **infinitesimal derived foliation** is like a derived foliation, except that* $\mathrm{dR}(\mathcal{F})$ *has a* $\Bbbk[\eta]$*-structure instead of a* $\Bbbk[\varepsilon]$*-structure, and the connectivity condition is on* $\mathbb{L}_{\mathcal{F}} = \mathrm{dR}(\mathcal{F})(1)[1]$.

Remark 1.9 (Towards a geometric interpretation). Recall that a grading on a derived stack X can be encoded, through the Rees algebra construction, as an action of the affine group scheme $\mathbb{G}_m$, spectrum of the Hopf algebra $\Bbbk[t, t^{-1}]$, so equivalently a morphism to the classifying stack $\mathcal{B}\,\mathbb{G}_m$. We would therefore like to express a mixed graded structure as an extension of this action to one by a larger group stack (in fact, it will be a semi-direct product group stack, so as to encode the compatibility of the mixed structure with the grading).

The derived algebra $\Bbbk[\varepsilon]$ is a Hopf algebra, with the usual comultiplication $\varepsilon \mapsto \varepsilon \otimes 1 + 1 \otimes \varepsilon$, and its dual Hopf algebra is $\Bbbk[\eta]$. As such, $\Bbbk[\varepsilon]$-module structures correspond to $\Bbbk[\eta]$-comodule structures and vice versa.

A $\Bbbk[\varepsilon]$-comodule structure is equivalently an action of the group derived scheme $\mathrm{Spec}(\Bbbk[\varepsilon]) =: S^1_{\mathrm{inf}}$ (thus named because $\Bbbk[\varepsilon] \simeq C_\bullet(S^1; \Bbbk)$), so infinitesimal derived foliations can be understood as natural objects of derived geometry. But $\Bbbk[\eta]$ lives in the wrong degrees for its "spectrum" to define a derived affine scheme, so the comparison with derived foliations requires a broader context to define $S^1_{\mathrm{dR}} := \mathrm{Spec}(\Bbbk[\eta])$ (where $\Bbbk[\eta] \simeq C^\bullet(S^1; \Bbbk)$).

Example 1.10 (Linear derived stacks). Let $\mathcal{M}$ be a quasicoherent derived module on a derived stack X. We define its "total space" as the derived stack $\mathbb{V}_X(\mathcal{M})$ mapping an affine X-scheme $\mathrm{Spec}\, A \xrightarrow{\varphi} X$ to

$$\mathbb{V}_X(\mathcal{M})(A) := \mathrm{hom}_{\mathfrak{dMod}_A}(\varphi^*\mathcal{M}, A).$$

When $\mathcal{M}$ is connective (*i.e.* has Tor-amplitude concentrated in non-positive degrees), this is clearly the functor of points of $\mathrm{Spec}_X(\mathcal{S}ym_{\mathcal{O}_X}\mathcal{M})$, so affine. But when $\mathcal{M}$ is not connective, this description no longer gives an affine scheme.

2 Derived affine stacks

By denormalisation, connective derived algebras can be modelled by animated (*i.e.* simplicial, modulo weak equivalences) algebras, and coconnective derived algebras correspond to cosimplicial algebras (with respect to a certain model structure). We will use a model structure on cosimplicial simplicial algebras that mixes these two orthogonal directions as capturing the coconnective and connective part of the same direction.

Construction 2.1 (Product totalisation of cosimplicial simplicial modules). We let $\mathfrak{csMod}$ denote the category of cosimplicial objects in simplicial modules. By convention, a cosimplicial simplicial module has components M_p^q where p is the simplicial index and q the cosimplicial one. The **product totalisation** of a cosimplicial simplicial module M is the cochain complex $\mathrm{Tot}^{\Pi}(M)$ given by

$$\mathrm{Tot}^{\Pi}(M)^n = \prod_{p \geq 0} M_p^{p+n},$$

with differential the alternated sum of simplicial faces and cosimplicial cofaces.

