# Categorified quasimap theory of derived Deligne–Mumford stacks

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# Plan

1. Derived thickenings and (relative) virtual classes
   - Derived moduli of quasimaps
   - Virtual classes and quantum Lefschetz
2. Constructing the Gromov–Witten action
   - Action by the operad of stacky curves
   - Brane actions for coloured $\infty$-operads

# Construction of Cohomological Field Theories

A CohFT is an algebra (in $\mathbb{Q}-\mathfrak{Mod}^{\heartsuit\,\mathrm{gr}}$) over the modular operad $(A_\bullet\overline{\mathscr{M}}_{g,n+1})_{g,n\in\mathbb{N}}$:

$$\Omega_{g,n}\colon A_\bullet\overline{\mathscr{M}}_{g,n+1}\otimes V^{\otimes n}\to V.$$

Problem: $X$ smooth $\Bbbk$-scheme. Quantise Frobenius algebra structure on $A^\bullet X$ to a CohFT.

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Idea:

$$\overline{\mathscr{M}}_{g,n+1}\xleftarrow{\ \mathrm{Stab}\ }\coprod_{\beta\in\mathrm{Eff}(X)}\overline{\mathscr{M}}_{g,n+1}(X,\beta)\xrightarrow{\ (\mathrm{ev}_1,\ldots,\mathrm{ev}_{n+1})\ }X^{n+1}$$

# Construction of Cohomological Field Theories

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Problem: $X$ smooth $\Bbbk$-scheme. Quantise Frobenius algebra structure on $A^\bullet X$ to a CohFT.

Idea:

$$\begin{array}{ccccc} & & \coprod_{\beta\in \mathrm{Eff}(X)} \overline{\mathscr{M}}_{g,n+1}(X,\beta) & & \\ & \overset{\mathrm{Stab}}{\swarrow} & & \overset{(\mathrm{ev}_1,\dots,\mathrm{ev}_{n+1})}{\searrow} & \\ \overline{\mathscr{M}}_{g,n+1} & & & & X^{n+1} \end{array}$$

$$A^\bullet \overline{\mathscr{M}}_{g,n+1} \xleftarrow{\mathrm{Stab}_*} A^\bullet\Big(\coprod\nolimits_\beta \overline{\mathscr{M}}_{g,n+1}(X,\beta)\Big) \xleftarrow{(\mathrm{ev}_1,\dots,\mathrm{ev}_{n+1})^*} (A^\bullet X)^{\otimes n+1}$$

$$\circlearrowleft \frown [\overline{\mathscr{M}}_{g,n+1}(X)]^{\mathrm{vir}}$$

# Moduli of stacky curves

$X$ a Deligne–Mumford $\Bbbk$-stack
$\rightsquigarrow$ Stab no longer proper unless $\overline{\mathscr{M}}_{g,n+1}(X,\beta)$ is replaced by maps from *stacky* curves.

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### Families of stacky curves (Abramovich, Graber and Vistoli 2008)

A (balanced) $n$-marked **stacky curve** over a base $S$ is a proper tame 1-dimensional flat family $C \to S$ such that $|C| \to S$ is a prestable curve with $n$ markings $p_i \colon S \to |C|$ and

1. at a marking, $C$ is of the form $\operatorname{Spec}\big(\Bbbk[x]\big)/\mu_r$
2. at a node, $C$ is of the form $\operatorname{Spec}\big(\Bbbk[x,y]/(xy)\big)/\mu_s$ with action $(x,y) \mapsto (\zeta \cdot x, \zeta^{-1} \cdot y)$

### Theorem (Olsson 2007)

The moduli stack $\mathfrak{M}_{g,n,(r_1,\ldots,r_n)}$ of genus-$g$ stacky curves with gerbes of orders $r_1,\ldots,r_n$ is a smooth Artin stack.

# A Gromov–Witten Geometric Field Theory

$\overline{\mathcal{Q}}^{\infty}_{g,n}(X,\beta)$ moduli stack of stable maps from stacky curves to $X$ with class $\beta$.

Key idea: Lift $\left[\overline{\mathcal{Q}}^{\infty}_{g,n}(X,\beta)\right]^{\mathrm{vir}}$ to derived thickening $\overline{\mathcal{Q}}^{\infty}_{g,n}(X,\beta) \hookrightarrow \mathbb{R}\overline{\mathcal{Q}}^{\infty}_{g,n}(X,\beta)$.

