Wall-crossing for quasimap Cohomological Field Theories

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Stable quasimaps generalise Gromov–Witten theory by allowing the stability condition to be parameterised by a rational line bundle on the target stack X. This has given rise to wall-crossing formulæ between the virtual structure sheaves of different stability chambers. The virtual sheaf of a moduli stack should be interpreted as the classical shadow of the structure sheaf of a derived enhancement, and indeed Mann and Robalo showed that its role as integral kernel for the construction of Cohomological Field Theories by Gromov–Witen invariants could be made more functorial by considering instead an action in the ∞-bicategory of spans in derived stacks. Similarly, in quasimap theory, any choice of stability parameter will endow the loop stack $\mathcal{L}X$ with a structure of lax algebra over the moduli spaces of curves $\overline{\mathcal{M}_{0,n}}$. We will explain how the wall-crossing phenomena can be interpreted as an action of a categorified Givental group on the space of $\overline{\mathcal{M}_{0,\bullet}}$-algebra structures.

1 Quasi-stable maps into Deligne–Mumford stacks

In this presentation, an **orbifold** will be a Deligne–Mumford stack presented by an orbifold groupoid, that is an étale and proper groupoid in algebraic spaces.

Let us denote Θ the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$, with its two points 0 and $1\colon \operatorname{Spec} k \to \Theta$ (we work over an algebraically closed field k of characteristic 0). A **close degeneration** of a geometric point x of an orbifold $\mathcal{X}$ is a map $f\colon \Theta \to \mathcal{X}$ such that $f1 = x$ et $f0 \neq x$.

Let $\mathscr{L}$ be an invertible sheaf on $\mathcal{X}$. A geometric point x is said to be $\mathscr{L}$-stable if the automorphism group stack of x in $\mathcal{X}$ is 0-dimensional and for every close degeneration f the weight of $f^*\mathscr{L}$ is negative.

Example 1. Suppose $\mathcal{X}$ is the quotient stack $[W/G]$ of a G-variety W, for some algebraic group G. The datum of an invertible sheaf on $\mathcal{X}$ is equivalent to that of a G-equivariant line bundle on W, that is a line bundle endowed with a G-linearisation. Then a geometric point of W is GIT-stable for this linearisation if and only if the induced point of $\mathcal{X}$ is stable relative to the line bundle.

The above definition can be extended to rational line bundles. Let $\mathscr{L} = \varepsilon \cdot \mathscr{L}_0 \in \operatorname{Pic}(\mathcal{X}) \otimes \mathbb{Q}$; we shall use it as a stability parameter for curves into $\mathcal{X}$.

We recall that a twisted curve with n markings over a base S is an orbifold $\mathcal{C} \to S$ which is étale-locally a nodal curve over S with n disjoint substacks $\Sigma_i \subset \mathcal{C}$ which are étale gerbes over S such that $\mathcal{C}$ is isomorphic to its coarse moduli space $C = |\mathcal{C}|$ over the complement of the nodes and marked gerbes. We may also consider the notion of very twisted curve, an étale gerbe over a twisted curve (so that the marked gerbes are banded by general group schemes rather than just roots of unity).

Definition 2. *An pre-$\mathscr{L}$-quasi-stable map into $\mathcal{X}$ is the data of a twisted curve $(\mathcal{C}, \Sigma_1, \dots, \Sigma_n)$ and a representable morphism $\mu\colon C \to X$ such that $\mu^{-1}(X^{\text{unstable}})$, called the base locus of μ, is of pure dimension 0 and contains no special gerbe.*

Construction 3. Given a quasimap $\mu\colon \mathcal{C} \to \mathcal{X}$, we can define its degree $\beta\colon \operatorname{Pic}(\mathcal{X})\otimes\mathbb{Q} \to \mathbb{Q}$ by $\mathscr{F} \mapsto \deg(\mu^*\mathscr{F})$. In particular, we have the number $\beta_0 = \beta(\mathscr{L})$.

Definition 4. *A pre-quasistable map is $\mathscr{L}$-quasi-stable if*

1. *the $\mathbb{Q}$-line bundle $\omega_{|\mathcal{C}|,\log} \otimes \mu^*\mathscr{L}$ on $C = |\mathcal{C}|$ is ample*
2. *for any geometric point x of $\mathcal{C}$, the order of vanishing of x along $\mathscr{L}$ is ≤ 1.*

Example 5. Explicitly, this means that for every irreducible component C_i of C (of genus g_i and with m_i special points) we have $2g_i - 2 + m_i + \epsilon \deg(\mu^*\mathscr{L}_0|_{C_i}) > 0$, and $\epsilon \operatorname{lngth}(x) \leq 1$.