Definition 2.2. *A morphism* $M \to N$ *in* $\mathfrak{csMod}$ *is a* ***completed quasi-isomorphism*** *if* $\mathrm{Tot}^{\Pi}(M) \to \mathrm{Tot}^{\Pi}(N)$ *is a quasi-isomorphism of cochain complexes.*

Theorem 2.3 ([TV23, Theorem 1.2])**.** *The completed quasi-isomorphisms are the weak-equivalences for a cofibrantly generated simplicial model structure on* $\mathfrak{csMod}$ *(whose fibrations are the epimorphisms).*

The simplicial enrichment comes from the cotensoring over $\mathfrak{sSet}$ defined by $(M^X)^q_p = (M^q_p)^{X_q}$ for $M \in \mathfrak{csMod}$ and $X \in \mathfrak{sSet}$. It is compatible with the totalisation, that is $\mathrm{Tot}^{\Pi}(M^X) \simeq \mathrm{Tot}^{\Pi}(M)^X = \mathrm{holim}_p \, \mathrm{Tot}^{\Pi}(M)^{X_p}$.

Lemma 2.4. *The functor* Tot^{Π} *is a Quillen equivalence, so induces an equivalence between the homotopy* ∞*-category of the completed model structure on* $\mathfrak{csMod}$ *and* $\mathfrak{dMod}$*.*

Proof. Tautological. □

Definition 2.5. *The category of cosimplicial simplicial algebras is* $\mathfrak{csAlg}$*; it has a forgetful functor to* $\mathfrak{csMod}$*. The* ∞*-category of* ***completed non-connective derived algebras*** $\widehat{\mathfrak{dAlg}}$ *is the homotopy* ∞*-category of* $\mathfrak{csAlg}$ *with the model structure transfered along this forgetful functor,* i.e. *the localisation along the completed quasi-isomorphisms of underlying cosimplicial simplicial modules.*

Example 2.6. Consider the free cdga $\Bbbk[u]$ on a generator of degree 2 (coconnective), denormalised to a cosimplicial (simplicially constant) algebra, and its completion $\Bbbk[\![u]\!]$. Its product totalisation has

$$\mathrm{Tot}^{\Pi}(\Bbbk[u])^n = \prod_{p\geq 0}(\Bbbk[u])^{p+n} = \begin{cases} \prod_{\ell\geq 0}(\Bbbk[u])^{2\ell+n} & \text{if } n \text{ is even} \\ \prod_{\ell\geq 0}(\Bbbk[u])^{2\ell+1+n} & \text{if } n \text{ is odd,} \end{cases}$$

which in cohomology looks like

$$\begin{aligned} H^n(\mathrm{Tot}^{\Pi}(\Bbbk[u])) &= \begin{cases} \prod_{\ell\geq 0} u^{\ell+n/2}\Bbbk = u^{n/2}\prod_{\ell\geq 0} u^{\ell}\Bbbk & \text{if } n \text{ is even} \\ \prod_{\ell\geq 0} u^{\ell+(n+1)/2}\Bbbk = u^{(n+1)/2}\prod_{\ell\geq 0} u^{\ell}\Bbbk & \text{if } n \text{ is odd.} \end{cases} \\ &= u^{\lceil n/2\rceil}\prod_{\ell\geq 0} u^{\ell}\Bbbk \end{aligned}$$

We thus see that the finiteness condition differentiating $\Bbbk[u]$ from $\Bbbk[\![u]\!]$ disappears after totalisation, so that the map $\Bbbk[u] \to \Bbbk[\![u]\!]$ is a completed quasi-isomorphism.

Remark 2.7. The totalisation $\mathrm{Tot}^{\Pi}(M^\bullet_\bullet)$ of a cosimplicial simplicial module is a model for $\mathrm{holim}_{q\in\Delta}\,\mathrm{hocolim}_{p\in\Delta^{op}} M^q_p$ where the homotopy colimits and limits are taken in dg-modules (seeing each module M^q_p as a dg-module). In particular, up to completed quasi-isomorphism any cosimplicial simplicial algebra $A^{\bullet}_{\bullet}$ can be seen as a homotopy limit of a cosimplicial diagram of connective derived algebras.

In fact, the canonical map $A^{\bullet}_{\bullet} \to \lim_{A^{\bullet}_{\bullet} \to B_{\bullet}} B_{\bullet}$ (for $B_{\bullet}$ connective, so a simplicial algebra seen as a cosimplicial simplicial algebra constant in the cosimplicial direction) to the "connective completion" is a completed quasi-isomorphism.

Remark 2.8 (Graded variant). The above constructions also admit a graded variant, where $\mathrm{Tot}^{\Pi,\mathrm{gr}}$ is defined by applying Tot^{Π} weight by weight.