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**Theorem (Mann and Robalo 2018)**

If $X$ is a scheme, then $X$ carries a lax $(\overline{\mathcal{M}}_{0,n+1})_{n\in\mathbb{N}}$ in correspondences in derived stacks, with components

$$\begin{array}{ccccc} & & \coprod_{\beta\in\mathrm{Eff}(X)} \mathbb{R}\overline{\mathcal{Q}}^{\infty}_{0,n+1}(X,\beta) & & \\ & \swarrow^{(\mathrm{ev}_1,\ldots,\mathrm{ev}_n,\mathrm{Stab})} & & \searrow^{\mathrm{ev}_{n+1}} & \\ X^n\times\overline{\mathcal{M}}_{0,n+1} & & & & X. \end{array}$$

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Key idea: Lift $\left[\overline{\mathcal{Q}}^{\infty}_{g,n}(X,\beta)\right]^{\text{vir}}$ to derived thickening $\overline{\mathcal{Q}}^{\infty}_{g,n}(X,\beta) \hookrightarrow \mathbb{R}\overline{\mathcal{Q}}^{\infty}_{g,n}(X,\beta)$.

## Theorem (K. 2021, following Mann and Robalo 2018)

The "rigidified cyclotomic loop stack" of $X$ carries a lax $(\overline{\mathcal{M}}_{0,n+1})_{n\in\mathbb{N}}$ in correspondences in derived stacks, with components

$$
\begin{array}{ccc}
 & \coprod_{\beta\in\mathrm{Eff}(X)} \mathbb{R}\overline{\mathcal{Q}}^{\infty}_{0,n+1}(X,\beta) & \\
\swarrow^{(\mathrm{ev}_1,\ldots,\mathrm{ev}_n,\mathrm{Stab})} & & \searrow^{\mathrm{ev}_{n+1}} \\
(\overline{\mathscr{L}}_{\mu}X)^n \times \overline{\mathcal{M}}_{0,n+1} & & \overline{\mathscr{L}}_{\mu}X.
\end{array}
$$

# Application to the quantum Lefschetz principle

$E \to X$ vector bundle, $s\colon X \to E$ section, $Z = Z(s) := s^{-1}(0) \subset X$.

## Motivation (Classical Lefschetz)

$$
\begin{array}{ccc}
Z & \longrightarrow & X \\
\downarrow & \lrcorner & \downarrow \zeta \\
X & \underset{s}{\longrightarrow} & E
\end{array}
\quad \Longrightarrow \quad
\begin{cases}
[\mathcal{O}_Z] = [\mathcal{O}_X] \otimes \bigwedge^{\bullet} E \\
\mathcal{O}_Z = \mathcal{O}_X \otimes \mathscr{K}os(s)
\end{cases}
$$

# Application to the quantum Lefschetz principle

$E \to X$ vector bundle, $s \colon X \to E$ section, $Z = Z(s) := s^{-1}(0) \subset X$.

Goal: Describe Gromov–Witten invariants of $Z$ in function of those of $X$

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# Application to the quantum Lefschetz principle

$E \to X$ vector bundle, $s \colon X \to E$ section, $Z = Z(s) := s^{-1}(0) \subset X$.

Goal: Describe Gromov–Witten invariants of $Z$ in function of those of $X$

**Theorem (Quantum Lefschetz, Kim, Kresch and Pantev 2003, Joshua 2010)**

$$\begin{array}{ccc} \overline{\mathcal{Q}}^{\infty}_{g,n}(Z) & \longrightarrow & \overline{\mathcal{Q}}^{\infty}_{g,n}(X) \\ \downarrow & \lrcorner & \downarrow \zeta \\ \overline{\mathcal{Q}}^{\infty}_{g,n}(X) & \xrightarrow[\overline{\mathcal{Q}}^{\infty}_{g,n}(s)]{} & \overline{\mathcal{Q}}^{\infty}_{g,n}(E) \end{array} \quad + \quad \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{Q}}^{\infty}_{g,n}(Z)}\right] = \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{Q}}^{\infty}_{g,n}(X)}\right] \otimes \bigwedge\nolimits^{\bullet} \overline{\mathcal{Q}}^{\infty}_{g,n}(E)$$

