Overview: Facets of spectral algebraic geometry

David KERN

KTH/SU Spectral algebraic geometry learning seminar
5th March 2024

0 Introduction: What are (ring) spectra?

The aim of this talk is to motivate spectral algebraic geometry both from the points of view of (derived) algebraic geometry and of homotopy theory. For this, we will view (ring) spectra in two different ways, geared towards the studies of arithmetic and cohomology theories. But, to introduce them, motivating them from a more categorical point of view shows why spectra are inevitable:

Slogan 0. *Spectra are to ∞-categories as abelian groups are to 1-categories.*

More precisely, we develop the analogy in the following table.

Category theory	Higher category theory
Abelian categories	Stable ∞-categories
Abelian category $\mathfrak{Ab}$ of abelian groups	Stable ∞-cat. $\mathfrak{Sp}$ of spectra = freely stable on $\infty\mathfrak{Grpd}$
Abelian group of integers $\mathbb{Z}$	Sphere spectrum $\mathbb{S}$ = free on the point
Rings = commutative algebras in $(\mathfrak{Ab}, \otimes_{\mathbb{Z}})$	$\mathscr{E}_\infty$-rings = htopy coherent comm. alg. in $(\mathfrak{Sp}, \otimes_{\mathbb{S}})$

In particular, the last two lines mean that $\mathbb{S}$ is the initial $\mathscr{E}_\infty$-ring spectrum.

From this, we obtain our first point of view on spectra:

Slogan 1. *Spectra are $\mathbb{S}$-modules, $\mathbb{S}$ being a "topological" base below $\mathbb{Z}$.*

This implies that doing geometry with ring spectra corresponds to moving from the study of the arithmetic of the integers to the deeper arithmetic of the sphere spectrum. We will explore the advantages of this point of view in section 1.

The second point of view on spectra, whose benefits we will reap in section 2, is as they relate to cohomology theories.

For this, let us list some of the salient features of spectra:

- a spectrum E has a homotopy $\mathbb{Z}$-graded abelian group $\pi_\bullet E$,
- a spectrum E admits shifts $\Sigma^n E$ for all $n \in \mathbb{Z}$ (with $\pi_\bullet(\Sigma^n E) = \pi_{\bullet-n} E$ as expected),

- a spectrum E has an underlying ∞-groupoid (aka type, or anima, *etc.*) denoted $\Omega^\infty E$ (its "infinite loop space"),
- between any two spectra E and F there is a mapping spectrum $\underline{\hom}(E, F)$ (making $\mathfrak{Sp}$ a self-enriched ∞-category), with $\Omega^\infty \underline{\hom}(E, F) \simeq \hom(E, F)$ receovering the standard hom ∞-groupoid,
- the ∞-functor $\Omega^\infty \colon \mathfrak{Sp} \to \infty\mathfrak{Grpd}$ admits a left-adjoint Σ^∞ ("infinite suspension") producing the free spectrum generated by a type — so that we can think of it as the "free $\mathbb{S}$-module" functor $X \mapsto \mathbb{S}^X$. In particular, $\mathbb{S} = \Sigma^\infty *$.

Thus, an informal way of thinking of slogan 1 is that spectra are a "topological" variant of chain complexes or dg modules.

Now, given a spectrum E, and implicitly viewing topological spaces through the localisation $\Pi_\infty \colon \mathfrak{TopSp} \to \infty\mathfrak{Grpd}$, the functor $E^\bullet \colon \mathfrak{TopSp} \to \mathfrak{Ab}^{\mathbb{Z}\text{-gr}}$ mapping a space X to

$$E^\bullet X \coloneqq \pi_\bullet \underline{\hom}(\Sigma^\infty X, E) \simeq \pi_0 \underline{\hom}(\Sigma^\infty X, \Sigma^\bullet E) \simeq \pi_0 \hom(X, \Omega^{\infty - \bullet} E) \tag{1}$$

satisfies the axioms for a generalised cohomology theory (*i.e.*, the Eilenberg–Steenrod axioms except the dimension axiom); we thus say it is the cohomology theory represented by E.

The Brown representability theorem states that every generalised cohomology theory arises this way, giving way to our second point of view on spectra:

Slogan 2. *Spectra represent cohomology theories.*

An important distinction to make, however, is that there is only a bijection between *equivalence classes* of spectra and isomorphy classes of generalised cohomology theories: a spectrum contains more information than the cohomology theory it represents. This is particularly apparent for $\mathscr{E}_\infty$-ring spectra: the algebra structure on E induces multiplicative operations on $E^\bullet$, but the simple datum of a multiplicative cohomology theory does not capture the higher coherences of an $\mathscr{E}_\infty$-algebra structure.