Definition 2.9 (Non-connective spectrum). *The non-connective spectrum ∞-functor is defined as a restricted Yoneda embedding: for any $A \in \widehat{\mathfrak{dAlg}}$, the derived stack $\mathrm{Spec}^{\mathrm{n.c.}}(A)$ is given by*

$$\mathrm{Spec}^{\mathrm{n.c.}}(A)\colon \mathfrak{dAlg} \ni B \mapsto \hom_{\widehat{\mathfrak{dAlg}}}(A, B).$$

Lemma 2.10 ([TV23, Proposition 1.7]). *The ∞-functor $\mathrm{Spec}^{\mathrm{n.c.}}\colon \widehat{\mathfrak{dAlg}}^{\mathrm{op}} \to \mathfrak{dSt}$ is fully faithful.*

Proof. The functor Spec admits a left-adjoint $\mathcal{O}(-)$, which at the level of the cosimplicial simplicial model sends a sheaf $K \times \mathrm{Spec}\, A$ (with $\mathrm{Spec}\, A = y_A$ representable) to $A^K \in \mathfrak{csAlg}^{\mathrm{op}}$ and is determined on general sheaves by colimit preservation. Thus $\mathrm{Spec}^{\mathrm{n.c.}}$ is fully faithful if and only if the counit of the adjunction $\mathcal{O}(\mathrm{Spec}^{\mathrm{n.c.}}) \to \mathrm{id}_{\mathfrak{csAlg}^{\mathrm{op}}}$ is a completed equivalence.

We now use the fact that $\mathrm{Spec}^{\mathrm{n.c.}}\, A$ is the homotopy colimit of the simplicial derived stack $[q] \mapsto \mathrm{Spec}(A^q_{\bullet})$ (with $A^q_{\bullet}$ a simplicial algebra viewed as a connective derived algebra). Thus the counit takes the form $A \to \mathrm{holim}_q\, A^q_{\bullet}$, which is an equivalence by definition of the completed quasi-isomorphisms. ☐

Likewise, the ∞-functor $\mathrm{Spec}^{\mathrm{n.c.,gr}}\colon \widehat{\mathfrak{dAlg}}^{\mathrm{gr,op}} \to \mathfrak{dSt}^{\mathrm{gr}} = \mathbb{G}_m\text{-}\mathfrak{dSt}$ of **graded non-connective spectrum** is fully faithful.

Remark 2.11 (Linear stacks revisited). It is now possible to say that for any derived module M, the linear stack $\mathbb{V}(M)$ of example 1.10 is the derived affine stack $\mathrm{Spec}^{\mathrm{n.c.}}(\mathrm{Sym}(M))$. It is shown in [Mon21, Corollary 2.9, Theorem 2.5] that, when restricted to eventually connective (aka bounded above) derived modules, the functor $\mathbb{V}^{\mathrm{gr}} = \mathrm{Spec}^{\mathrm{n.c.,gr}}\,\mathrm{Sym}$ is fully faithful.

Example 2.12. For $M = \Bbbk[-n]$, we get $\mathbb{V}(\Bbbk[-n]) = K(\mathbb{G}_a, n) = \mathcal{B}^n\, \mathbb{G}_a$. In fact, it can be shown that the ∞-category of derived affine stacks is the smallest full sub-∞-category of $\mathfrak{dSt}$ containing the $K(\mathbb{G}_a, n)$s and stable by small limits.

Note however that there is an equivalence $\mathrm{Sym}(\Bbbk[-1]) \simeq \Bbbk[\eta]$ only in characteristic zero (due to the cohomology of the symmetric groups with coefficients in $\Bbbk$ then vanishing).

3 Infinitesimal derived foliations and their cohomology

Construction 3.1 (Graded circles). The **infinitesimal circle** is $S^1_{\mathrm{inf}} \coloneqq \mathrm{Spec}(\Bbbk[\varepsilon])$. Note that it can be seen geometrically as $\Omega_0\mathbb{A}^1 = \{0\} \times_{\mathbb{A}^1} \{0\}$.

The **de Rham circle** is $S^1_{dR} \coloneqq \mathrm{Spec}^{n.c.}(\Bbbk[\eta])$. It can be obtained by [MRT19, Theorem 3.4.17] as $\mathcal{B}\,K$ where K is the intersection of the kernels of the p-Frobenius operators on big Witt vectors, or equivalently (cf. [MRT19, § 6.4.1]) the free divided power algebra on one generator.