Under assumptions: $g = 0$, $s$ regular, $E$ convex

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**Theorem (Geometric and categorified quantum Lefschetz, K. 2020)**

$$\begin{array}{ccc} \mathbb{R}\overline{\mathcal{Q}}^{\infty}_{g,n}(Z) & \longrightarrow & \mathbb{R}\overline{\mathcal{Q}}^{\infty}_{g,n}(X) \\ \downarrow & \lrcorner & \downarrow \zeta \\ \mathbb{R}\overline{\mathcal{Q}}^{\infty}_{g,n}(X) & \xrightarrow[\mathbb{R}\overline{\mathcal{Q}}^{\infty}_{g,n}(s)]{} & \mathbb{R}\overline{\mathcal{Q}}^{\infty}_{g,n}(E) \end{array} \implies \mathcal{O}_{\mathbb{R}\overline{\mathcal{Q}}^{\infty}_{g,n}(Z)} = \mathcal{O}_{\mathbb{R}\overline{\mathcal{Q}}^{\infty}_{g,n}(X)} \otimes \mathcal{K}os(\mathbb{R}\overline{\mathcal{Q}}^{\infty}_{g,n}(s))$$

Under assumptions: none

# Contents – Section 1: Derived thickenings and (relative) virtual classes

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# Stable loci of polarised stacks

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Purpose: Parameter for the stability condition

# Stable loci of polarised stacks

A rational **polarisation** of $X$ 1-Artin (quasiprojective) is $\mathcal{P} = \mathcal{P}_0 \otimes \varepsilon \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ ample

Purpose: Parameter for the stability condition

## Stability function (Halpern-Leistner 2018, Alper, Heinloth, ...)

- A filtered point of $X$ is a map $\Theta := \mathbb{A}^1/\mathbb{G}_m \xrightarrow{\lambda} X$; its underlying point is $\lambda(1)$
- $x$ point of $X$, $\lambda$ filtration on $x$:

$$\mu(\lambda) = -\mathrm{wt}(\lambda(0)^*\mathcal{P})$$

- $x$ is $\mathcal{P}$**-unstable** if it admits a filtration with positive weight.

$X^{\mathcal{P}\text{-st}} \simeq X^{\mathcal{P}_0\text{-st}}$ the locus of stable points (assumed 1-DM).

# Quasi-stability

Source: $(C, \Sigma_1, \ldots, \Sigma_n)$ a prestable stacky curve, irreducible components $C_i$

$f\colon C \to X$ representable is **pre-$\mathcal{P}$-quasistable** if it maps the generic points $\eta_i \in C_i$ to $X^{\mathcal{P}\text{-st}}$ and its basepoints are disjoint from the special points of $C$.

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**Definition (stable quasimaps to $X^{\mathcal{P}\text{-st}}$, Cheong, Ciocan-Fontanine and Kim 2015)**

A pre-$\mathcal{P}$-quasistable $C \to X$ is $\mathcal{P}$**-quasistable** if

1. $$\omega_{|C|}\left(\sum_{i=1}^{n} |\Sigma_i|\right) \otimes \left(f^*\mathcal{P}_0^e\right)^{\varepsilon/e}$$

   is ample ($e$ the least common multiple of $\mathrm{ord}(\mathcal{A}ut(x))$, $x \in X$)
2. $\forall x \in X, \varepsilon\ell(x) \leqslant 1$

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   is ample ($e$ the least common multiple of $\operatorname{ord}(\mathcal{A}ut(x))$, $x\in X$)
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If $\varepsilon>2$ (denoted $\varepsilon=\infty$): stable maps to $X^{\mathcal{P}\text{-st}}$

# Derived moduli stacks of quasimaps

$\overline{\mathbb{Q}}^{\mathcal{P}}_{g,n}(X,\beta) \overset{\text{op.}}{\subset} t_0\,\mathscr{M}or_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$ classical moduli stack of quasimaps

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**Proposition (Cheong, Ciocan-Fontanine and Kim 2015)**

If $X$ is 1-Artin and quasi-smooth, and $X^{\mathcal{P}\text{-st}}$ is 1-DM and smooth, $\mathbb{R}\overline{\mathbb{Q}}^{\mathcal{P}}_{g,n}(X,\beta)$ is a proper quasi-smooth derived 1-DM stack.

