Master's thesis

M2 Mathématiques fondamentales — UPMC

Categorification of Gromov–Witten invariants and derived algebraic geometry

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Abstract

Following [MR18a] and its companion [MR18b], we describe a categorification of genus 0 Gromov–Witten theory. The moduli stacks $\overline{\mathcal{M}_{0,n}}(X, \beta)$ of stable maps to a variety X allow one to exhibit a structure of Cohomological Field Theory on the Chow ring $A^\bullet X$, seen as a structure of algebra over the operad $(A_\bullet \overline{\mathcal{M}_{0,n}})_n$. Introducing the derived enhancements $\mathbb{R}\overline{\mathcal{M}_{0,n}}(X, \beta)$, we lift this structure to a lax algebra over $(\overline{\mathcal{M}_{0,n}})_n$ seen as an operad in correspondences in derived stacks.

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Contents

Introduction

Quantum field theory in a given space X studies integrals of certain geometrically relevant quantities over a space parameterising the possible "paths" in X. In the language of algebraic geometry, this is recognisable as intersection theory over moduli spaces, that is to say as enumerative geometry. A case under particular study is the so-called non-linear topological σ-model, modelling the propagation of a topological quantum string in the algebraic variety X. In this case, the geometric problem studied is that of embeddings of projective curves in X. This is the basis for Gromov–Witten theory.

The Gromov–Witten invariants of a proper scheme X can be defined through intersection theory on a moduli stack $\overline{\mathcal{M}_{g,n}}(X, \beta)$ parameterising maps from nodal curves of genus g with n marked points to X mapping the fundamental class of the source curve to the cycle class $\beta \in A_1 X$, required to satisfy a stability condition. The Gromov–Witten invariants are the degrees of certain cohomology classes in $A^\bullet X$, which by the projection formula can be seen as the degrees of classes in the Chow rings of the moduli stacks $\overline{\mathcal{M}_{g,n}}$ of stable curves of genus g with n marked points. These classes are in fact induced by inverse and direct images along a universal diagram

$$\begin{tikzcd} & \overline{\mathcal{M}_{g,n}}(X, \beta) \arrow[dl, "\mathrm{Stab}"'] \arrow[dr, "ev=(ev_1,\dots,ev_n)"] & \\ \overline{\mathcal{M}_{g,n}} & & X^n \end{tikzcd}, \tag{0.1}$$

where the map Stab corresponds to forgetting the map to X and stabilising its source curve, and the maps ev_i, $i = 1, \dots, n$ correspond to evaluating the map at the curve's n marked points.

Similar invariants can be defined in the G-theory of the schemes, by applying the functor of Grothendieck K-groups to the universal diagram. In fact, this procedure suggests that Gromov–Witten invariants should be defined in all motivic contexts, which by Ayoub's formalism of crossed 2-functors are equipped with Grothendieck's six operations providing the required direct and inverse images. It was suggested by Manin (see for example [Man17]) that Gromov–Witten invariants should exist at the level of motives. In particular, following the route suggested by the G-theoretic invariants, we would like to define a categorification of Gromov–Witten theory by lifting the invariants from operators on G-groups to functors between (suitable enrichments of) derived categories of coherent sheaves.

However, it is not possible to directly generalise the above construction to derived categories as the "direct image" along the forgetful stabilisation morphism must in fact be a *virtual* direct image, that is the pushforward must be preceded by a twisting by a virtual class (in Chow homology the virtual fundamental class, and in G-theory the virtual structure sheaf) on $\overline{\mathcal{M}_{g,n}}(X, \beta)$, whose construction is not natural. This necessity comes from Kontsevich's "hidden smoothness" philosophy, according to which the highly singular stack $\overline{\mathcal{M}_{g,n}}(X, \beta)$ can be only seen as being quasi-smooth in a context which naturally incorporates the higher cohomological information coming from the singularities. This context has been realised by derived algebraic geometry, and so in order to have the virtual classes appear naturally we must treat $\overline{\mathcal{M}_{g,n}}(X, \beta)$ as nothing but the classical truncation of a true derived moduli stack $\mathbb{R}\overline{\mathcal{M}_{g,n}}(X, \beta)$ whose structure sheaf induces the virtual structure sheaf.