By the standard $\mathbb{G}_m$-action on the shifted copy of $\Bbbk$ (since ε and η were declared of weight 1), both of these circles upgrade to (affine) graded derived stacks. We further let $\mathcal{H}_{dR} = \mathbb{G}_m \ltimes S^1_{dR}$ and $\mathcal{H}_{inf} = \mathbb{G}_m \ltimes S^1_{inf}$ be the de Rham and infinitesimal Hilbert group stacks.

Remark 3.2. The de Rham circle $S^1_{dR} \simeq \mathcal{B}\,K$ is, by [MRT19, Theorem 1.2.1], the associated graded of a filtration on the affinisation of the topological circle, that is on the non-connective spectrum $\mathrm{Spec}^{n.c.}(C^\bullet(S^1)) = \mathrm{Spec}^{n.c.}(\mathcal{O}(S^1_B))$ of chains on the circle (equivalently of global functions on its Betti shape, the constant stack S^1_B), appearing as intersection of the fixed points of the Frobenius operators on Witt vectors.

Meanwhile, S^1_{inf} is the associated graded of a filtration (realised as $\mathcal{B}\,\mathbb{G}_m \to \mathbb{A}^1/\mathbb{G}_m$) on the empty scheme.

Lemma 3.3. *There are (symmetric monoidal) equivalences of ∞-categories $\mathfrak{QCoh}(\mathcal{B}\,\mathcal{H}_{dR}) \simeq \varepsilon\text{-}\mathfrak{dMod}^{gr}$ and $\mathfrak{QCoh}(\mathcal{B}\,\mathcal{H}_{inf}) \simeq \eta\text{-}\mathfrak{dMod}^{gr}$ — and so the redshift equivalence can be seen as $\mathfrak{QCoh}(\mathcal{B}\,\mathcal{H}_{dR}) \simeq \mathfrak{QCoh}(\mathcal{B}\,\mathcal{H}_{inf})$.*

We can thus define more generally an ε-mixed graded (resp. η-mixed graded) derived stack to be a derived stack over $\mathcal{B}\,\mathcal{H}_{dR}$ (resp. over $\mathcal{B}\,\mathcal{H}_{inf}$).

Example 3.4 (Graded loop stacks). Let X be a derived stack. The de Rham graded loop stack of X is the derived stack of morphisms $\mathscr{L}^{dR}X = \mathcal{M}or(S^1_{dR}, X)$, and the infinitesimal graded loop stack of X is $\mathscr{L}^{inf}X = \mathcal{M}or(S^1_{inf}, X)$. They carry respective actions of $\mathcal{H}_{dR}$ and $\mathcal{H}_{inf}$, induced by the canonical such actions on the relevant graded circles.

In fact, $\mathscr{L}^{dR}X$ (resp. $\mathscr{L}^{inf}X$) is the cofree $\mathcal{H}_{dR}$-equivariant (resp. $\mathcal{H}_{inf}$-equivariant) derived stack on the graded derived stack $X \times \mathcal{B}\,\mathbb{G}_m$ (X with trivial grading).

By [HAG2, Proposition 1.4.1.6], one can also see that $\mathscr{L}^{dR}X \simeq T[-1]X = \mathbb{V}(\mathbb{L}_X[1])$ is the (-1)-shifted tangent bundle of X and $\mathscr{L}^{inf}X \simeq T[1]X = \mathbb{V}(\mathbb{L}[-1])$ is its 1-shifted tangent bundle.

Definition 3.5. *Let X be a derived Deligne–Mumford stack.*

*A **derived foliation** on X is an $\mathcal{H}_{dR}$-equivariant derived X-stack $\mathcal{F}$, over $\mathscr{L}^{dR}X$ as an ε-mixed graded derived stack, such that its underlying graded derived stack is of the form $\mathbb{V}(\mathbb{L}_{\mathcal{F}}[1])$ for a (uniquely determined by remark 2.11) perfect $\mathbb{L}_{\mathcal{F}}$.*

*An **infinitesimal derived foliation** on X is an $\mathcal{H}_{inf}$-equivariant derived X-stack $\mathcal{F}$, over $\mathscr{L}^{inf}X$ as an η-mixed graded derived stack, such that its underlying graded derived stack is of the form $\mathbb{V}(\mathbb{L}_{\mathcal{F}}[-1])$ for a (uniquely determined) perfect $\mathbb{L}_{\mathcal{F}}$.*

Example 3.6. We can now give the foliations of example 1.7 a geometric interpretation, in any characteristic.

