Enrichments for higher categories

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Sommaire - Section 1: The algebraic structure of free objects

The free category functor

Let $Q = (\{\text{vertices}\}, \{\text{arrows}\})$ be a quiver. What is the "best" way to turn it into a category?

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  objects the vertices of Q

  morphisms $\hom_{\mathcal{F}Q}(V_0, V_1) = \{\text{strings of composable arrows joining } V_0 \text{ to } V_1\}$

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Adjunction property

Let $\mathfrak{C}$ be a category. Any functor $\mathcal{F}Q \to \mathfrak{C}$ is determined by its action on Q, and any graph morphism $Q \to \mathfrak{C}$ extends uniquely (by concatenation):

$$\hom_{\mathfrak{Cat}}(\mathcal{F}Q, \mathfrak{C}) \simeq \hom_{\mathfrak{Quiver}}(Q, \mathcal{U}\mathfrak{C}) \qquad \text{(with } \mathcal{U} \text{ the "underlying quiver" functor)}$$

Free objects in general

Let $\mathcal{U}\colon \mathfrak{D} \to \mathfrak{C}$ be a "forgetful" functor.
For any object $X \in \mathfrak{C}$, the *free object of* $\mathfrak{D}$ on X is $\mathcal{F}X \in \mathfrak{D}$ with $\eta_X\colon X \to \mathcal{U}\mathcal{F}X$ (in $\mathfrak{C}$) such that:

for any $Y \in \mathfrak{D}$ and any morphism $f\colon X \to \mathcal{U}Y$ in $\mathfrak{C}$, there is a unique $\overline{f}\colon \mathcal{F}X \to Y$ (in $\mathfrak{D}$) making the diagram commute (in $\mathfrak{C}$)

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & \mathcal{U}\mathcal{F}X \arrow[d, dashed, "\mathcal{U}\overline{f}"] \\ & \mathcal{U}Y \end{tikzcd}$$

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Examples

- $\mathcal{U}\colon \mathfrak{Vect}_k \to \mathfrak{Set}$ the forgetful "underlying set" functor;
  the free vector space $k[S]$ on a set S is the k-linear span of its elements
- $\mathcal{U}\colon \mathfrak{CAlg}_k \to \mathfrak{Vect}_k$; the free commutative k-algebra on V is the symmetric tensor algebra: $\mathcal{U}\left(\mathrm{Sym}^\bullet V\right) \cong k \oplus V \oplus (V^{\otimes 2})_{\mathbb{S}_2} \oplus (V^{\otimes 3})_{\mathbb{S}_3} \oplus \cdots$

Remark: $\mathcal{F}X$ is functorial in X, giving $\mathcal{F}\colon \mathfrak{C} \to \mathfrak{D}$

Adjoint functors

Definition

An *adjunction* $\mathcal{F} \dashv \mathcal{G}$ between categories $\mathfrak{C}$ and $\mathfrak{D}$ is $\mathcal{F}\colon \mathfrak{C} \to \mathfrak{D}$ and $\mathcal{G}\colon \mathfrak{D} \to \mathfrak{C}$ with bijections

$$\phi_{X,Y}\colon \hom_{\mathfrak{D}}(\mathcal{F}X, Y) \simeq \hom_{\mathfrak{C}}(X, \mathcal{G}Y)$$

natural in $X \in \mathfrak{C}$ and $Y \in \mathfrak{D}$.

$\mathcal{U}\colon \mathfrak{D} \to \mathfrak{C}$ forgetful functor

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & \mathcal{U}\mathcal{F}X \arrow[d, dashed, "\exists!\,\mathcal{U}\bar{f}"] \\ & \mathcal{U}Y \end{tikzcd}$$

$$\phi_{X,Y}\colon \hom(\mathcal{F}X, Y) \ni g \mapsto \mathcal{U}g \circ \eta_X \in \hom(X, \mathcal{U}Y)$$

bijective by free property

$$\Longrightarrow \mathcal{F}\colon \begin{tikzcd} \mathfrak{C} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathfrak{D} \arrow[l, bend left] \end{tikzcd} :\mathcal{U}$$

The algebra of adjunctions

$\phi_{X,\mathcal{F}X}\colon \hom_{\mathfrak{D}}(\mathcal{F}X, \mathcal{F}X) \xrightarrow{\simeq} \hom_{\mathfrak{C}}(X, \mathcal{G}\mathcal{F}X)$ maps $\mathrm{id}_{\mathcal{F}X} \mapsto \left(\eta_X\colon X \to \mathcal{G}\mathcal{F}X\right)$