Thanks to this construction, it then becomes possible to lift the structure of Gromov–Witten classes not only to the level of derived categories, but even of correspondences between derived stacks, the action being given directly by the derived enhancement of (0.1). This is what has been achieved in [MR18a], and what we will study in this text.

Since structures in higher categories are given with an infinite tower of coherences, it is often very difficult to write down functors between ∞-categories and should rather be obtained by universal properties from some elementary ∞-functors. In our case, we cannot simply check that the derived correspondences carry the coherences defining the structure of Gromov–Witten theory. When Gromov–Witten classes were first defined, it was soon recognised that they could be organised in the structure of a Cohomological Field Theory, which itself can be summed up as nothing but an algebra over the operad formed by the Chow groups of the moduli stacks of stable curves. We will use this principle to exhibit the Gromov–Witten action as an algebra over an appropriate ∞-operad.

It was discovered by Toën[Toë13] that this algebra structure is a particular case of a very general phenomenon, called brane action, which implies that the space of binary operations $\mathfrak{O}(2)$ of any coherent ∞-operad $\mathfrak{O}$ carries a structure of $\mathfrak{O}$-algebra in co-correspondences, which after application of the ∞-fuctor represented by a required space X gives an $\mathfrak{O}$-algebra structure on X in correspondences. As the ∞-operad governing the Gromov–Witten moduli stacks is not coherent, this theorem cannot be applied directly; however the algebra structure can still be constructed, albeit in a lax form, for non coherent ∞-operads. This means that our action will in fact take place in the ∞-bicategory of spans of spaces (or rather derived stacks), with non-invertible coherences. It is known from [Hau17, Corollary 12.5] that any (∞, n)-category of iterated spans is fully dualisable, so by the cobordism hypothesis, the objects appearing in our correspondences must correspond to two-dimensional fully extended topological field theories, that is topological string theory: from this point of view, the appearance of the ∞-bicategory of spans in Gromov–Witten theory was inevitable.

Organisation of the text

To study Gromov–Witten theory in its natural setting, we must thus work with derived algebraic geometry and higher categories. To motivate this language, we open chapter 1 by a discussion of the construction of the virtual structure sheaf of a Deligne–Mumford stack, focusing especially on the role of the cotangent complex. We then introduce in section 1.2 some basic notions and results about ∞-categories and derived geometry that we will need, and finally in section 1.3 we bring these constructions together by showing how to reinterpret the virtual structure sheaf of a classical algebraic stack as a shadow of the true structure sheaf of a derived enhancement.

In chapter 2, we discuss ∞-operads, the structure that will be used to classify the genus 0 moduli spaces. We first present, in section 2.1, two models for higher operads, and their generalisation for operads enriched not simply in spaces but in an arbitrary ∞-topos of derived stacks. We then describe, in section 2.2, the general phenomenon of brane action for a coherent ∞-operad giving rise to the Gromov–Witten action: since we shall need the lax version for non coherent operads, we spend some time describing of to think of lax morphisms of $(\infty, 1)$-operads, then we introduce the relevant definitions and construct the brane action.

In chapter 3, we turn our attention towards Gromov–Witten theory. In a first time, we present in section 3.1 the moduli spaces of stable curves with their (modular) operadic structure, which we complement in section 3.2 by the algebra structures induced by the moduli spaces of stable maps. In an independent section 3.3, we introduced the more general notion of ε-stable quasimaps to a GIT quotient. Although we have not yet studied them further, we intend to extend the results of the next chapter to this setting.

Finally, in the as yet unfinished chapter 4, we specialise the results of chapter 1 and chapter 2 to the moduli stacks studied in chapter 3.

Prerequisites

Since this text is its author's Master's thesis, it is adapted to the Master's courses followed during the corresponding year. This means that we take the following subjects as prerequisites: étale cohomology, algebraic operads[LV12], intersection theory[Ful98], derived categories of sheaves and their Verdier duality, homotopy theory of model categories.