- The terminal (resp. infinitesimal) derived foliation 1_X is given by $\mathscr{L}^{dR}X \xrightarrow{id} \mathscr{L}^{dR}X$ (resp. $\mathscr{L}^{inf}X \xrightarrow{id} \mathscr{L}^{inf}X$).

- The initial (resp. infinitesimal) derived foliation 0_X is given by the $\mathcal{H}_{dR}$-equivariant $X \to \mathscr{L}^{dR}X$ (resp. the $\mathcal{H}_{inf}$-equivariant $X \to \mathscr{L}^{inf}X$) mapping X to "constant loops" — or, in the interpretation as shifted tangent bundles, the zero section.

Construction 3.7 (Crystals). Let $\mathcal{F} \to \mathscr{L}^{dR}X$ be a derived foliation. Recall that $\mathfrak{QCoh}([\mathcal{F}/\mathcal{H}_{dR}])$ is the ∞-category of $\mathcal{H}_{dR}$-equivariant quasi-coherent modules on $\mathcal{F} = \operatorname{Spec}^{n.c.}(dR(\mathcal{F}))$, that is ε-mixed graded $dR(\mathcal{F})$-modules.

The projection $p\colon \mathcal{F} \to X$, while not $\mathscr{H}_{dR}$-equivariant, is $\mathbb{G}_m$-equivariant (where X is acted upon trivially), so that it lifts to $[\mathcal{F}/\mathbb{G}_m] \to X$ and induces $p^*\colon \mathfrak{QCoh}(X) \to \mathfrak{QCoh}([\mathcal{F}/\mathbb{G}_m])$, producing the free graded $dR(\mathcal{F})$-module $E = dR(\mathcal{F}) \otimes_{dR(\mathcal{F})(0) \simeq \mathcal{O}_X} \mathcal{E}(0)$ on a quasicoherent $\mathcal{O}_X$-module $E(0)$.

A quasicoherent **crystal** along $\mathcal{F}$ is an object of $\mathfrak{QCoh}([\mathcal{F}/\mathcal{H}_{dR}])$ whose underlying graded (pullback to $[\mathcal{F}/\mathbb{G}_m]$) is free.

We define crystals along an infinitesimal foliation by replacing all the de Rham objects by their infinitesimal versions in the above construction.

Definition 3.8. *Let $\mathcal{F}$ be a derived foliation on X, and $\mathcal{E}$ a crystal along $\mathcal{F}$. The (derived) de Rham cohomology of X along $\mathcal{F}$ with coefficients in $\mathcal{E}$ is*

$$\widehat{C}^\bullet_{dR}(\mathcal{F}, \mathcal{E}) \coloneqq \Gamma([\mathcal{F}/\mathcal{H}_{dR}], \mathcal{E}).$$

If $\mathcal{F}$ is an infinitesimal derived foliation on X, and $\mathcal{E}$ a crystal along $\mathcal{F}$, the (derived) infinitesimal cohomology of X along $\mathcal{F}$ with coefficients in $\mathcal{E}$ is

$$\widehat{C}^\bullet_{inf}(\mathcal{F}, \mathcal{E}) \coloneqq \Gamma([\mathcal{F}/\mathcal{H}_{inf}], \mathcal{E}).$$

Reminder 3.9 (Infinitesimal cohomology). For X any derived stack, its **infinitesimal shape** is the sheaf

$$X_{inf}\colon \mathfrak{dAlg} \ni A \mapsto X(A_{red}).$$

Note that by construction, X_{inf} is étale over $\operatorname{Spec} \mathbb{k}$, and in particular classical. When X is a classical scheme, the cohomology of X_{inf} relates to the infinitesimal cohomology of X in the following way.

