Towards wall-crossing for categorified quasimap CohFTs

David Kern | LAREMA, Université d'Angers | kern@math.univ-angers.fr

The usual notion of a tree-level CohFT is captured by that of an algebra over the operad $(A_\bullet \overline{\mathcal{M}}_{0,n+1})_{n\geq 0}$. Gromov–Witten classes furnish such a structure on the cyclotomic inertia stack $\mathcal{I}_\mu X$ of any Deligne–Mumford stack X by pullback–virtual pushforward along the correspondence

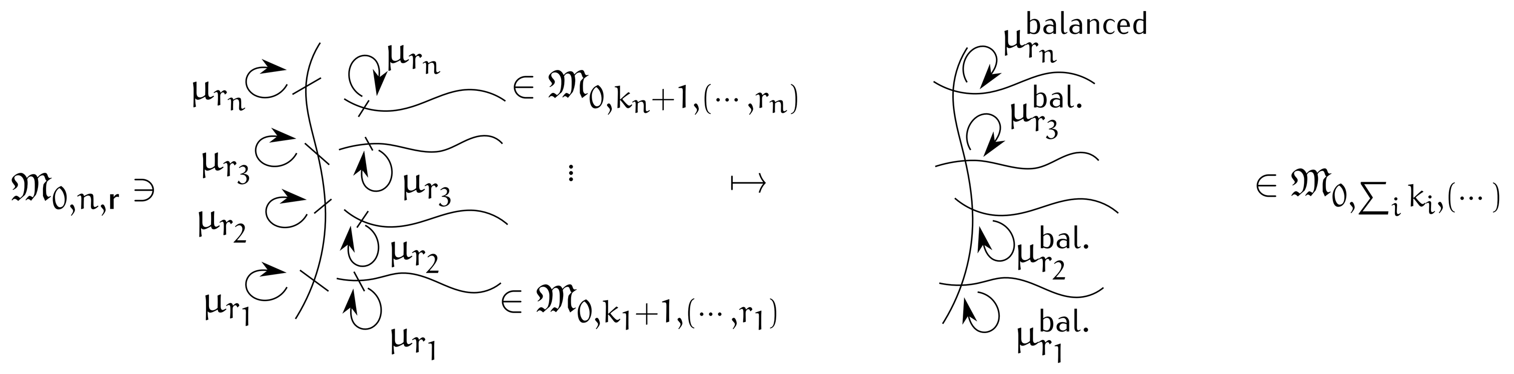
$$\overline{\mathcal{M}}_{0,n+1}\times(\mathcal{I}X)^n \xleftarrow{(\mathrm{Stab},\mathrm{ev}_1,\dots,\mathrm{ev}_n)} \coprod_{\beta\in A_1X}\overline{\mathcal{M}}_{0,n+1}(X,\beta) \xrightarrow{\mathrm{ev}_{n+1}} \mathcal{I}X. \tag{1}$$

A similar structure can be obtained in G-theory, replacing the twisting by the virtual fundamental class $\frown[\overline{\mathcal{M}}_{0,n+1}(X,\beta)]^{\mathrm{vir}}$ by a virtual structure sheaf $\otimes[\mathcal{O}_{\overline{\mathcal{M}}_{0,n+1}(X,\beta)}]^{\mathrm{vir}}$. For the natural question of the categorification of the GW classes from operators on $G_0(X)=K_0(\mathfrak{Perf}_{\mathcal{O}_X})$ to dg-functors on $\mathfrak{Perf}_{\mathcal{O}_X}$, one needs a natural lift of the G-theoretic virtual sheaf. This virtual sheaf can be realised as the "shadow" of the structure sheaf of a derived moduli stack enhancing $\overline{\mathcal{M}}_{0,n+1}(X,\beta)$. Furthermore, at the geometric level, all the motivic operations originate from the (∞-)bicategory of correspondences in (derived) algebraic stacks. We will thus study Motivic Field Theories (MotFTs), defined as lax algebras in correspondences over the ∞-operad $\overline{\mathcal{M}}_0 := (\overline{\mathcal{M}}_{0,n+1})_n$.