If $\epsilon > 1$, then there are no basepoints and we obtain exactly (twisted) stable maps into the stable locus; we write this chamber as $\epsilon = +\infty$. If $\epsilon < \beta_0$, written $\epsilon = 0^+$, no rational tails are allowed. Between these two chambers, for a fixed $\mathscr{L}_0$, the stability condition remains constant for $\frac{1}{d+1} < \epsilon < \frac{1}{d}$, given a decomposition into chambers of the space of stability parameters.

The quasi-stability conditions are open conditions, so the moduli stack $\mathfrak{Q}^{\varepsilon\mathscr{L}_0}_{g,n}(\mathcal{X}, \beta)$ is an algebraic substack of the stack of morphisms $\mathfrak{Hom}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathfrak{C}^{\mathrm{tw}}_{g,n}, \mathfrak{M}^{\mathrm{tw}}_{g,n} \times \mathcal{X})$.

If $\mathcal{X}^{\mathrm{st}}$ is a complete intersection in $\mathbb{P}^N$, for a fixed $\mathscr{L}_0$ and two rationals $\varepsilon^+ > \frac{1}{d} > \varepsilon^-$ (with d integer), there is a morphism $c\colon \mathfrak{Q}^{\varepsilon^+}_{g,n}(\mathcal{X}, \beta) \to \mathfrak{Q}^{\varepsilon^-}_{g,n}(\mathcal{X}, \beta)$ which contracts the unstable rational tails of the source curve and replaces them by basepoints for the map, as well as a morphism $b_{\beta_0,\cdots,\beta_k}\colon \mathfrak{Q}^{\varepsilon^+}_{g,n+k}(\mathcal{X}, \beta_0 - \sum_{i=1}^k \beta_i) \to \mathfrak{Q}^{\varepsilon^-}_{g,n}(\mathcal{X}, \beta_0)$ which replaces the last k markings with basepoints.

Theorem 6 (Projective wall-crossing)**.** *There is a wall-crossing formula relating the virtual classes*

$$\sum_{\beta} q^{\beta}[\mathfrak{Q}^{\varepsilon^-}_{g,n}(\mathcal{X}, \beta)]^{\mathrm{vir}} = \sum_{\beta_0,\ldots,\beta_k} \frac{q^{\beta_0}}{k!} b_{\beta,*} c_* \left(\prod_{i=1}^{k} q^{\beta_i} \operatorname{ev}^*_{n+i} \mu^{\varepsilon^-}_{\beta_i}(-\psi_{n+i}) \frown [\mathfrak{Q}^{\varepsilon^+}_{g,+k}(\mathcal{X}, \beta_0)]^{\mathrm{vir}} \right) \tag{1}$$

where μ_β is the coefficient of q^β in Givental's J-function and ψ_i the tautological class.

We wish to understand this wall-crossing forumla (and the mirror parameter $\mu_\beta(-\psi_i)$) through the interpretation of the virtual structure sheaf of the moduli stack as the shadow of the moduli stack of a derived enhancement.

2 Derived enhancements and circle action

As the moduli space of quasi-stable maps is an open in a stack of morphisms, it admits a canonical derived enhancement. Let us recall the ∞-adjunction $\tau_0 \dashv \iota\colon \mathfrak{dSt} \leftrightarrows \mathfrak{St}$, whose right adjoint ι is fully faithful (and shall thus be omitted from notations). It follows that, for any stacks, $\mathcal{F}$ and $\mathcal{G}$, the derived mapping stack $\mathbb{R}\mathfrak{Map}(\mathcal{F}, \mathcal{G})$ is a derived enhancement of $\mathfrak{Hom}(\mathcal{F}, \mathcal{G})$. Concomitantly, there is a bijection between Zariski opens of a derived Artin stack and those of its truncation, allowing any open of an Artin stack to be extended to a derived enhancement compatible with the substack structure.

In the case of the moduli stacks of quasi-stable maps, we may thus define a derived (Artin) stack $\mathbb{R}\mathfrak{Q}^{\varepsilon\mathscr{L}_0}_{g,n}(\mathcal{X}, \beta) \subset \mathbb{R}\mathfrak{Map}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathfrak{C}^{\mathrm{tw}}_{g,n}, \mathfrak{M}^{\mathrm{tw}}_{g,n} \times \mathcal{X})$, with $\tau_0(\mathbb{R}\mathfrak{Q}^{\varepsilon\mathscr{L}_0}_{g,n}(\mathcal{X}, \beta)) = \mathfrak{Q}^{\varepsilon\mathcal{L}_0}_{g,n}(\mathcal{X}, \beta)$. If the algebraic space of objects of the presenting groupoid for $\mathcal{X}$ is a local complete intersection (out of the stable locus), this derived moduli stack is quasi-smooth.