Recall that the infinitesimal site $\mathfrak{Inf}(X)$ has objects the pairs of an X-scheme $Y^\circ \to X$ and a closed immersion with nilpotent kernel $Y^\circ \hookrightarrow Y$. There is then a morphism of sites $\mathfrak{Sch}_{/X_{inf}} \to \mathfrak{Inf}(X)$ sending $Y \to X_{dR}$ (corresponding to a map $Y_{red} \to X$) to the pair $(Y_{red} \to X, Y_{red} \hookrightarrow Y)$. This is not an equivalence, but the induced morphism of topoï can be seen to be an equivalence, and so for the purposes of computing cohomology, there is no harm in also denoting X_{inf} the topos presented by $\mathfrak{Inf}(X)$: in either case, the infinitesimal cohomology of X is $\Gamma(X_{inf}, \mathcal{O})$.

Theorem 3.10 ([Toë20, Proposition 3.2.2],[TV23, Theorem 3.2])**.** *Let X be a smooth affine scheme.*

1. *If $1_{X,dR}$ denotes the terminal derived foliation $\mathcal{L}^{dR}X \to \mathcal{L}^{dR}X$, then $\widehat{C}^\bullet_{dR}(1_{X,dR}, \mathcal{O})$ is equivalent to the (Hodge-completed, derived) de Rham cohomology of X.*

2. *If $1_{X,inf}$ denotes the terminal infinitesimal derived foliation $\mathcal{L}^{inf}X \to \mathcal{L}^{inf}X$, then there is an equivalence $\widehat{C}^\bullet_{inf}(1_{X,inf}, \mathcal{O}) \simeq \Gamma(X_{inf}, \mathcal{O})$.*

4 Integrability and comparison

Definition 4.1 (Smooth infinitesimal foliation)**.** *An infinitesimal derived foliation $\mathcal{F}$ on X is* ***smooth*** *if $\mathbb{L}_{\mathcal{F}}$ has perfect Tor-amplitude 0, and* ***shifted smooth*** *if $\mathbb{L}_{\mathcal{F}} \simeq V[1]$ with V of perfect Tor-amplitude 0.*

If X is smooth, shifted smoothness means that $\mathcal{F} = \mathrm{Spec}(\mathrm{dR}(\mathcal{F})) = \mathbb{V}_X(\mathbb{L}_{\mathcal{F}}[-1])$ is a (smooth) vector bundle on X.

Lemma 4.2 ([TV23, Proposition 4.1])**.** *Let X be a smooth scheme. The category of shifted smooth infinitesimal derived foliations is equivalent to that of formal thickenings of X (formal schemes Z with $Z_{\mathrm{red}} \simeq X$) locally equivalent to $X \times \widehat{\mathbb{A}^r}$, where $r = \mathrm{rk}(\mathbb{L}_{\mathcal{F}}[-1])$.*

Proof. The formal scheme corresponding to $\mathcal{F}$ is $\mathrm{Spf}(\widehat{C}_{\mathrm{inf}}(\mathcal{F}))$, where $\widehat{C}_{\mathrm{inf}}(\mathcal{F})$ is a sheaf on X of (discrete) complete filtered algebras augmented to $\mathcal{O}_X$, with associated graded isomorphic to $\mathcal{O}_X[\![t_1, \ldots, t_r]\!]$ by the smoothness assumption. This construction is seen to be an equivalence thanks to Tate realisation of remark 1.3 inducing an equivalence between complete filtered modules and mixed graded modules. □

Proposition 4.3 ([TV23, Corollary 4.2])**.** *The category of smooth infinitesimal derived foliations on X is equivalent to that of formally smooth groupoids on X.*

Proof. Let $\mathcal{F}$ be a smooth infinitesimal derived foliation, and recall the initial infinitesimal foliation 0_X given by $X \to \mathscr{L}^{\mathrm{inf}}X$. The loop object $\Omega_X\mathcal{F} = 0_X \times_{\mathcal{F}} 0_X$ has cotangent complex given by $\mathbb{L}_{\mathcal{F}}[1]$, so is shifted smooth. The same is true of the further stages of the higher kernel (a.k.a. nerve) of the map $0_X \to \mathcal{F}$, so that it produces a groupoid in shifted smooth foliations, corresponding by the above lemma to a groupoid in formal schemes (whose 0th stage is X).