Operadic structure of moduli of twisted curves

For any $\mathbf{r}=(r_i)\in\mathbb{N}^n$, there is a moduli stack $\mathfrak{M}_{g,n,\mathbf{r}}$ parameterising stacky curves of genus g with n marked gerbes s_i banded respectively by μ_{r_i}.

Proposition. The genus 0 moduli assemble into a coloured (cyclic) operad $\mathfrak{M}_0$ with set of colours $\mathbb{N}$, stacks of n-ary multimorphisms $\mathfrak{M}_{0,n+1,\mathbf{r}} =: \hom_{\mathfrak{M}_0}((r_1,\dots,r_n);r_{n+1})$, and the operadic composition given by the gluing maps



The stabilisation maps $\mathfrak{M}_{0,n,\mathbf{r}}\to\overline{\mathcal{M}}_{0,n}$ provide a morphism of operads $\mathfrak{M}_0\to\overline{\mathcal{M}}_0$.

Brane actions for coloured ∞-operads

Correspondences and categorical ∞-operads

To any ∞-category $\mathfrak{C}$ with pullbacks, one associates an $(\infty,2)$-category $\mathfrak{Corr}(\mathfrak{C})$ whose 1-morphisms are correspondences (or spans) between objects of $\mathfrak{C}$. A cartesian monoidal structure $\mathfrak{C}^\times$ induces a symmetric monoidal structure $\mathfrak{Corr}^\times(\mathfrak{C})$, which should make it a special case of $(\infty,2)$-operad. In the dendroidal model for ∞-operads, we model such **categorical ∞-operads** as presheaves of ∞-categories on the category of trees Ω which satisfy the Segal conditions.

If $\mathfrak{T}=\mathfrak{HSh}_\tau(\mathfrak{S})$ is a hypercomplete ∞-topos, a (categorical) ∞-operad enriched in $\mathfrak{T}$ corresponds to a hypersheaf of (categorical) ∞-operads on the ∞-site $(\mathfrak{S},\tau)$.

Example. There is a categorical ∞-operad $\mathfrak{Corr}^\times(\mathfrak{T}_{/-})$, $S\mapsto\mathfrak{Corr}^\times(\mathfrak{T}_{/S})$.

Viewing a categorical ∞-operad as its Grothendieck construction (over $\mathfrak{T}\times\Omega^{\mathrm{op}}$), we can use the cartesian lifts to define the notion of **lax morphism**, preserving the operadic compatibilities only up to non-invertible natural transformation.

Coloured brane action

Let $\mathfrak{O}$ be an ∞-operad in $\mathfrak{T}=\mathfrak{HSh}_\tau(\mathfrak{S})$. For any multimorphism α of arity n (over an object $S\in\mathfrak{S}$), the space of **extensions** $\mathrm{Ext}(\alpha)$ contains the choices of an $(n+1)$-ary multimorphism extending α along an additional colour.

Theorem. There is a lax morphism of categorical ∞-operads $\mathfrak{O}\to\mathfrak{Corr}^{\amalg}(\mathfrak{T}^{\mathrm{op}}_{/-})$, which sends any colour C of $\mathfrak{O}(S)$ to the space $\mathrm{Ext}(\mathrm{id}_C)$. For any hypersheaf $\mathcal{X}\in\mathfrak{T}$, composition with the "internal hom" ∞-functor $\mathbb{R}\mathfrak{Map}(-,\mathcal{X})$ induces $\mathfrak{O}\to\mathfrak{Corr}^\times(\mathfrak{T}_{/-})$.