In the general situation of an algebraic stack $\mathcal{M}$ with a quasi-smooth derived enhancement $j = \iota_{\mathbb{R}\mathcal{M}} \colon \mathcal{M} \hookrightarrow \mathbb{R}\mathcal{M}$, the induced morphism $j^*\mathbb{L}_{\mathbb{R}M} \to \mathbb{L}_M$ is a (2-coconnective) perfect obstruction theory on $\mathcal{M}$ and therefore allows the construction of a virtual sheaf by (derived) intersection of its zero section with the intrinsic normal cone. This G-theoretic class does in fact coincide with $(j_*)^{-1}[\mathscr{O}_{\mathbb{R}\mathcal{M}}]$, where j_* is an isomorphism by dévissage and the theorem of the heart.

The obstruction theory commonly used in the study of moduli spaces of quasimaps is indeed the one induced by the mapping stack, along the universal correspondence

$$\begin{array}{ccc} & \mathcal{C} \times \mathbb{R}\mathfrak{Map}(\mathcal{C}, \mathcal{X}) & \\ \swarrow & & \searrow^{\mathrm{ev}} \\ \mathcal{C} & & \mathcal{X} \end{array} , \tag{2}$$

where ev is the canonical evaluation map.

Recall however that in the case of twisted curves, as the markings have non-trivial gerbe structures, the morphisms ev_i evaluating the (quasi)map at the ith marking take value not in $\mathcal{X}$ itself but rather in its inertia stack $\mathcal{I}_{\mathcal{X}}$ (at least over $k = \mathbb{C}$, where it can be identified with the cyclotomic inertia stack). Since it satisfies the universal property of the 2-fibred product of stacks $\mathcal{I}_{\mathcal{X}} = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$, it is naturally enhanced to the derived free loop stack $\mathcal{L}\mathcal{X} = \mathcal{X} \times^{\mathbb{R}}_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$.

Hence, the spans

$$\begin{array}{ccc} & \mathbb{R}\mathfrak{Q}^{\varepsilon\mathscr{L}_0}_{0,n+1}(\mathcal{X}, \beta) & \\ {}^{(\mathrm{Stab},\mathrm{ev}_1,\ldots,\mathrm{ev}_n)}\swarrow & & \searrow^{\mathrm{ev}_{n+1}} \\ \overline{\mathcal{M}_{0,n+1}} \times \mathcal{L}\mathcal{X}^n & & \mathcal{L}\mathcal{X} \end{array} \tag{3}$$

endow $\mathcal{L}\mathcal{X}$, by brane action, with a structure of lax algebra over the ∞-operad of the moduli stacks of genus 0 stable curves in the ∞-bicategory of spans in derived stacks, which we may call a motivic field theory (MotFT).

One may observe that the free loop stack also apears naturally from the setting of the ∞-bicategory of spans. Indeed, $\mathfrak{Corr}_1(\mathfrak{St}) = \hom_{\mathfrak{Corr}_2(\mathfrak{St})}(*, *)$ by definition, where we recall that all (∞, k)-categories of k-iterated spans are fully dualisable. The image of the identity span $(\mathcal{X} = \mathcal{X} = \mathcal{X})$ in $\mathfrak{Corr}_1$ by the trace functor (also called the dimension of $\mathcal{X}$) is then $\mathcal{L}\mathcal{X}$. In particular, seeing as the objects of $\mathfrak{Corr}_1$ are not only dualisable but even Calabi–Yau, we recover the S^1-action on the loop stack $\mathcal{L}\mathcal{X}$.

3 Wall-crossing phenomena

The classical wall-crossing formula explicitly computes the difference between two virtual classes, and uses a mirror transformation parameter. Both those elements cannot directly be interpreted in the (derived) geometric setting. We will first explain the geometric origin of the wall-crossing phenomenon, and thanks to an operadic reinterpretation give a way of comparing the moduli spaces related by the wall-crossing.

3.1 Geometric wall-crossing by variations of stability in the master space

To simplify matters, we will focus on the case of $\mathcal{X}$ a quotient stack $[W/G]$, and $\mathscr{L}_\pm$ two fractional polarisations of the G-action.