If on the other hand we have a formally smooth formal groupoid $G_\bullet$, with $G_0 = X$, passing to global functions produces a cosimplicial object $\mathcal{O}(G_\bullet)$ in complete filtered algebras, whose associated graded is the cosimplicial algebra of functions on the simplicial scheme $\mathcal{B}_\bullet V$ (presenting the stack $\mathcal{B}V = \mathbb{V}(V[1])$), where $V = \Omega^1_{G_1/X}|_X$. Since the Tate construction provided an equivalence between complete filtered modules and mixed graded modules, this now corresponds to a simplicial mixed graded algebra. Taking degreewise spectra, and the limit of the simplicial $\mathcal{H}_{\mathrm{inf}}$-scheme thereby obtained, we get $\mathbb{V}(V[1])$ with an $\mathcal{H}_{\mathrm{inf}}$-action, so a smooth infinitesimal foliation. □

We interpret the quotient of a formal groupoid integrating an infinitesimal derived foliation $\mathcal{F}$ as its formal leaf space.

It is expected that infinitesimal derived foliations should be formally integrable in full generality, so equivalent to derived formal groupoids. This is in contrast with derived foliations, which can be seen to not always be formally integrable. Thus, to understand the integrability of infinitesimal derived foliations, it will be useful to understand their precise comparison with derived foliations.

At the level of underlying derived modules, derived foliations and infinitesimal derived foliations appear to be only slight variants of one another, related through the redshift equivalence. However, the behaviour of the redshift on derived algebras is a lot more subtle, and will lead to infinitesimal derived foliations being seen not just as a variant of derived foliations, but as derived foliations equipped with an additional infinitesimal structure, which is what allows them to be formally integrated.

Construction 4.4 (Redshift for algebras). We let $\mathbb{Z}\langle u, v\rangle = \mathbb{Z}[u] \times_{\mathbb{Z}} \mathbb{Z}[v]$, where u is a generator in (cohomological) degree 2 and weight 1 and v in degree -2 and weight -1. Note that this is very different from $\mathbb{Z}[u, v]$ as it only contains monomials purely in u or v; in particular $u \cdot v = 0$. Note also that $\mathbb{Z}[v] = \mathrm{Sym}(\mathbb{Z}[2]) \simeq \bigoplus_{n\geq 0} \Gamma^n(\mathbb{Z})[2n]$ where $\Gamma^\bullet$ denotes the free divided power algebra — so that $H^\bullet(\mathrm{Tot}^\Pi \mathbb{Z}[v])$ in weight p is $\mathbb{Z} \cdot \frac{v^p}{p!}$.

We consider the Hadamard product $\otimes_H$ on graded completed non-connective derived algebras, given by degree-wise and weight-wise tensor product of rings. We then define the **redshift** endofunctor of $\widehat{\mathfrak{dAlg}}$ as $\mathscr{RS}(-) = - \otimes_H \mathbb{Z}\langle u, v\rangle$.

Via the $\mathrm{Spec}^{\mathrm{n.c.}}$ functor, this provides the redshift as an endofunctor of the ∞-category of graded derived affine stacks.

Lemma 4.5. *For any* $A \in \widehat{\mathfrak{dAlg}}$*, there is an equivalence* $\mathscr{RS}(\mathrm{Tot}^\Pi(A)) \simeq \mathrm{Tot}^\Pi(\mathscr{RS}(A))$.

In other words, the redshift introduced here is indeed a lift of the redshift on graded derived modules introduced in construction 1.4. However, unlike the redshift on graded derived modules, the redshift on graded derived algebras is not an equivalence.

Proposition 4.6. [*TV23, Lemma 5.3*] *If* X *and* Y *are graded derived affine stacks concentrated in weights* $[-1, 0]$*, then the canonical map* $\mathscr{RS}(X \times Y) \to \mathscr{RS}(X) \times \mathscr{RS}(Y)$ *is an equivalence away from* $(2) \in \mathrm{Spec}\,\mathbb{Z}$.

Proof. Note first that if a graded algebra A is concentrated in negative weights, then $\mathscr{RS}(A) \simeq A \otimes_H \mathbb{Z}[v]$. We then observe that the image by Tot^Π of the multiplication map $\mathbb{Z}[v](-1) \otimes \mathbb{Z}[v](-1) \to \mathbb{Z}[v](-2)$ is equivalent to $\mathbb{Z}[2] \otimes \mathbb{Z}[2] \simeq \mathbb{Z}[4] \xrightarrow{\times 2} \mathbb{Z}[4]$, which is indeed an equivalence once restricted to $\mathrm{Spec}\,\mathbb{Z}[1/2]$. □

It follows that $\mathscr{RS}(S^1_{\mathrm{dR}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[1/2])$ is again a group object, which an easy calculation shows to be equivalent to $S^1_{\mathrm{inf}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[1/2]$.