Part I

Preliminary constructions

Chapter 1

Virtual sheaves and derived stacks

1.1 The virtual fundamental class

1.1.1 The cotangent complex, deformations and obstructions

1.1.1.1 The relative cotangent complex of a ring map

Let $\mathfrak{T}$ be a topos, and A a ring object of the category $\mathfrak{T}$ (this general setup allows us to consider the case $\mathfrak{T} = \mathfrak{Set}$, where A represents an affine scheme, and $\mathfrak{T} = \mathfrak{Shv}(X)$ with $A = \mathcal{O}_X$). Let $A \to B$ be an A-algebra; then the **cotangent complex** is the object $\mathbb{L}^\bullet_{B/A} \in \mathbb{D}^{\leq 0}(\mathfrak{Mod}_B)$ defined by

$$\mathbb{L}^\bullet_{B/A} \coloneqq \Omega_{P(B)_\bullet/A} \underset{P(B)_\bullet}{\otimes} B \tag{1.1}$$

where $B \to P(B)_\bullet$ is a free resolution of B, typically the standard simplicial free A-resolution $P(B)_n = A[\cdots[A[B]]\cdots]$ with its augmentation morphism, and where $\Omega_{B/A}$ is the cotangent module, representing A-derivations of B. Since we will only consider $\mathbb{L}^\bullet_{B/A}$ as an object of the derived category, the choice of resolution does not matter up to isomorphism. The augmentation $\mathbb{L}^\bullet_{B/A} \to \Omega_{B/A}$ induces an isomorphism $\mathcal{H}^0(\mathbb{L}^\bullet_{B/A}) \xrightarrow{\simeq} \Omega_{B/A}$. The universal property of $\mathbb{L}^\bullet_{B/A}$, in terms of derivations, is only seen at the derived level, as in section 1.2.1.2. We also define the **tangent complex** $\mathbb{T}^\bullet_{B/A} \coloneqq (\mathbb{L}^\bullet_{B/A})^{\mathbb{R}\vee} = \mathbb{R}\mathcal{H}om(\mathbb{L}^\bullet_{B/A}, A)$.

Let $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, giving a homomorphism of rings $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ in $\mathfrak{Shv}(Y)$, which corresponds by adjunction to $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ in $\mathfrak{Shv}(X)$. We then define the relative cotangent complex of X over Y as

$$\mathbb{L}^\bullet_{X/Y} \coloneqq \mathbb{L}^\bullet_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y} \in \mathbb{D}^{\leq 0}\mathfrak{Mod}_{\mathcal{O}_X}. \tag{1.2}$$

If $U \subset (X, \mathcal{O}_X)$ and $V \subset (Y, \mathcal{O}_Y)$ are affine open subschemes such that $f(U) \subset V$, respectively $\operatorname{Spec} B$ and $\operatorname{Spec} A$, then $\mathbb{L}^\bullet_{X/Y}\Big|_U = \widetilde{\mathbb{L}^\bullet_{B/A}}$ ([Stacks, Tag 08T3]), where the functor $\widetilde{\bullet}$ is the defining equivalence $\mathfrak{Mod}_B \xrightarrow{\sim} \mathfrak{QCoh}_{\operatorname{Spec} B}$. If $Y = \operatorname{Spec} k$ and $X \to Y$ is the structure map of a k-scheme, then $\mathbb{L}^\bullet_{X/k}$ is simply written $\mathbb{L}^\bullet_X$, and called the *absolute* cotangent complex of X.

Proposition 1.1.1.1.1 (Functoriality properties).

1. *[Stacks, Tag 08QX] Let* $A \to B \to C$ *be ring maps in* $\mathfrak{T}$. *The triangle*

$$\mathbb{L}^\bullet_{B/A} \overset{\mathbb{L}}{\underset{B}{\otimes}} C \to \mathbb{L}^\bullet_{C/A} \to \mathbb{L}^\bullet_{C/B} \to \left(\mathbb{L}^\bullet_{B/A} \overset{\mathbb{L}}{\underset{B}{\otimes}} C[1]\right) \tag{1.3}$$

   is a distinguished triangle in $\mathbb{D}^{\leq 0}\mathfrak{Mod}_C$. *In particular, morphisms of schemes* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *induce a distinguished triangle* $f^*\mathbb{L}^\bullet_{Y/Z} \to \mathbb{L}^\bullet_{X/Z} \to \mathbb{L}^\bullet_{X/Y} \to f^*\mathbb{L}^\bullet_{Y/Z}[1]$.