- By functoriality of $\phi \implies$ natural transformation $\eta\colon \mathrm{id}_{\mathfrak{C}} \Rightarrow \mathcal{G}\mathcal{F}$

Similarly, by $\phi^{-1}_{\mathcal{G}Y,Y}\colon \hom(\mathcal{G}Y, \mathcal{G}Y) \xrightarrow{\simeq} \hom(\mathcal{F}\mathcal{G}Y, Y)$, get $\varepsilon\colon \mathcal{F}\mathcal{G} \Rightarrow \mathrm{id}_{\mathfrak{D}}$

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Notation: η and ε are called the *unit* and *counit* of the adjunction

ε and η determine the adjunction ϕ, that is:

$\mathcal{F}$ and $\mathcal{G}$ are adjoint iff there are $\varepsilon \colon \mathcal{F}\mathcal{G} \Rightarrow \mathrm{id}_{\mathfrak{D}}$ and $\eta \colon \mathrm{id}_{\mathfrak{C}} \Rightarrow \mathcal{G}\mathcal{F}$ with the compatibility condition

$$\begin{tikzcd}[column sep=huge] \mathfrak{C} \arrow[r, bend left=60, "\mathcal{F}"] \arrow[r, "\mathcal{F}\mathcal{G}\mathcal{F}" description] \arrow[r, bend right=60, "\mathcal{F}"'] \arrow[r, phantom, shift left=5, "\Downarrow \mathcal{F}\eta"] \arrow[r, phantom, shift right=5, "\Downarrow \varepsilon\mathcal{F}"] & \mathfrak{D} \end{tikzcd} = \begin{tikzcd} \mathfrak{C} \arrow[r, bend left=50, "\mathcal{F}"] \arrow[r, bend right=50, "\mathcal{F}"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_{\mathcal{F}}"] & \mathfrak{D} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=huge] \mathfrak{D} \arrow[r, bend left=60, "\mathcal{G}"] \arrow[r, "\mathcal{G}\mathcal{F}\mathcal{G}" description] \arrow[r, bend right=60, "\mathcal{G}"'] \arrow[r, phantom, shift left=5, "\Downarrow \eta\mathcal{G}"] \arrow[r, phantom, shift right=5, "\Downarrow \mathcal{G}\varepsilon"] & \mathfrak{C} \end{tikzcd} = \begin{tikzcd} \mathfrak{D} \arrow[r, bend left=50, "\mathcal{G}"] \arrow[r, bend right=50, "\mathcal{G}"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_{\mathcal{G}}"] & \mathfrak{C} \end{tikzcd}$$

From adjunctions to monads

Using the counit, we get a natural transformation

$$\mu \coloneqq \mathcal{G}\varepsilon\mathcal{F} \colon (\mathcal{G}\mathcal{F}) \circ (\mathcal{G}\mathcal{F}) = \mathcal{G} \circ (\mathcal{F}\mathcal{G}) \circ \mathcal{F} \Rightarrow \mathcal{G}\mathcal{F}.$$

- The endofunctor $\mathcal{G}\mathcal{F}$ on $\mathfrak{C}$ has a "multiplication" $\mu\colon (\mathcal{G}\mathcal{F})^{\circ 2} \Rightarrow \mathcal{G}\mathcal{F}$ and a "unit" $\eta\colon \mathrm{id}_{\mathfrak{C}} \Rightarrow \mathcal{G}\mathcal{F}$ (associative and unital by compatibility condition)

Example: $\mathcal{U}\colon \mathfrak{D} \to \mathfrak{C}$ forgetful functor

$\mathcal{U}\mathcal{F}$ creates a free object and forgets the added structure
μ remembers reduction along this structure (*e.g.* $\mathcal{F} = k[-]$; μ is linear dependence)

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Similarly, $\mathcal{F}\eta\mathcal{G}$ is a comultiplication $\mathcal{F}\mathcal{G} \Rightarrow (\mathcal{F}\mathcal{G})^{\circ 2}$ on the counital $\mathcal{F}\mathcal{G}$ on $\mathfrak{D}$

$\Longrightarrow$ $\mathcal{G}\mathcal{F}$ and $\mathcal{F}\mathcal{G}$ are an algebra and a cogebra for the composition products on $\mathfrak{EndoFun}(\mathfrak{C})$ and $\mathfrak{EndoFun}(\mathfrak{D})$

Sommaire - Section 2: Monoids and enrichments

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Monoidal categories

Definition

A monoidal structure on a category $\mathfrak{V}$ is a bifunctor $\otimes \colon \mathfrak{V} \times \mathfrak{V} \to \mathfrak{V}$ and a choice of object $1 \in \mathfrak{V}$, with natural transformations expressing associativity and unitality