What is not clear is the following

Conjecture 4.7. *The redshift can be extended* (*away from* (2)) *to a functor from* S^1_{dR}*-equivariant,* i.e. ε*-mixed, graded derived affine stacks to* S^1_{inf}*-equivariant,* i.e. η*-mixed graded, ones.*

We can still, however, use this observation to get a comparison from derived foliations to derived infinitesimal foliations.

Construction 4.8. Let $\mathcal{F}$ be a derived foliation on X, given as $\mathbb{V}_X(\mathbb{L}_{\mathcal{F}}[1])$ with an $\mathcal{H}_{\mathrm{dR}}$-action. The algebra of functions on $\mathscr{RS}(\mathbb{V}_X(\mathbb{L}_{\mathcal{F}}[1]))$ is by definition $\mathscr{RS}(\mathcal{S}ym(\mathbb{L}_{\mathcal{F}}[1]))$. Then the canonical inclusion of modules $\mathbb{L}_{\mathcal{F}}[1] \to \mathcal{S}ym(\mathbb{L}_{\mathcal{F}}[1])$, combined with the equivalence $\mathscr{RS}(\mathbb{L}_{\mathcal{F}}[1]) \simeq \mathbb{L}_{\mathcal{F}}[-1]$, gives a map $\mathbb{L}_{\mathcal{F}}[-1] \to \mathscr{RS}(\mathcal{S}ym(\mathbb{L}_{\mathcal{F}}[1]))$, which through the adjunction property of $\mathcal{S}ym$ corresponds to a map of graded derived stacks

$$\mathscr{RS}(\mathbb{V}_X(\mathbb{L}_{\mathcal{F}}[1])) \to \mathbb{V}_X(\mathbb{L}_{\mathcal{F}}[-1]).$$

Assume the above conjecture is true (and work still over $\mathbb{Z}[1/2]$), so that $\mathscr{RS}(\mathbb{V}_X(\mathbb{L}_{\mathcal{F}}[1]))$ inherits an action of $\mathcal{H}_{\mathrm{inf}}$.

Definition 4.9. *An **infinitesimal structure** on a derived foliation $\mathscr{F}$ is an extension of the $\mathcal{H}_{\mathrm{inf}}$-action on $\mathscr{RS}(\mathbb{V}_X(\mathbb{L}_{\mathcal{F}}[1]))$ to $\mathbb{V}_X(\mathbb{L}_{\mathcal{F}}[-1])$ along the morphism constructed.*

Conjecture 4.10. *There is a forgetful functor from derived infinitesimal foliations to derived foliations, generalising the forgetting of an infinitesimal structure.*

This functor should work in the following way: $\mathscr{RS}(\mathbb{V}(\mathbb{L}[1]))$ is the divided power completion of $\mathbb{V}(\mathbb{L}[-1])$ along its zero section, and it is expected that $\mathscr{RS}(-)$ induces an equivalence between graded affine derived stacks and graded affine derived stacks equipped with divided power structures. Thus, taking the divided power completion of an infinitesimal derived foliation should, through this equivalence, recover a derived foliation.

Remark 4.11. Thanks to [BCN21], the classical Koszul duality between commutative algebras and Lie algebras refines to pair of Koszul dualities between, on the one hand, divided power algebras and Lie algebras, and on the other hand, commutative algebras and partition Lie algebras (the homotopical version of restricted Lie algebras).

The putative forgetful functor from infinitesimal derived foliations should thus ressemble, up to Koszul duality, the forgetful functor from partition Lie algebras to Lie algebras, or more generally, thanks to the definitions of [Fu≥23], from partition Lie algebroids to Lie algebroids.

More precisely, [Fu≥23] constructs a Chevalley–Eilenberg algebra functor on partition Lie algebroids, whose resulting commutative algebras should be equipped with an η-mixed graded structure induced by the Lie bracket.

Conjecture 4.12. *The Chevalley–Eilenberg functor induces an equivalence of ∞-categories between perfect partition Lie algebroids and perfect infinitesimal derived foliations.*

The formal integrability of infinitesimal derived foliations would then be a consequence of the equivalence between partition Lie algebroids and (derived) formal moduli problems.

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