Remark: $\mathbb{L}_{\mathbb{R}\overline{\mathbb{Q}}^{\mathcal{P}}_{g,n}(X,\beta)} \simeq p_! \, \mathrm{ev}^* \mathbb{L}_X$
with

$$\mathbb{R}\overline{\mathbb{Q}}^{\mathcal{P}}_{g,n}(X,\beta) \xleftarrow{\;p\;} \mathbb{R}\overline{\mathscr{C}}^{\mathcal{P}}_{g,n}(X,\beta) \xrightarrow{\;\mathrm{ev}\;} X$$

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**Proposition (Cheong, Ciocan-Fontanine and Kim 2015)**

If $X$ is 1-Artin and quasi-smooth, and $X^{\mathcal{P}\text{-st}}$ is 1-DM and smooth, $\mathbb{R}\overline{\mathbb{Q}}^{\mathcal{P}}_{g,n}(X,\beta)$ is a proper quasi-smooth derived 1-DM stack.

Remark: $\mathbb{L}_{\mathbb{R}\overline{\mathbb{Q}}^{\mathcal{P}}_{g,n}(X,\beta)} \simeq p_! \operatorname{ev}^* \mathbb{L}_X$

**Evaluation maps**

$$\operatorname{ev}_i \colon \mathbb{R}\overline{\mathbb{Q}}^{\mathcal{P}}_{g,n}(X,\beta) \to \overline{\mathscr{L}}_{\mu} X^{\mathcal{P}\text{-st}} := \coprod_{r \in \mathbb{N}} \mathscr{M}or(\mathcal{B}\mu_r, X^{\mathcal{P}\text{-st}})/\mathcal{B}\mu_r$$
$$(C, \Sigma_1, \ldots, \Sigma_n; f) \mapsto f(\Sigma_i)$$

# Contents – Section 1: Derived thickenings and (relative) virtual classes

# Virtual pullbacks

**Lemma**

$\jmath_M\colon t_0 M \hookrightarrow M$ locally noetherian derived Artin thickening. Then $\jmath_{M,*}\colon G(t_0 M) \xrightarrow{\simeq} G(M)$.

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$f \colon M \to B$ quasi-smooth, so $f^* \colon \mathfrak{Coh}^{\mathrm{b}}(B) \to \mathfrak{Coh}^{\mathrm{b}}(M)$

**Definition**

$$f^{!,\mathrm{virt}} \colon G(t_0 B) \xrightarrow{\jmath_{B,*}} G(B) \xrightarrow{f^*} G(M) \xrightarrow{\jmath_{M,*}{}^{-1}} G(t_0 M)$$

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**Lemma (Schürg, Toën and Vezzosi 2015)**

$M$ quasi-smooth $\implies \mathbb{L}_M$ is a $[0,1]$-perfect obstruction theory on $t_0 M$.

**Proposition (Mann and Robalo 2018, K. 2020)**

$f^{!,\mathrm{virt}}$ coincides with the virtual pullback induced by the perfect obstruction theory $\mathbb{L}_f|_{t_0 M}$.

# The quantum Lefschetz principle

$E = \mathbb{V}_X(\mathcal{E}) \underset{\curvearrowleft s}{\longrightarrow} X$ vector bundle with section, $i\colon Z := Z(s) \hookrightarrow X$

$$(\mathbb{R})\overline{\mathscr{C}}^{\mathcal{P}}_{g,n}(X,\beta)$$

$$(\mathbb{R})\overline{\mathbb{Q}}^{\mathcal{P}}_{g,n}(X,\beta) \xleftarrow{\;p\;} (\mathbb{R})\overline{\mathscr{C}}^{\mathcal{P}}_{g,n}(X,\beta) \xrightarrow{\;\mathrm{ev}\;} X$$

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Transgressed derived bundle
$\mathfrak{E}_{g,n} := \mathbb{V}_{\overline{\mathbb{Q}}^{\mathcal{P}}_{g,n}(X,\beta)}(\mathfrak{p}_* \mathrm{ev}^* \mathcal{E})$

If $\pi_1(\mathfrak{p}^* \mathrm{ev}_* \mathcal{E}) = 0$ (so $\mathfrak{p}^* \mathrm{ev}_* \mathcal{E} \simeq \pi_0(\mathfrak{p}^* \mathrm{ev}_* \mathcal{E})$)