- A *braiding* is a natural isomorphism $\tau_{X,Y} \colon X \otimes Y \xrightarrow{\simeq} Y \otimes X$
- *Symmetric* structure if $\tau_{X,Y}^{-1} = \tau_{Y,X}$

A *monoidal functor* $(\mathfrak{V}, \otimes, 1) \to (\mathfrak{N}, \boxtimes, I)$ is $\mathcal{F} \colon \mathfrak{V} \to \mathfrak{N}$ with $\mathcal{F}(X \otimes Y) \xrightarrow{\simeq} \mathcal{F}X \boxtimes \mathcal{F}Y$ (natural) and $\mathcal{F}(1) \xrightarrow{\simeq} I$

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Examples

- $(\mathfrak{Mod}_R, \otimes_R, R)$

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- $(\mathfrak{EndoFun}(\mathfrak{C}), \circ, \mathrm{id}_{\mathfrak{C}})$ is *strict* monoidal (strictly associative)

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Examples

- $(\mathfrak{Mod}_R,\otimes_R,R)$
- $(\mathfrak{EndoFun}(\mathfrak{C}),\circ,\mathrm{id}_{\mathfrak{C}})$ is *strict* monoidal (strictly associative)
- Any category with finite products (1 is the final object)

Enriched categories

Definition

A $(\mathfrak{V}, \otimes, 1)$-enriched category $\mathfrak{C}$ has a set of objects $\mathrm{Ob}(\mathfrak{C})$ and, for any pair (A, B) of objects, an object $\underline{\mathrm{hom}}(A, B)$ of $\mathfrak{V}$, along with $\underline{\mathrm{hom}}(A, B) \otimes \underline{\mathrm{hom}}(B, C) \longrightarrow \underline{\mathrm{hom}}(A, C)$, "$(f, g) \mapsto g \circ f$" and $\mathrm{id}_A \colon 1 \longrightarrow \underline{\mathrm{hom}}(A, A)$, all following the category axioms.

A $\mathfrak{V}$-functor $\mathcal{F} \colon \mathfrak{C} \longrightarrow \mathfrak{D}$ is a map of sets $\mathrm{Ob}(\mathfrak{C}) \longrightarrow \mathrm{Ob}(\mathfrak{D})$ and morphisms $\underline{\mathrm{hom}}_{\mathfrak{C}}(A, B) \longrightarrow \underline{\mathrm{hom}}_{\mathfrak{D}}(\mathcal{F}A, \mathcal{F}B)$ in $\mathfrak{V}$ with functoriality.

Examples

- A category is a $(\mathfrak{Set}, \times)$-enriched category
- A $\mathfrak{Mod}_R$-enriched category is called an R-linear category.
  An R-algebra is an R-category with one object

Closure and self-enrichment

A *closed symmetric monoidal category* is $(\mathfrak{V}, \otimes, 1)$ such that, $\forall Y \in \mathfrak{V}$, the (endo)functor $- \otimes Y$ has a right adjoint $[Y, -]$:

$$\hom(X \otimes Y, Z) = \hom(X, [Y, Z])$$

Notation: the object $[Y, Z]$ is called the *internal hom* from Y to Z

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Proposition

A closed monoidal structure gives a self-enrichment of $\mathfrak{V}$ with $\underline{\hom}(-,-) = [-,-]$

- $(\mathfrak{Mod}_R, \otimes_R, R)$, and the dg-derived category $(\mathrm{Ho}(\mathfrak{dgMod}_R), \otimes_R^{\mathbb{L}}, R[0])$
- $(\mathfrak{Top}, \times, \mathrm{pt})$, and $\mathrm{Ho}(\mathfrak{Top})$
- $(\mathfrak{Cat}, \times, \mathrm{pt})$ cartesian closed $\implies$ $\mathfrak{Cat}$-enriched category, *i.e.* (strict) 2-category

Sommaire - Section 2: Monoids and enrichments

Monoids in a monoidal category

Definition

A *monoid* in $(\mathfrak{V}, \otimes, 1)$ is $M \in \mathfrak{V}$ with $\mu\colon M \otimes M \to M$ and a generalised element $\eta\colon 1 \to M$, satisfying associativity and unitality

$$\begin{tikzcd} M \otimes M \otimes M \arrow[r, "\mu \otimes \mathrm{id}_M"] \arrow[d, "\mathrm{id}_M \otimes \mu"'] & M \otimes M \arrow[d, "\mu"] \\ M \otimes M \arrow[r, "\mu"'] & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \otimes 1 \arrow[r, "\mathrm{id}_M \otimes \eta"] \arrow[dr, "\simeq"'] & M \otimes M \arrow[d, "\mu"] & 1 \otimes M \arrow[l, "\eta \otimes \mathrm{id}_M"'] \arrow[dl, "\simeq"] \\ & M & \end{tikzcd}$$