Example of twisted curves: For $\mathfrak{M}_0$, we have $\mathrm{Ext}(\mathrm{id}_n)=\coprod_{r\in\mathbb{N}}\mathfrak{M}_{0,3,(n,r,n)}$.
We set $\mathcal{L}_\mu X=\coprod_{n\geq 0}\mathbb{R}\mathfrak{Map}(\mathrm{Ext}(\mathrm{id}_n),X)$ the **cyclotomic loop stack** of a target stack X.
Note also that $\mathfrak{M}_{0,n+1}\to\mathfrak{M}_{0,n}$ is the universal curve $\mathfrak{C}_{0,n}\to\mathfrak{M}_{0,n}$, so the brane action for $\mathfrak{M}_0$ is given by analogues of the GW correspondence (1) with the derived mapping stack (2).

Derived enhancements and virtual pullbacks

Obstruction theories and derived thickenings

Let $f\colon Y\to X$ be a quasi-smooth morphism of derived algebraic stacks, that is $\mathbb{L}_{f\colon Y/X}$ is of perfect amplitude in $[-1,0]$. The closed immersions of the truncations $j_X\colon t_0X\hookrightarrow X$, $j_Y\colon t_0Y\hookrightarrow Y$ into their derived thickenings induce isomorphisms in G-theory, and we define the virtual pullback

$$(t_0f)^! = (j_{Y,*})^{-1}f^*j_{X,*}\colon G_0(t_0X)\xrightarrow{\simeq}G_0(X)\to G_0(Y)\xrightarrow{\simeq}G_0(t_0Y).$$

The virtual structure sheaf of t_0X is the virtual pullback of k along the structure map $X\to\operatorname{Spec}k$, and $(t_0f)^!$ preserves virtual sheaves. It also coincides with Manolache's virtual pullback built from the perfect obstruction theory (POT) $j_Y^*\mathbb{L}_f\to\mathbb{L}_{t_0f}$.

Derived mapping stacks and prestable curves

Definition. Let $\mathfrak{C}_{g,n,\mathbf{r}}\to\mathfrak{M}_{g,n,\mathbf{r}}$ denote the universal curve, and let X be a target algebraic stack. We define the derived mapping stack

$$\mathbb{R}\mathfrak{Map}_{/\mathfrak{M}_{g,n,\mathbf{r}}}(\mathfrak{C}_{g,n,\mathbf{r}},X\times\mathfrak{M}_{g,n,\mathbf{r}}), \tag{2}$$

which is a derived thickening of the classical mapping stack.

The POT coming from its cotangent complex coincides with the usual POT obtained by pulling back $\mathbb{T}_X$ along the universal evaluation map $\mathfrak{C}_{g,n,\mathbf{r}}\times\mathfrak{Map}_{/\mathfrak{M}_{g,n,\mathbf{r}}}(\mathfrak{C}_{g,n,\mathbf{r}},X\times\mathfrak{M}_{g,n,\mathbf{r}})\to X$.
By imposing open stability conditions, we will define open derived substacks which are thickenings of the moduli stacks of stable maps and more generally quasi-stable maps.

Stability conditions and quasimaps

Stable points of algebraic stacks

Let $\mathbb{G}_m$ be the multiplicative group k-scheme, and let $\mathrm{B}\mathbb{G}_m=[*/\mathbb{G}_m]$ and $\Theta=[\mathbb{A}^1/\mathbb{G}_m]$ be the moduli stacks for line bundles and line bundles with a section. For any algebraic stack X, a close degeneration of a point $x\colon\operatorname{Spec}k\to X$ is a morphism $\tilde{x}\colon\Theta\to X$ such that $\tilde{x}(1)=x$ and $\tilde{x}(0)\neq x$.
If X is endowed with a line bundle $\mathcal{L}_0$, we can define a point x to be $\mathcal{L}_0$**-stable** if its automorphisms are finite over k and the pullback of $\mathcal{L}_0$ along any close degeneration of x has negative weight. This condition naturally extends to rational line bundles $\mathcal{L}=\varepsilon\mathcal{L}_0\in\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$. For any choice of such stability parameter $\mathcal{L}=\varepsilon\mathcal{L}_0$, we denote X^{st} the locus of stable points in X.