1 Homotopy theory for commutative algebra: deeper arithmetic over the sphere spectrum

Let A be an associative ring. The group $K_0(A)$ extends to a family of higher K-theory groups $K_n(A)$, which are in fact the homotopy groups of a spectrum $K(A)$. It is however very difficult to compute, so certain approximations are used to work with it.

Construction 1.1 (Negative cyclic homology). The **Hochschild complex** of A is

$$\mathrm{HH}(A) = A \underset{A \underset{\mathbb{Z}}{\otimes} A}{\otimes} A \tag{2}$$

where the tensor product is implicitly derived. Seeing it as the function algebra $S^1 \otimes_{\mathbb{Z}} A$ of the derived loop stack $\mathscr{L}\,\mathrm{Spec}\,A = \mathrm{Spec}\,A \times_{\mathrm{Spec}\,A \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,A} \mathrm{Spec}\,A$, it has a canonical S^1-action.

We then define the **negative cyclic complex** of A as the (homotopy) fixed points

$$\mathrm{HC}^-(A) = \mathrm{HH}(A)^{S^1}. \tag{3}$$

There is a known **Dennis trace** $K(A) \to \mathrm{HH}(A)$, which lifts to a **cyclotomic trace** $K(A) \to \mathrm{HC}^-(A)$.

Rationally, it helps with understanding K-theory:

Proposition 1.2 ([Goo86]). *Let $A \to \overline{A}$ be a quotient with nilpotent kernel. Then the square*

$$\begin{array}{ccc} K(A) \otimes_{\mathbb{S}} \mathbb{Q} & \longrightarrow & \mathrm{HC}^-(A \otimes_{\mathbb{Z}} \mathbb{Q}) \\ \downarrow & \lrcorner & \downarrow \\ K(\overline{A}) \otimes_{\mathbb{S}} \mathbb{Q} & \longrightarrow & \mathrm{HC}^-(\overline{A} \otimes_{\mathbb{Z}} \mathbb{Q}) \end{array} \tag{4}$$

is (homotopy) cartesian.

However, without rationalisation, the result becomes false. Changing the base from $\operatorname{Spec} \mathbb{Z}$ to $\operatorname{Spec} \mathbb{S}$ solves this issue.

Definition 1.3. *The **topological Hochschild homology** of A is*

$$\mathrm{THH}(A) = A \underset{A \underset{\mathbb{S}}{\otimes} A}{\otimes} A \simeq S^1 \underset{\mathbb{S}}{\otimes} A \simeq \mathcal{O}\big(\operatorname{Spec} A \underset{\operatorname{Spec} A \times_{\operatorname{Spec} \mathbb{S}} \operatorname{Spec} A}{\times} \operatorname{Spec} A\big), \tag{5}$$

and its topological negative cyclic homology is $\mathrm{TC}^-(A) = \mathrm{THH}(A)^{S^1}$.

The Tate topological cyclic homology $\mathrm{TC}^t(A)$ is a refinement of $\mathrm{TC}^-(A)$ where the fixed points are taken respectively to all the Frobenius of the cyclic subgroups of S^1.

Theorem 1.4 ([DGM13, NS18]). *Let $A \to \overline{A}$ be a quotient with nilpotent kernel (or more generally, a map of connective ring spectra such that $\pi_0 A \to \pi_0 \overline{A}$ is surjective with nilpotent kernel). Then the square*

$$\begin{array}{ccc} K(A) & \longrightarrow & \mathrm{TC}^t(A) \\ \downarrow & \lrcorner & \downarrow \\ K(\overline{A}) & \longrightarrow & \mathrm{TC}^t(\overline{A}) \end{array} \tag{6}$$

is (homotopy) cartesian.

2 Geometry for homotopy theory: moduli stacks and cohomology theories

Consider the diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{\text{canon.}} & K(\mathbb{Z}, 2) \simeq \mathcal{B} S^1 \\ \simeq\downarrow & & \downarrow\simeq \\ \mathbb{CP}^1 & \hookrightarrow & \mathbb{CP}^\infty \end{array} \tag{7}$$

Definition 2.1. *A multiplicative cohomology theory* $E^\bullet$ *is* ***complex-orientable*** *if* $E^2(\mathcal{B}\, S^1) \to E^2(S^2)$ *is surjective.*

A ***complex orientation*** *for* E *is a lift* $c_1^E \in \widetilde{E}^2(\mathcal{B}\, S^1)$ *of* $1 \in E^0(*) \simeq \widetilde{E}^0(S^0) \simeq \widetilde{E}^2(S^2)$ *in reduced* E*-cohomology.*

Example 2.2. Any even (*i.e.* concentrated in even degrees) multiplicative cohomology theory is complex-orientable.