Comonoid = monoid in $\mathfrak{V}^{\mathrm{op}}$: comultiplication $\delta\colon M \to M \otimes M$, counit $\varepsilon\colon M \to 1$, coassociative and counital

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Examples

- Monoids in $(\mathfrak{Set}, \times)$ are monoids. Monoids in monoids are commutative monoids.
- Monoids in $\mathfrak{Mod}_R$ are R-algebras. (Commutative) monoids in $\mathfrak{dgMod}_R$ are R-(c)dgas
- Monoids in $(\mathfrak{EndoFun}(\mathfrak{C}), \circ, \mathrm{id}_{\mathfrak{C}})$ are monads on $\mathfrak{C}$ (comonoids are comonads)

Remark: Define similarly modules and algebras over a monoid

Eilenberg–Moore category

An *algebra* over $(\mathcal{T}, \mu, \eta)$ is a functor $A\colon \mathrm{pt} \to \mathfrak{C}$ (equivalently, $A \in \mathfrak{C}$) with $\rho\colon \mathcal{T}A \to A$ such that

$$
\begin{array}{ccc}
\mathcal{T}\mathcal{T}A & \xrightarrow{\mu_A} & \mathcal{T}A \\
{\scriptstyle \mathcal{T}\rho}\downarrow & & \downarrow{\scriptstyle \rho} \\
\mathcal{T}A & \xrightarrow[\rho]{} & A
\end{array}
\qquad \text{and} \qquad
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, equal, "\mathrm{id}_A"'] & \mathcal{T}A \arrow[d, "\rho"] \\
& A
\end{tikzcd}
\qquad \text{commute.}
$$

A morphism of $\mathcal{T}$-algebras $(A, \rho) \to (A', \rho')$ is $f\colon A \to A'$ compatible with ρ, ρ'

Monads and adjunctions

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The forgetful functor $\mathfrak{Alg}_{\mathcal{T}} \to \mathfrak{C}, (A, \rho) \mapsto A$ has a "free" left adjoint $C \mapsto (\mathcal{T}C, \mu_C)$. The monad for the adjunction is (universally) $\mathcal{T}$.

Remark: An adjunction $\mathcal{F}\colon \begin{tikzcd} \mathfrak{C} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathfrak{D} \arrow[l, bend left] \end{tikzcd} :\mathcal{G}$ is said *monadic* if $\mathfrak{Alg}_{\mathcal{G}\mathcal{F}} \simeq \mathfrak{D}$

$\Longrightarrow$ $\mathfrak{Alg}_{\mathcal{G}\mathcal{F}}$ measures how free $\mathcal{F}$ is

Example: the ultrafilter monad I

Ultrafilters form a monad

$\beta\colon \mathfrak{Set} \to \mathfrak{Set}, X \mapsto \{\text{ultrafilters on } X\} \simeq \hom_{\mathfrak{Bool}}(\mathcal{P}(X), 2)$

- $\eta_X(x)$ is the principal ultrafilter $\varpi_x = \{U \subset X \mid x \in U\}$
- $\mu_X\colon \beta\beta X \ni F \mapsto \big\{U \subset X \mid \{G \in \beta X \mid U \in G\} \in F\big\} \in \beta X$

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Ultrafilters form a monad

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Algebras over β

Let $\rho\colon \beta X \to X$ be a β-algebra. Topologise X with $U \subset X$ open if

$$\forall x \in U, \forall F \in \beta X, \rho(F) = x \implies U \in F$$

- The space X is a compactum (quasicompact–Hausdorff)

Example: the ultrafilter monad II

Properties (X a topological space)

A **net** in X is $\nu\colon I \to X$, I directed set. Converges to x if $\forall U \ni x, \exists i, \forall j \geq i, \nu_j \in U$

- X is quasicompact iff every net has a convergent subnet
- X is Hausdorff iff no net can converge to two distinct limits

For X a compactum, define β-algebra structure $\rho\colon \beta X \to X$ by

$$\rho(F) = x \text{ unique limit point of } F \text{ (\textit{i.e.} filter } \{\text{neighbourhoods of } x\} \subset F)$$

In fact $\mathfrak{Alg}_\beta \simeq \mathfrak{CHaus}$

Remarks (β regularises infinite sets)

- On a finite set every ultrafilter is principal, so the free β-algebra $\beta[n] \xleftarrow{\simeq} [n]\colon \eta_{[n]}$
- For any set X, βX is the (Stone–Čech) compactification of $(X, \tau_{\text{discrete}})$