Quasi-stable maps

A representable morphism $f\colon C\to X$ from a stacky curve $(C,s_1,\dots,s_r)$ to X is pre-$\mathcal{L}$-stable if it maps generically to X^{st}: only isolated basepoints are mapped to the unstable locus.
A **quasi-$\mathcal{L}$-stable map** into X is a pre-$\mathcal{L}$-stable map $f\colon C\to X$ such that the order of vanishing of f along $\mathcal{L}$ at any point c of C is ≤ 1, and $\omega_{C,\log}\otimes f^*\mathcal{L}$ is ample. Note that the stable locus only depends on $\mathcal{L}_0$. If $\varepsilon>2$, a quasi-$\mathcal{L}$-stable map to X is a stable map to X^{st}.
There is an open sub-derived stack $\mathbb{R}\mathcal{Q}^{\mathcal{L}}_{g,n,\mathbf{r}}(X,\beta)\subset\mathbb{R}\mathfrak{Map}_{/\mathfrak{M}_{g,n,\mathbf{r}}}(\mathfrak{C}_{g,n,\mathbf{r}},X\times\mathfrak{M}_{g,n,\mathbf{r}})$, which is a derived thickening of the moduli stack of quasi-$\mathcal{L}$-stable maps.

Quasimap MotFTs

By left extending the brane action $\mathfrak{M}_0\to\mathfrak{Corr}^\times(\mathfrak{dSt}_{/-})$ along $\mathfrak{M}_0\to\overline{\mathcal{M}}_0$, one obtains a lax morphism $\overline{\mathcal{M}}_0\to\mathfrak{Corr}^\times(\mathfrak{dSt}_{/-})$, sending the unique colour to the cyclotomic loop stack $\mathcal{L}_\mu X$.
For any stability bundle $\mathcal{L}\in\mathrm{Pic}(X)\otimes\mathbb{Q}$, restricting the mapping stacks appearing in the correspondences to $\coprod_{\mathbf{r},\beta}\mathbb{R}\mathcal{Q}^{\mathcal{L}}_{0,n,\mathbf{r}}(X,\beta)$ gives rise to a new MotFT on $\mathcal{L}_\mu X$.

References

References

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Future directions

Wall-crossing in categorified Givental group

Our main goal is to categorify the wall-crossing formulæ for the virtual classes of quasimap moduli stacks and lift them to the non-linear (derived) geometric setting. Rather than comparing individual derived enhancements, we compare the MotFTs $\left(\coprod_{\mathbf{r},\beta}\mathbb{R}\mathcal{Q}^{\varepsilon\mathcal{L}_0}_{0,n,\mathbf{r}}(X,\beta)\right)_n$ induced by the families of derived moduli stacks associated to each stability parameters.
Classically, CohFT structures on $A_\bullet X$ are classified by a dg-Lie algebra which integrates to the Givental group. The difference between MotFTs should then lie in a formal group derived stack (in correspondences), obtained from an $\mathcal{L}_\infty$-algebra (in correspondences) classifying lax $\overline{\mathcal{M}}_0$-algebras.

Higher genus and quantisation

Although we have considered the moduli stacks of genus 0 twisted curves as an operad, they actually possess the further structure of a cyclic operad, which is the genus 0 part of a modular operad formed by higher genus moduli stacks. As of yet there is no theory of modular ∞-operads, but once appropriately defined they ought to admit brane actions allowing the construction of higher genus MotFTs.
In the classical theory, a full CohFT is determined by its genus 0 part, and is obtained from the latter by a process of quantisation. We expect that this quantisation should come from a relationship between (the Feynman categories for) modular operads and topological recursion operads.