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Transgressed derived bundle
$\mathfrak{E}_{g,n} := \mathbb{V}_{\overline{\mathbb{Q}}^{\mathcal{P}}_{g,n}(X,\beta)}(p_* \mathrm{ev}^* \mathcal{E})$

**Theorem (Kim, Kresch and Pantev 2003, Joshua 2010)**

If $\pi_1(p^* \mathrm{ev}_* \mathcal{E}) = 0$ (so $p^* \mathrm{ev}_* \mathcal{E} \simeq \pi_0(p^* \mathrm{ev}_* \mathcal{E})$), then

$$\sum_{i_*\gamma=\beta} u_{\gamma,*}\left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathbb{Q}}^{\mathcal{P}}_{g,n}(Z,\gamma)}\right] = \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathbb{Q}}^{\mathcal{P}}_{g,n}(X,\beta)}\right] \otimes \bigwedge\nolimits^{\bullet} p_* \mathrm{ev}^* \mathcal{E} \qquad \in G(\overline{\mathbb{Q}}^{\mathcal{P}}_{g,n}(X,\beta))$$

# Categorification of the quantum Lefschetz principle

**Lemma**

Equivalence of bundles

$$\mathfrak{E}_{g,n} = \mathbb{R}\mathbb{Q}^{\mathcal{P}}_{g,n}(E, \beta)$$

Quasi-stable maps to $E$ projecting to $\beta$

# Categorification of the quantum Lefschetz principle

**Lemma**

Equivalence of bundles

$$\mathfrak{E}_{g,n} = \mathbb{RQ}^{\mathcal{P}}_{g,n}(E,\beta)$$

Quasi-stable maps to $E$ projecting to $\beta$

**Theorem**

$$\begin{array}{ccc}
\coprod\limits_{i_*\gamma=\beta} \mathbb{RQ}^{i^*\mathcal{P}}_{g,n}(Z,\gamma) & \xrightarrow{(u_\gamma)_\gamma} & \mathbb{RQ}^{\mathcal{P}}_{g,n}(X,\beta) \\
\downarrow & \lrcorner & \downarrow \mathfrak{s} \\
\mathbb{RQ}^{\mathcal{P}}_{g,n}(X,\beta) & \xrightarrow{\zeta} & \mathfrak{E}_{g,n}
\end{array}$$

**Corollary**

$$\bigoplus_{i_*\gamma=\beta} u_{\gamma,*}\mathcal{O}_{\mathbb{RQ}^{i^*\mathcal{P}}_{g,n}(Z,\gamma)} \simeq \mathscr{Kos}(\mathfrak{s}) := \Big( \underbrace{\mathcal{S}ym\,\mathrm{cofib}(\mathfrak{s})}_{\mathcal{O}_{\mathbb{RQ}^{\mathcal{P}}_{g,n}(X,\beta)}[t]\text{-mod.}} \Big)/(t-1) \qquad \text{in } \mathfrak{QCoh}(\mathbb{RQ}^{\mathcal{P}}_{g,n}(X,\beta))$$

# Categorification of the quantum Lefschetz principle

## Lemma

Equivalence of bundles

$$\mathfrak{E}_{g,n} = \mathbb{R}\mathbb{Q}^{\mathcal{P}}_{g,n}(E,\beta)$$

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## Theorem

$$\begin{array}{ccc} \coprod_{i_*\gamma=\beta} \mathbb{R}\mathbb{Q}^{i^*\mathcal{P}}_{g,n}(Z,\gamma) & \xrightarrow{(u_\gamma)_\gamma} & \mathbb{R}\mathbb{Q}^{\mathcal{P}}_{g,n}(X,\beta) \\ \downarrow & \lrcorner & \downarrow \mathfrak{s} \\ \mathbb{R}\mathbb{Q}^{\mathcal{P}}_{g,n}(X,\beta) & \xrightarrow{\zeta} & \mathfrak{E}_{g,n} \end{array}$$

## Corollary

$$\bigoplus_{i_*\gamma=\beta} u_{\gamma,*}\mathcal{O}_{\mathbb{R}\mathbb{Q}^{i^*\mathcal{P}}_{g,n}(Z,\gamma)} \simeq \mathscr{K}os(\mathfrak{s}) := \Big( \underbrace{\mathscr{S}ym\,\mathrm{cofib}(\mathfrak{s})}_{\mathcal{O}_{\mathbb{R}\mathbb{Q}^{\mathcal{P}}_{g,n}(X,\beta)}[t]\text{-mod.}} \Big)/(t-1) \qquad \text{in } \mathfrak{QCoh}(\mathbb{R}\mathbb{Q}^{\mathcal{P}}_{g,n}(X,\beta))$$

Remark: Gives a class in $\mathfrak{Coh}^b$ iff $u_\gamma$ quasi-smooth: classical assumptions.