In that case, Thaddeus's variations of GIT give a way of comparing the quasi-stability conditions. Serre's twisting sheaf $\mathscr{O}(1)$ on $\mathbb{P}_W(\mathscr{L}_+ \oplus \mathscr{L}_-)$ admits a canonical linearisation of the induced G-action, and the GIT quotient is called the master space. It inherits an action by the algebraic torus $T^1 = \mathrm{Spec}(k[t_1, t_2]/(t_1 t_2))$ together with a family of linearisations indexed by the algebraic simplex $\Delta^1 = \mathrm{Spec}(\mathbb{Q}[\delta_1, \delta_2]/(\delta_1 + \delta_2))$ (corresponding to $\mathscr{L}_+^{\otimes \delta_1} \otimes \mathscr{L}_-^{\otimes \delta_2}$), such that the quotient along the linearisation $(1, 0)$ gives back $W//_+G$ while that along $(0, 1)$ gives $W//_-G$. Let us denote $Z \to \Delta^1$ the (fibrewise) quotient stack of the trivial family (with appropriate, non-trivial, linearisation as mentioned).

The moduli stack of quasimaps to Z can then be endowed with an action of the torus, so that its fixed locus corresponds to maps into the fibres $(1, 0)$ et $(0, 1)$. Hence, we get a monic cospan between the two moduli stacks to compare. (We may also apply equivariant localisation to obtain explicit comparison formulæ, once the output of derived geometric equivariant localisation is clear.)

3.2 Operadic interpretation for CohFTs

The MotFT correspondence, after applying the functor $\mathfrak{Coh}$, passing to the K-group, and taking the Chern character, is used to obtain a CohFT structure on the graded algebra $A = A^\bullet \mathcal{X}$. Any operator R in the Givental group $\mathrm{End}(A)[[z]]$ verifying the symplectic constraint $R(z)R^*(-z) = 1$ (and $R(z) \equiv 1 \mod z$) can be used to transform a CohFT into another one.

Theorem 7 (Teleman)**.** *Let* A *be a semisimple cdga of rank* n*. The Givental group acts transitively on the set of CohFTs on* A*. Any semi-simple CohFT on* A *is thus determined by an* $R(z) \in \mathrm{End}(A)[[z]]$ *as above.*

When A is the cohomology of a space with a circle action, the formal part of $\mathrm{End}(A)[[z]]$ corresponds to a splitting given by a trivialisation of the circle action by R. In our derived setting, we always have a circle action as A is the orbifold cohomology of $\mathcal{X}$, that is the cohomology of the derived loop stack $\mathcal{L}\mathcal{X}$.

The Givental group action was further examined by Dostenko–Shadrin–Vallette, who showed that it admits a purely operadic interpretation. Quantisation of gauge theories is governed by the Batalin–Vilkovisky operad (which is also equivalent to the homology of the framed little disks operad). It is a quadratic–linear Koszul operad, and its reduced Koszul dual cooperad splits as $\overline{\mathscr{BV}^{\text{i}}} = A_\bullet(\overline{\mathcal{M}_{0,\bullet}})^{\text{i}} \oplus \overline{T}^c(\delta)$, where the cofree cogebra $T^c(\delta)$ is Koszul dual to $H^\bullet(S^1)$. In particular this decomposition carries over to the $\mathscr{L}_\infty$-algebra $\mathfrak{g}_{\mathscr{BV}}$ governing $\mathscr{BV}$-algebra structures on A, which splits along $\mathfrak{g}_\Delta$ (which is exactly $z\,\mathrm{End}(A)[[z]]$, the dg-Lie algebra of the Givental group). A (homotopy) CohFT on A is a $\mathscr{BV}$-algebra structure together with a homotopy trivialisation of the circle action. More precisely:

Theorem 8 (Dostenko–Shadrin–Vallette)**.** *Deformations of structures coming from* $\mathfrak{g}_{A^\bullet(\overline{\mathcal{M}_{0,\bullet}})} \subset \mathfrak{g}_{\mathscr{BV}}$ *in the direction of* $\mathfrak{g}_\Delta$ *still give CohFTs. Furthermore, the Givental group action corresponds exactly to the choice of trivialisations of the circle action.*

Remark 9. A similar structure result is also true at the topological level. The operad of (topological) moduli spaces of genus 0 stable curves was proven by Drummond-Cole to be the homotopy pushout of the framed little disks operad and the trivial operad along S^1, that is the homotopy trivialisation of the circle action in the framed little disks operad.

Every choice of stability parameters gives a quasimap MotFT on $\mathcal{L}\mathcal{X}$, that is a lax morphism of sheaves of categorical ∞-operads $\overline{\mathcal{M}_{0,\bullet}} \to \mathfrak{d}\mathfrak{S}\mathfrak{t}^{\text{corr},\times}_{/-}$, and variation of GIT allows us to compare the components (of the spans) across different chambers.