Example 2.3. There is a universal complex-oriented spectrum, denoted MU and known as complex cobordism: for any $\mathscr{E}_\infty$-ring spectrum E, (homotopy classes of) homomorphisms $MU \to E$ are in bijection with complex orientations on E.

Lemma 2.4. *For any complex orientation* c_1^E*, there is a isomorphism* $E^\bullet(\mathbb{CP}^\infty) \simeq E^\bullet(*)[[c_1^E]]$*.*

Under this isomorphism, the natural group law on $\mathcal{B}\, S^1$ *becomes a formal group law over* $E^\bullet(*)$*.*

In particular, a morphism $MU \to E$ induces a 1-dimensional formal group over $\pi_\bullet E$. What about going in the other direction?

Theorem 2.5 ([Laz55, Mil60, Qui69])**.** *Let* L *be Lazard's universal ring, such that morphisms* $L \to R$ *(for* R *a classical ring) are the same as* 1*-dimensional formal groups over* R*. There is an isomorphism of graded rings* $\pi_\bullet(MU) \simeq L$*.*

In other words, putting a formal group structure on $\pi_\bullet E = E^\bullet(*)$ is the same thing as giving $E^\bullet(*)$ an $MU^\bullet(*)$-algebra structure. We may then hope to recover $E^\bullet$ by the formula

$$E^\bullet(X) = MU^\bullet(X) \underset{MU^\bullet(*)}{\otimes} E^\bullet(*). \tag{8}$$

Theorem 2.6 ([Lan76])**.** *For every prime* p*, there is a sequence of elements* $v_1^{(p)}, \cdots \in \pi_\bullet MU$*, such that eq.* (8) *defines a cohomology theory if and only if* $(p, v_1^{(p)}, \cdots, v_n^{(p)})$ *is regular in* $\pi_\bullet E = E^\bullet(*)$ *for any* p *and any* n*.*

This condition becomes much clearer under the light of algebraic geometry: $\operatorname{Spec} L$ provides an affine atlas for the moduli stack $\mathscr{M}_{fg}^\heartsuit$ of formal groups (as they are locally given by formal group laws), and $L \to R$ is Landweber-exact if and only if $\operatorname{Spec} R \to \mathscr{M}_{fg}^\heartsuit$ is flat.

How do we now understand $L \simeq \pi_\bullet MU$ in terms of spectral geometry? Note that MU is even, so $\pi_\bullet MU = \pi_{2\bullet} MU$, and in turn

$$\pi_{2\bullet} MU \simeq \pi_0 \underbrace{\left(\sum_{n \in \mathbb{Z}} \Sigma^{2n} MU \right)}_{=: MP}, \tag{9}$$

where MP is then the 2-periodisation of MU, the universal 2-periodic complex-oriented spectrum.

Theorem 2.7 ([Gre21])**.** *The* $\mathscr{E}_\infty$*-ring spectrum* MP *provides an affine chart for the (non-connective) spectral moduli stack of oriented spectral formal groups (where an orientation on a formal group* G *over* R *is* $\omega_G \simeq \Sigma^{-2} R$ *in* $\mathfrak{Mod}_R$*).*

3 What happens

Some things remain essentially the same as in usual (derived) algebraic geometry: we can define a theory of étale morphisms of $\mathcal{E}_\infty$-rings, and spectral schemes and DM stacks, following the same general pattern.

In particular, we may define spectral schemes and Deligne–Mumford stacks in two ways, as explained in the following table (which is meant to be read clockwise, starting from the top left).

	Geometric picture	Functor of points
Spectral schemes	$(\lvert X \rvert, \mathcal{O}_X)$	$\hom(\mathrm{Spec}(-), X)$
Spectral DM stacks	$((\mathcal{A}\mathfrak{ff}_{/\mathcal{F}})_{\text{ét}}, \mathcal{O})$	$\mathcal{F}$

That is, a "geometric" spectral scheme given by a topological space $\lvert X \rvert$ with a structure sheaf of $\mathcal{E}_\infty$-rings can be seen as its functor of points through the Yoneda embedding, and a more general functor of points defining a spectral DM stack can be seen geometrically by passing to its étale topos with its canonical structure sheaf.