# Contents - Section 2: Constructing the Gromov–Witten action

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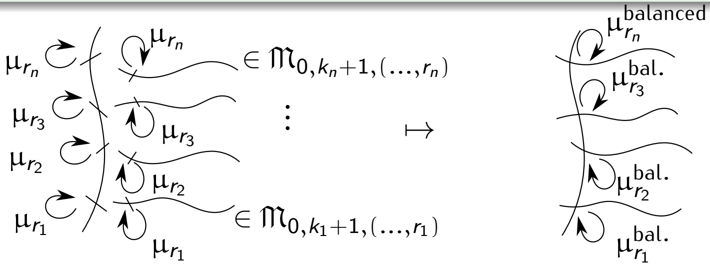
# Gluing of curves

## Gluing curves along marked gerbes(Abramovich, Graber and Vistoli 2008)

There are representable gluing maps

$$\mathfrak{M}_{g,n+1,(r_1,\dots,r_n,s)} \underset{\mathcal{B}^2\mu_s}{\times} \mathfrak{M}_{h,p+1,(\bar{s},t_1,\dots,t_p)} \to \mathfrak{M}_{g+h,n+p,(r_1,\dots,r_n,t_1,\dots,t_p)}$$

$\mathfrak{M}_{0,n,\mathbf{r}} \ni$




$\in \mathfrak{M}_{0,\sum_i k_i,(\dots)}$

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There are representable gluing maps

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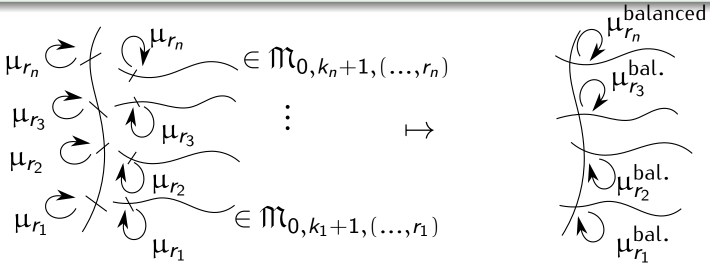


$\in \mathfrak{M}_{0,\sum_i k_i,(\dots)}$

$\implies$ $\mathcal{B}^2\mu$-Coloured modular operad $\mathfrak{M}$ (and operad $\mathfrak{M}_0$), where $\mathcal{B}^2\mu = \coprod_{r\in\mathbb{N}+1} \mathcal{B}^2\mu_r$

# Universal curve and unitality

**Lemma (Costello 2006, Abramovich, Graber and Vistoli 2008)**

The universal curve $\mathfrak{C}_{g,n,(r_1,\ldots,r_n)} \to \mathfrak{M}_{g,n,(r_1,\ldots,r_n)}$ is $\mathrm{frgt}_{n+1}\colon \mathfrak{M}_{g,n+1,(r_1,\ldots,r_n,1)} \to \mathfrak{M}_{g,n,(r_1,\ldots,r_n)}$

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**Operadic interpretation**

$\mathrm{Mult}_{\mathfrak{M}_0}(\emptyset; r) = \mathcal{B}^2 \mu_r$: the nullary morphism has $\mathcal{B}\mu_r$ automorphisms

$\implies$ Not unital, except for the colour 1

We say the pointed operad $(\mathfrak{M}_0, 1)$ is hapaxunital

# Universal curve and unitality

**Lemma (Costello 2006, Abramovich, Graber and Vistoli 2008)**

The universal curve $\mathfrak{C}_{g,n,(r_1,\ldots,r_n)} \to \mathfrak{M}_{g,n,(r_1,\ldots,r_n)}$ is $\mathrm{frgt}_{n+1} \colon \mathfrak{M}_{g,n+1,(r_1,\ldots,r_n,1)} \to \mathfrak{M}_{g,n,(r_1,\ldots,r_n)}$