2. *[Stacks, Tag 08QQ](Base change) A commutative square*

$$\begin{tikzcd} B' & B \arrow[l] \\ A' \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd} \tag{1.4}$$

   of ring maps induces a morphism $\mathbb{L}^\bullet_{B/A} \otimes^{\mathbb{L}}_B B' \to \mathbb{L}^\bullet_{B'/A'}$, *which is a (quasi-)isomorphism if the square is cocartesian.*

The cotangent complex controls the deformation theory of A-algebras in the following way. A typical problem of deformation theory has the following form: let

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d] & \overline{B} \arrow[r, two heads] \arrow[d, dashed] & B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r, dashed] & J \arrow[r, dashed] & {} \arrow[r, dashed] & C \arrow[r, dashed] & 0 \end{tikzcd} \tag{1.5}$$

be the data of a square-zero extension of an A-algebra B by the A-module I (which becomes a nilpotent ideal of $\overline{B}$), for example corresponding to an infinitesimal thickening of A-schemes $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} \overline{B}$, and a morphism from the pair (B, I) to the pair (C, J); we seek a square-zero extension of C by J and a morphism of extensions inducing the diagram. Then the group $\operatorname{Ext}^2_A(\mathbb{L}^\bullet_{C/B}, J)$ contains a canonical obstruction whose vanishing is equivalent to the existence of a solution to the lifting problem. In that case, the set of solutions forms a torsor under the group $\operatorname{Ext}^1_A(\mathbb{L}^\bullet_{C/B}, J)$, and the automorphism group of a given solution is canonically identified with $\operatorname{Ext}^0_A(\mathbb{L}^\bullet_{C/B}, J)$.

Definition 1.1.1.1.2 (Obstruction theory)**.** Let X be an S-scheme; an **obstruction theory** for X is a morphism $\mathcal{K}^\bullet \to \mathbb{L}^\bullet_{X/S}$ in the derived category $\mathbb{D}^{\leq 0}(\mathfrak{Mod}_{\mathcal{O}_X})$ such that

1. $\mathcal{H}^i(\mathcal{K}^\bullet)$ is coherent, for $i = -1, 0$;

2. $\mathcal{H}^0(\mathcal{K}^\bullet) \to \mathcal{H}^0(\mathbb{L}^\bullet_{X/S})$ is an isomorphism and $\mathcal{H}^{-1}(\mathcal{K}^\bullet) \to \mathcal{H}^{-1}(\mathbb{L}^\bullet_{X/S})$ is an epimorphism.

Equivalently, it is given by a morphism $\mathbb{T}^\bullet_{X/S} \to (\mathcal{K}^\bullet)^{\mathbb{R}\vee}$. An obstruction theory $\mathcal{K}^\bullet \to \mathbb{L}^\bullet_{X/S}$ is said to be **perfect** if in addition $\mathcal{K}^\bullet$ is locally, over an open U, isomorphic to an object $[\mathcal{E}^{-1} \to \mathcal{E}^0]$ of $\mathbb{D}^{[-1,0]}(\mathfrak{Mod}_{\mathcal{O}_U}) \subset \mathbb{D}^{\leq 0}(\mathfrak{Mod}_{\mathcal{O}_U})$ whose components $\mathcal{E}^i$ are locally free sheaves of finite rank, in which case we say that $\mathcal{K}^\bullet$ is of perfect amplitude in $[-1, 0]$. If $\mathbb{L}^\bullet_{X/S}$ is of perfect amplitude in $[-1, 0]$, we say that X is **quasi-smooth** (or virtually smooth) over S.

Example 1.1.1.1.3. By [Avr99, (1.2)], a morphism of nœtherian rings is a local complete intersection if and only if it is quasi-smooth.

The **virtual dimension** of X relatively to a perfect obstruction theory $\mathcal{E}^\bullet$ is the locally constant number $\operatorname{vdim}_{\mathcal{E}^\bullet} X = \operatorname{rk}\mathcal{E}^0 - \operatorname{rk}\mathcal{E}^{-1}$. A perfect obstruction theory is to be understood[Beh14] as a shadow on X of the (natural) cotangent complex of a derived enhancement of X, in a way that will be made precise in section 1.3.

1.1.1.2 Algebraic stacks

While the cotangent complex explains the infinitesimal deformations of algebras, global deformations, or moduli problems, often require too much information to be captured by schemes and necessitate a degree of categorification. An infinitesimal deformation problem for a scheme S is usually expressed as a functor on the category of artinian algebras. The global formulation is a sheaf on the category of S-schemes with an appropriate topology, associating to each S-scheme a set of equivalence classes of families over it. If the moduli functor is a scheme, it is called a **fine moduli space** for the moduli problem. When this is not the case, a better approach is to replace the sets of equivalence classes of families by the corresponding groupoids, conserving the information of how families can be equivalent. This requires the study of sheaves of categories up to isomorphisms, or stacks.