Sommaire – Section 3: Categories up to homotopy

The augmented simplex category

Category Δ_+ with

- **objects** $[n] = \{0 \leq 1 \leq \cdots \leq n\}$ (seen as a category), with $[-1] = \emptyset$
- **arrows** order-preserving, *i.e.* non-decreasing, functions (or simply functors)
  Note $[-1]$ is initial

Also (non augmented) simplex catgory Δ without $[-1]$

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Monoidal structure: $[n] \boxplus [m] := [n+m+1]$ (strict, but not symmetric!)
$[0]$ is a monoid: $[0] \boxplus [0] = [1] \xrightarrow{0,1 \mapsto 0} [0]$

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Universal property: cobar construction

Let $(\mathfrak{C}, \otimes)$ be a strict monoidal category with a monoid M. There is a unique monoidal functor $\mathcal{B}_{\mathfrak{C}}(M)_\bullet \colon \Delta_+ \to \mathfrak{C}$ sending $[0]$ to M (and generally $[n]$ to $M^{\otimes(n+1)}$)

(Co)simplicial objects

Face maps $\delta_i \colon [n] \rightarrowtail [n+1], \{0 \leq \cdots \leq n\} \mapsto \{0 \leq \cdots \leq i-1 \leq i+1 \leq \cdots \leq n+1\}$

Degeneracy maps $\sigma_j \colon [n] \twoheadrightarrow [n-1], \{0 \leq \cdots \leq n\} \mapsto \{0 \leq \cdots \leq j \leq j \leq \cdots \leq n-1\}$

Every morphism in Δ factors uniquely as degeneracies followed by faces

An augmented (co)simplicial object in $\mathfrak{C}$ is a functor $X_\bullet \colon {\Delta_+}^{\mathrm{op}} \to \mathfrak{C}$ (or $\Delta_+ \to \mathfrak{C}$):

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Given by diagram of $X_n = X_\bullet([n])$ with faces $d^n = X_\bullet(\delta_n)$ (resp. cofaces d_n) and degeneracies $s^n = X_\bullet(\sigma_n)$ (resp. codegeneracies s_n)

Cobar resolution

$$
\mathcal{B}_{\mathfrak{C}}(M)_\bullet(\Delta) = \begin{tikzcd}[column sep=large]
1 \arrow[r, "\eta"] & M \arrow[r, shift left=2, "\mathrm{id}\otimes\eta"] \arrow[r, shift right=2, "\eta\otimes\mathrm{id}"'] & M^{\otimes 2} \arrow[l, "\mu"'] \arrow[r, shift left=4, "\mathrm{id}\otimes\mathrm{id}\otimes\eta"] \arrow[r, "\mathrm{id}\otimes\eta\otimes\mathrm{id}"] \arrow[r, shift right=4, "\eta\otimes\mathrm{id}\otimes\mathrm{id}"'] & \cdots \arrow[l, shift right=2, "\mathrm{id}\otimes\mu"'] \arrow[l, shift left=2, "\mu\otimes\mathrm{id}"]
\end{tikzcd}
$$

Simplicial categories and simplicial enrichments

- $\mathfrak{C}$ a simplicially enriched category.
  $\implies$ Define simplicial object $\mathfrak{C}_\bullet \colon \Delta^{\mathrm{op}} \to \mathfrak{Cat}$ with $\mathrm{Ob}(\mathfrak{C}_i) = \mathrm{Ob}(\mathfrak{C})$ and $\hom_{\mathfrak{C}_i}(X, Y) = \underline{\hom}_{\mathfrak{C}}(X, Y)_i$
- Conversely, $\Delta^{\mathrm{op}} \to \mathfrak{Cat}$ with constant objects gives an $\mathfrak{sSet}$-category

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The nerve problem

$\mathfrak{C}$ a category. Its nerve $N_\bullet(\mathfrak{C}) \in \mathfrak{sSet}$ has $N_i(\mathfrak{C}) = \hom_{\mathfrak{Cat}}([i], \mathfrak{C})$: strings of i composable morphisms. It is 2-coskeletal: determined by $N_{\leq 1}$ (as $\mathfrak{Qvr}^{\text{unit.}} = \mathfrak{s}_{\leq 1}\mathfrak{Set}$)

E.g.: M a monoid, $\mathbb{B}M$ its 1-object category. Then $N_\bullet(\mathbb{B}M) = \mathcal{B}^{\mathfrak{Set}}(M)_\bullet$ is the bar resolution

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The left adjoint of $N_\bullet \colon \mathfrak{Cat} \to \mathfrak{sSet}$ ("fundamental category") only depends on 0-, 1- and 2-simplices

$\implies$ How to define the nerve of an $\mathfrak{sSet}$-category, or the "correct" fundamental category of a simplicial set?