**Operadic interpretation**

$\mathrm{Mult}_{\mathfrak{M}_0}(\emptyset; r) = \mathcal{B}^2 \mu_r$: the nullary morphism has $\mathcal{B}\mu_r$ automorphisms

$\Longrightarrow$ Not unital, except for the colour 1

We say the pointed operad $(\mathfrak{M}_0, 1)$ is hapaxunital

$\rightsquigarrow$ Correct extensions $\mathrm{Ext}(C) = \mathfrak{M}_{0,n+1,(r_1,\ldots,r_n,1)} \underset{\mathfrak{M}_{0,n,(r_1,\ldots,r_n)}}{\times} \{C\}$

# Existence of the algebra structure

**Theorem (Mann and Robalo 2018, K. 2021)**

There is a lax morphism of $(\infty, 2)$-operads (internal to $\mathfrak{dSt}$)

$$\begin{aligned} \mathfrak{M}_0 &\xrightarrow{\mathscr{B}_{\mathfrak{M}}} \mathscr{Cospan}(\mathfrak{dSt}_{/\_})^{\amalg} \\ r &\mapsto \mathrm{Ext}(\mathrm{id}_r) \simeq \mathcal{B}\,\mu_r \end{aligned}$$

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**Corollary**

There is a lax morphism of $(\infty, 2)$-operads in $\mathfrak{dSt}$

$$\begin{gathered}\overline{\mathscr{M}}_0 \xrightarrow{\mathscr{GW}} \mathscr{Span}(\mathfrak{dSt}_{/\_})^{\times} \\ * \mapsto \overline{\mathscr{L}}_{\mu} X^{\mathcal{P}\text{-st}}\end{gathered}$$

# Contents - Section 2: Constructing the Gromov–Witten action

# Brane actions

**Theorem (Toën 2013, Mann and Robalo 2018, K. 2021)**

Let $(\mathfrak{P}, P_0)$ be a hapaxunital $\infty$-operad in an $(\infty,1)$-topos $\mathfrak{T}$. There is a lax morphism of $(\infty,2)$-operads in $\mathfrak{T}$

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$$\mathfrak{P} \to \mathscr{Cospan}(\mathfrak{T}_{/-})^{\amalg}$$
$$P \mapsto \mathrm{Ext}(\mathrm{id}_P)$$

An operation $\sigma \in \mathrm{Mult}_{\mathfrak{P}}(P_1, \ldots, P_n; P_{n+1})$ acts by the cospan

$$\coprod_{i=1}^{n} \mathrm{Ext}(\mathrm{id}_{P_i})^n \xrightarrow{\sigma \underset{i}{\circ} -} \mathrm{Ext}(\sigma) \xleftarrow{- \underset{1}{\circ} \sigma} \mathrm{Ext}(\mathrm{id}_{P_{n+1}})$$

# Plan of construction

By descent: construct for $\mathfrak{T} = \infty\!-\!\mathfrak{Grpd}$

1. $\mathfrak{P} \to \mathscr{Cospan}(\infty\!-\!\mathfrak{Grpd})^{\amalg}$ of $\infty$-operads
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# Proof of the corollary

$$\begin{array}{ccc} \mathfrak{M}_0 & \xrightarrow{\mathscr{B}_{\mathfrak{M},X}} & \mathcal{Span}(\mathfrak{dSt}_{/\_})^{\times} \\ {\scriptstyle \mathrm{Stab}}\downarrow & \nearrow_{\mathscr{GW}} & \\ \overline{\mathscr{M}}_0 & & \end{array}$$

Construct $\mathscr{GW}$ as oplax extension:
$\mathscr{GW} = \mathrm{Opex}_{\mathrm{Stab}} \mathscr{B}_{\mathfrak{M},X}$

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$$\mathscr{GW}(*) = \varinjlim_{\mathrm{Stab}(r) \to *} \mathscr{B}_{\mathfrak{M},X}(r)$$

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□

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- $X$ a 1-algebraic stack $\implies \overline{\mathscr{M}}_0$-action on $\overline{\mathscr{L}}_\mu X$ by $\overline{\mathbb{Q}}_{g,n}(X)$
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