A **stack** on a site $(\mathfrak{S}, \tau)$ is a pseudo-functor $\mathcal{F}\colon \mathfrak{S}^{\mathrm{op}} \to \mathfrak{Cat}$ with values in the 2-category of categories, such that the topology of universal effective $\mathcal{F}$-descent is finer than τ; more explicitly, for any τ-covering sieve $\mathcal{R}$ of any object S of $\mathfrak{S}$, the functor

$$\mathcal{F}(S) \simeq \mathfrak{hom}(\mathfrak{S}/S, \mathcal{F}) \to \mathfrak{hom}(\mathcal{R}, \mathcal{F}) =: \mathfrak{Desc}(\mathcal{R}; \mathcal{F}) \tag{1.6}$$

is an equivalence. By the Grothendieck construction, a pseudo-functor $\mathfrak{S}^{\mathrm{op}} \to \mathfrak{Cat}$ is equivalently given by a fibered category $\int \mathcal{F} \to \mathfrak{S}$ with a cleavage. The topology on $\mathfrak{S}$ can be transfered to this category $\int \mathcal{F}$, and this allows one to talk of sheaves and stacks over a stack.

An important example of stack is the **quotient stack** $[X/G]$ of a scheme X by the action of an algebraic group G. Over a scheme U, its category of sections is the category of diagrams $U \leftarrow P \to X$ where $P \to U$ is a G-torsor over U and $P \to X$ a G-equivariant map. Let $E^\bullet$ be a complex of abelian sheaves; then the map $(\tau_{[0,1]}E^\bullet)^0 \to (\tau_{[0,1]}E^\bullet)^1$ induces an action of the abelian group $(\tau_{[0,1]}E^\bullet)^0$, and we can define the stack

$$\mathcal{H}^1/\mathcal{H}^0(E^\bullet) = \left[(\tau_{[0,1]}E^\bullet)^1/(\tau_{[0,1]}E^\bullet)^0\right]. \tag{1.7}$$

This is an example of a Picard stack, a stack in symmetric 2-groups (categories endowed with associative and commutative bifunctors). The rule $\mathcal{H}^1/\mathcal{H}^0$ is functorial, and in fact induces[AGV72] an equivalence between $\mathbb{D}^{[0,1]}(\mathfrak{Mod}_{\mathcal{O}_X})$ and the category of Picard stacks and isomorphism classes of additive functors between them.

The infinitesimal lifting property characterising the cotangent complex of a morphism of schemes can be reformulated in a global way in terms of Picard stacks. Let $f\colon X \to Y$ be a morphism of schemes and let $\mathcal{I}$ be a quasicoherent $\mathcal{O}_X$-module. Let

$\mathfrak{Exal}_Y(X, \mathcal{I})$ denote the category of square-zero extensions of X over Y with ideal sheaf $\mathcal{I}$; it forms a Picard stack and there is an equivalence of Picard stacks[Ols07, Theorem 8.2]

$$\mathcal{H}^1/\mathcal{H}^0\left(\mathbb{R}\mathcal{H}om(\mathbb{L}^\bullet_{X/Y}, \mathcal{I})[1]\right) \simeq \mathfrak{Exal}_Y(X, \mathcal{I}). \tag{1.8}$$

A **geometric stack** is a stack equivalent to the "quotient stack" of an internal groupoid $[G_1 \rightrightarrows G_0]$, defined tautologically using the Yoneda lemma. Equivalently, its diagonal morphism is representable and it admits a surjection, called an **atlas**, from a representable stack, *i.e.* from an object of the base site. A geometric stack in the smooth topology on the category of (affine) S-schemes is called an **Artin stack** over the scheme S. An Artin stack whose atlas can be taken to be étale is a **Deligne-Mumford (DM) stack**; a stack is DM if and only if it is an Artin stack and has unramified diagonal[LM00, Theorem 8.1]. Both Artin and DM stacks are occasionally called algebraic. Many properties of schemes and their morphisms can be adapted to algebraic stacks, and we shall do so implicitly in the following. In particular, an algebraic stack $\mathfrak{X}$ has an étale site with "structure sheaf" $\mathcal{O}_\mathfrak{X}$ and an associated category of modules, and admits a cotangent complex $\mathbb{L}^\bullet_{\mathfrak{X}/S} \in \mathbb{D}(\mathfrak{Mod}_{\mathcal{O}_\mathfrak{X}})$.