Homotopy coherent diagrams

Back to the free category construction $\mathcal{F}$: $\begin{tikzcd} \mathfrak{Qvr} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathfrak{Cat} \arrow[l, bend left] \end{tikzcd} : \mathcal{U}$

- Comonad $\mathcal{FU}$ on $\mathfrak{Cat}$, with bar resolution $\mathcal{B}^{\mathfrak{EndoFun}(\mathfrak{Cat})}(\mathcal{FU})_\bullet \colon \Delta^{\mathrm{op}} \to \mathfrak{EndoFun}(\mathfrak{Cat})$
- Both $\mathcal{U}$ and $\mathcal{F}$ are identity on objects $\implies$ simplicial categories $\mathfrak{C}_\bullet^+ \coloneqq \mathcal{B}^{\mathfrak{EndoFun}(\mathfrak{Cat})}(\mathcal{FU})_\bullet(\mathfrak{C})$

Definition

$\mathfrak{I}$ a category, $\mathfrak{C}$ an $s\mathfrak{Set}$-category.
A *homotopy coherent diagram* of shape $\mathfrak{I}$ in $\mathfrak{C}$ is an $s\mathfrak{Set}$-functor $\mathfrak{I}_\bullet^+ \to \mathfrak{C}$

$$[2] = \left(\begin{tikzcd} 0 \arrow[r, "0<1"'] \arrow[rr, bend left, "(0<2)=(1<2)\circ(0<1)"] & 1 \arrow[r, "1<2"'] & 2 \end{tikzcd} \right)$$

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The homotopy coherent nerve

- By abstract nonsense (density theorem), any $X_\bullet \in \mathfrak{sSet}$ is $\varinjlim_{\Delta^n_\bullet \to X_\bullet} \Delta^n_\bullet$, where $\Delta^n_\bullet \colon \mathbb{\Delta}^{\mathrm{op}} \longrightarrow \mathfrak{Set}$ is the representable $\hom(-,[n])$
- We define the fundamental $\mathfrak{sSet}$-category $\varpi_\infty(X_\bullet) \coloneqq \varinjlim_{\Delta^n_\bullet \to X_\bullet} [n]^+_\bullet$

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- We define the fundamental $s\mathfrak{Set}$-category $\varpi_\infty(X_\bullet) \coloneqq \varinjlim_{\Delta^n_\bullet \to X_\bullet} [n]^+_\bullet$
- By the Yoneda lemma, $X_n = \hom(\Delta^n_\bullet, X_\bullet)$
- To have $\varpi_\infty \dashv N_\Delta$, we set $N_\Delta(\mathfrak{C})_n = \hom(\Delta^n_\bullet, N_\Delta(\mathfrak{C})_\bullet) \coloneqq \hom(\varpi_\infty(\Delta^n_\bullet), \mathfrak{C})$

$\implies$ an n-simplex in $N_\Delta(\mathfrak{C})$ is a string of n homotopy composable morphisms of $\mathfrak{C}$

Interpretation

$N_\Delta\mathfrak{C}$ is a combinatorial representation of $\mathfrak{C}$ exhibiting its structure of ∞-category

Bonus

Nerve and realisation

$\mathcal{I}\colon \mathfrak{D} \to \mathfrak{C}$ a functor. Define an adjoint pair $|{-}|\colon$
$$\begin{tikzcd} \mathfrak{Fun}(\mathfrak{D}^{\mathrm{op}}, \mathfrak{Set}) \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathfrak{C} \arrow[l, bend left] \end{tikzcd}$$
$: \mathcal{N}_{\mathcal{I}}$ with:

$$\mathcal{N}_{\mathcal{I}}(C)\colon \mathfrak{D}^{\mathrm{op}} \xrightarrow{\mathcal{I}^{\mathrm{op}}} \mathfrak{C}^{\mathrm{op}} \xrightarrow{\hom_{\mathfrak{C}}(-,C)} \mathfrak{Set}$$

Realisations of simplicial sets: $\mathfrak{D} = \mathbb{\Delta}$

- $\mathcal{I}\colon \mathbb{\Delta} \to \mathfrak{Cat}$ the inclusion: $\mathcal{N}_{\mathcal{I}}$ is the nerve
- $\mathcal{I}\colon \mathbb{\Delta} \to \mathfrak{Top}, [n] \mapsto |\Delta^n|$: $\mathcal{N}_{\mathcal{I}}$ is the singular complex, $|{-}|$ geometric realisation