However, certain phenomena will defy our algebraic intuition: in particular, polynomial algebras are no longer free as ring spectra, so there are two reasonable definitions of the affine line: $\mathbb{A}^1_R = \mathrm{Spec}(R\{x\})$ where $R\{x\}$ means the free $\mathcal{E}_\infty$-algebra on one element over R, and $\mathbb{A}^1_{R,\flat} = \mathrm{Spec}(R[x])$. The first is differentially smooth but not flat over R, while the second is flat but not differentially smooth — indeed, differential smoothness is based on projectivity of the topological cotangent complex.

The difference between the two can be understood as a matter of strictness of the underlying additive $\mathcal{E}_\infty$-group: for any spectrum X, the ∞-groupoid $\Omega^\infty X$, as an infinite loop space, admits a structure of grouplike $\mathcal{E}_\infty$-monoid (and in fact this restricts to an equivalence between connective spectra and grouplike $\mathcal{E}_\infty$-monoids in ∞-groupoids). Thus, an $\mathcal{E}_\infty$-ring spectrum also has an "underlying weak (*i.e.* $\mathcal{E}_\infty$) abelian group", but for derived (aka animated) rings such as polynomial rings, this underlying weak group is *strictly* commutative.

We illustrate this idea in a final example.

Definition 3.1 (Hopkins)**.** *Let X be a spectrum. The type of* ***weak elements*** *of X is*

$$\Omega^\infty X = \hom_{\mathfrak{Sp}}(\mathbb{S}, X). \tag{10}$$

The type of ***strict elements*** *of X is* $\hom_{\mathfrak{Sp}}(\mathbb{Z}, X)$.

Remark 3.2. We have

$$\hom_{\mathfrak{Sp}}(\mathbb{S}, X) \simeq \hom_{\mathfrak{Sp}}(\Sigma^\infty *, X) \simeq \hom_{\infty\mathfrak{Grpd}}(*, \Omega^\infty X) \simeq \hom_{\mathcal{E}_1\mathfrak{Grp}}(\mathbb{Z}, \Omega^\infty X) \tag{11}$$

and (since $\mathbb{Z}$ is connective)

$$\hom_{\mathfrak{Sp}}(\mathbb{Z}, X) \simeq \hom_{\mathfrak{Sp}^{\text{connective}}}(\mathbb{Z}, \tau_{\geq 0} X) \simeq \hom_{\mathcal{E}_\infty\mathfrak{Grp}}(\mathbb{Z}, \Omega^\infty X), \tag{12}$$

where $\mathcal{E}_n\mathfrak{Grp}$ denotes the ∞-category of grouplike $\mathcal{E}_n$-monoids (in ∞-groupoids). We can thus see weak elements as "$\mathcal{E}_1$-elements" and strict elements as "$\mathcal{E}_\infty$-elements".

Definition 3.3 ([GN23])**.** *The type of* $\mathscr{E}_n$***-elements*** *of* X *is* $\hom_{\mathscr{E}_n\mathfrak{Grp}}(\mathbb{Z}, \Omega^\infty X)$.

Theorem 3.4 ([GN23])**.** *For any ring spectrum* R*, complex orientations on* R *are equivalent to* $\mathscr{E}_2$*-strictifications (*i.e. *lifts from an* $\mathscr{E}_1$*- to an* $\mathscr{E}_2$*-element) of the weak element* $1 \in \pi_0 R$.

References

[DGM13] Bjørn Ian Dundas, Thomas G. Goodwillie, Randy McCarthy, "The local structure of algebraic K-theory", DOI: 10.1007/978-1-4471-4393-2 (2013)

[Goo86] Thomas Goodwillie, "Relative algebraic K-theory and cyclic homology", DOI: 10.2307/1971283 (1986)

[Gre21] Rok Gregoric, "Moduli stack of oriented formal groups and periodic complex bordism", arχiv: 2107.08657 (2021)

[GN23] Doron Grossman-Naples, "Complex Orientations are Partial Strictifications of the Unit", arχiv: 2311.01663 (2023)

[Lan76] Peter S. Landweber, "Homological Properties of Comodules Over $MU_*(MU)$ and $BP_*(BP)$", DOI: 10.2307/2373808

[Laz55] Michel Lazard, " Sur les groupes de Lie formels à un paramètre", DOI: 10.24033/bsmf.1462 (1955)

[Mil60] John Milnor, "On the Cobordism Ring Ω^* and a Complex Analogue, Part I", DOI: 10.2307/2372970

[NS18] Thomas Nikolaus, Peter Scholze, "On topological cyclic homology", DOI: 10.4310/ACTA.2018.v221.n2.a1, arχiv: 1707.01799 (2018)

[Qui69] Daniel Quillen, "On the formal group laws of unoriented and complex cobordism theory", project Euclid: euclid.bams/1183530915 (1969)