Remark $|{-}|$ is the left Kan extension of $\mathcal{I}$ along the Yoneda embedding:

$$\begin{tikzcd} \mathfrak{D} \arrow[r, "\mathcal{I}"] \arrow[d, hook] & \mathfrak{C} \\ \mathfrak{Fun}(\mathfrak{D}^{\mathrm{op}}, \mathfrak{Set}) \arrow[ur, bend right, "|{-}|"'] \arrow[u, phantom, "\Downarrow \text{univ.}", shift left=20] \end{tikzcd}$$

The codensity monad I

Dense and codense functors

- A functor $\mathcal{F}\colon \mathfrak{C} \to \mathfrak{D}$ is *dense* if every object D of $\mathfrak{D}$ is canonically a colimit $\varinjlim_{\mathcal{F}(\mathfrak{C})_{/D}} \mathcal{F}C$, with $\mathcal{F}(\mathfrak{C})_{/D}$ the comma category of morphisms $\mathcal{F}C \to D$, $C \in \mathfrak{C}$
- Equivalently, $\mathcal{N}_{\mathcal{F}}\colon \mathfrak{D} \to \mathfrak{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Set})$ is fully faithful

Density theorem: the Yoneda embedding is dense

- $\mathcal{F}$ is *codense* if every D is $\varprojlim_{D/\mathcal{F}(\mathfrak{C})} \mathcal{F}C$

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Density theorem: the Yoneda embedding is dense

- $\mathcal{F}$ is *codense* if every D is $\varprojlim_{D/\mathcal{F}(\mathfrak{C})} \mathcal{F}C$
- There is a functor $\mathcal{N}^{\mathcal{F}}\colon \mathfrak{D} \to \mathfrak{Fun}(\mathfrak{C}, \mathfrak{Set})^{\mathrm{op}}$ with $\mathcal{N}^{\mathcal{F}}(D)\colon C \mapsto \hom_{\mathfrak{D}}(D, \mathcal{F}C)$
- Right adjoint $- \pitchfork \mathcal{F}\colon \mathfrak{Fun}(\mathfrak{C}, \mathfrak{Set})^{\mathrm{op}} \to \mathfrak{D}$

Definition

The *codensity monad* of $\mathcal{F}$ is the monad $\mathcal{N}^{\mathcal{F}} \circ (- \pitchfork \mathcal{F})$ of the adjunction

Main properties

- $\mathcal{F}$ is codense iff its codensity monad is the identity
- The codensity monad is the right Kan extension of $\mathcal{F}$ along itself
- If $\mathcal{F}$ has a left adjoint $\mathcal{L}$, its codensity monad is $\mathcal{F}\mathcal{L}$

The codensity monad II

Properties and examples

Main properties

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- The codensity monad is the right Kan extension of $\mathcal{F}$ along itself
- If $\mathcal{F}$ has a left adjoint $\mathcal{L}$, its codensity monad is $\mathcal{F}\mathcal{L}$
- The functor $\mathfrak{Cat}^{\mathrm{op}}_{/\mathfrak{D}} \to \mathfrak{Monad}(\mathfrak{D})$ is right adjoint to the Eilenberg–Moore algebra functor
- The forgetful inclusion $\mathfrak{Cat}^{\text{monadic}}_{/\mathfrak{D}} \hookrightarrow \mathfrak{Cat}^{\text{w/ cod. mon.}}_{/\mathfrak{D}}$ has a left adjoint mapping $\mathcal{F}\colon \mathfrak{C} \to \mathfrak{D}$ to the forgetful functor $\mathfrak{Alg}(\mathcal{N}^{\mathcal{F}} \circ (- \pitchfork \mathcal{F})) \to \mathfrak{D}$

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- The codensity monad of $\mathfrak{FinSet} \hookrightarrow \mathfrak{Set}$ is the ultrafilter monad β
- The codensity monad of $\mathfrak{Vect}^{\text{fin.dim.}}_k \hookrightarrow \mathfrak{Vect}_k$ is the double dualisation monad $(-)^{\vee\vee}$

Sommaire - Section 5: Comparison of models

Homotopy categories and derived functors

Goal

$\mathfrak{M}$ a category with $\mathcal{W}$ a wide class of morphisms ("weak equivalences", written $\xrightarrow{\sim}$)
The *localisation* $\mathfrak{M} \xrightarrow{\ell} \mathfrak{M}[\mathcal{W}^{-1}]$ is the universal (initial) functor sending the weak equivalences to isomorphisms
Want to understand the homotopy category $\mathrm{Ho}(\mathfrak{M}, \mathcal{W}) \coloneqq \mathfrak{M}[\mathcal{W}^{-1}]$

Also, for $\mathcal{F} \colon (\mathfrak{M}, \mathcal{W}) \to \mathfrak{N}$, compute the derived functors

$$\mathbb{R}\mathcal{F} = \mathrm{Lan}_\ell \mathcal{F} \colon \mathrm{Ho}(\mathfrak{M}, \mathcal{W}) \to \mathfrak{N} \quad \text{or} \quad \mathbb{L}\mathcal{F} = \mathrm{Ran}_\ell \mathcal{F} \colon \mathrm{Ho}(\mathfrak{M}, \mathcal{W}) \to \mathfrak{N}$$

General construction

$\mathrm{hom}_{\mathrm{Ho}(\mathfrak{M},\mathcal{W})}(M, M') = \left\{ M \xleftarrow{\in\mathcal{W}} M_1 \to M_2 \xleftarrow{\in\mathcal{W}} \cdots \xleftarrow{\in\mathcal{W}} M' \right\}$
Very unwieldy model (possibly not even locally small)

Model structures

A model structure on $(\mathfrak{M}, \mathcal{W})$ consists of two classes $\mathcal{C}$ (cofibrations) and $\mathcal{F}$ (fibrations) satisfying conditions ($(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{W} \cap \mathcal{C}, \mathcal{F})$ must be weak factorisation systems)

In particular

- Say $M \in \mathfrak{M}$ is cofibrant if $\emptyset \to M$ is a cofibration, fibrant if $M \to *$ is a fibration
- For every M there is a cofibrant $\mathcal{Q}M$ with $\mathcal{Q}M \xrightarrow{\sim} M$ and a fibrant $\mathcal{R}M$ with $M \xrightarrow{\sim} \mathcal{R}M$, both functorially in M

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Provides a left deformation $\mathcal{Q} \colon \mathfrak{M} \to \mathfrak{M}_{\mathrm{cof}} \subset \mathfrak{M}$ and a right deformation $\mathcal{R} \colon \mathfrak{M} \to \mathfrak{M}_{\mathrm{fib}} \subset \mathfrak{M}$, with $\mathcal{Q} \overset{\sim}{\Rightarrow} \mathrm{id}_{\mathfrak{M}}$ and $\mathrm{id}_{\mathfrak{M}} \overset{\sim}{\Rightarrow} \mathcal{R}$

Theorem

- $\mathrm{Ho}(\mathfrak{M}, \mathcal{W}) \simeq \{$"homotopy" classes of maps in $\mathfrak{M}_{\mathrm{cf}}$ full subcat. on fibrant–cofibrant$\}$
- $\mathbb{L}\mathcal{F} = \mathcal{F} \circ \mathcal{Q}$ and $\mathbb{R}\mathcal{F} = \mathcal{F} \circ \mathcal{R}$

Models for higher categories

Spaces as ∞-groupoids

- Model structure on $\mathfrak{Top}$ with $\mathcal{W}$ the weak homotopy equivalences
- Similar model structure on $s\mathfrak{Set}$ whose fibrant objects are the Kan complexes

Nerve and realisation induce an equivalence of homotopy theories

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Joyal There is a model structure on $\mathfrak{sSet}$ whose fibrant objects are quasi-categories (weak Kan complexes)

Bergner Model structure on $\mathfrak{Cat}_{\mathfrak{sSet}}$ with fibrant objects locally Kan $\mathfrak{sSet}$-categories, $\mathcal{W}$ functors inducing equivalences of $\mathrm{Ho}(\mathfrak{sSet}, \mathcal{W}_{\mathrm{Kan}})$-categories

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Lurie The adjunction $\varpi_\infty \dashv N_\Delta$ induces an equivalence of homotopy categories

Model-independence in ∞-cosmoi

An ∞-cosmos $\mathfrak{K}$ is (roughly) a $\mathfrak{QCat}$-enriched category of fibrant objects

Much of ∞-category theory can be done in the homotopy 2-category of $\mathfrak{K}$

Virtual profunctor equipment

∞-cosmoi admit a calculus of "bimodules" $\mathfrak{C} \nrightarrow \mathfrak{D}$, called profunctors, which are (fibrations defining) ∞-functors "$\mathfrak{D}^{\mathrm{op}} \times \mathfrak{C} \longrightarrow \mathfrak{Grpd}_\infty$"

- Abstracts the internal hom functors

Model-independence Any notion that can be encoded in terms of the virtual equipment can be transported along cosmological 2-equivalences

$\Longrightarrow$ Constructions appropriately performed in any ∞-cosmos equivalent to $\mathfrak{QCat}$ are valid for all models of ∞-categories