1.1.2 The intrinsic normal cone

In order to construct a virtual fundamental class (or in our case a virtual structure sheaf K-class) for a Deligne–Mumford stack, we wish to perform a deformation to the normal cone. However, there is *a priori* no canonical embedding of a given DM stack, so we need to consider all possible (étale-)local embeddings. This requires a closer examination of cones over DM stacks.

1.1.2.1 Cone stacks

We generally call a **cone** over a Deligne–Mumford stack X the relative spectrum $\mathcal{S}pec(\mathcal{A})$ of a quasicoherent sheaf $\mathcal{A} = \bigoplus_{i \geq 0} \mathcal{A}_i$ of graded $\mathcal{O}_X$-algebras. A cone $\mathcal{S}pec(\mathcal{A})$ is called **abelian** if $\mathcal{A}$ is of the form $\operatorname{Sym}\mathcal{M}$ with $\mathcal{M}$ a coherent $\mathcal{O}_X$-module. Any cone $\mathcal{S}pec \bigoplus_i \mathcal{A}_i$ has an **abelian hull** $\mathcal{S}pec \operatorname{Sym}\mathcal{A}_1$ of which it is a closed subcone. The main example of a cone is the normal cone $C_{U/X} \coloneqq \mathcal{S}pec \bigoplus_n \mathcal{I}^n/\mathcal{I}^{n+1}$ of the embedding of a closed subscheme $U \hookrightarrow X$ with ideal sheaf $\mathcal{I}$; its abelian hull is the normal sheaf $\mathcal{N}_{U/X} \coloneqq \mathcal{S}pec \operatorname{Sym}\mathcal{I}/\mathcal{I}^2$.

A cone C over X naturally has a section $0\colon X \to C$ as well as an $\mathbb{A}^1$-action. We call a **cone stack** over X an Artin stack $\mathfrak{C}$ over X endowed with a section $X \to \mathfrak{C}$ and an $\mathbb{A}^1$-action, which étale-locally admits an $\mathbb{A}^1$-equivariant smooth surjection from a cone over (an étale open of) X, which is called a **local presentation** of $\mathfrak{C}$. Any cone stack $\mathfrak{C}$ with a local presentation $C \to \mathfrak{C}$ is then locally given ($\mathbb{A}^1$-equivariantly) as $[C/(C \times_\mathfrak{C} X)]$. We define similarly abelian cone stacks and vector bundle stacks by requiring that a (equivalently, any) presentation be smooth.

The structure sheaf $\mathcal{O}_X$ induces ring objects $\mathcal{O}_{X,\text{fl}}$ and $\mathcal{O}_{X,\text{ét}}$ in the big fppf (faithfully-flat-and-of-finite-presentation) and the small étale topoi X_{fl} and $X_{\text{ét}}$ of X respectively.

The embedding of the étale site into the fppf site induces a morphism of topoi $\nu_{\mathrm{fl}}\colon X_{\mathrm{fl}} \to X_{\text{ét}}$. If an object $\mathcal{K}^\bullet \in \mathbb{D}^{\leq 0}(\mathfrak{Mod}_{\mathcal{O}_{X,\text{ét}}})$ respects condition 1 of the definition 1.1.1.1.2 of an obstruction theory, then its associated Picard stack $\mathcal{H}^1/\mathcal{H}^0((\mathbb{L}\nu_{\mathrm{fl}}^*\mathcal{K}^\bullet)^{\mathbb{R}\vee})$ (where the "right derived dual" is taken by the functor $\mathbb{R}\mathcal{H}om(\bullet, \mathcal{O}_{X,\mathrm{fl}})$) is an Artin stack over X, and in fact an abelian cone stack ([BF97, proposition 2.4]), which is furthermore a vector bundle stack if $\mathcal{K}^\bullet$ is of perfect amplitude in $[-1, 0]$.

Let X be a DM stack locally of finite type over k; its absolute cotangent complex $\mathbb{L}_X^\bullet = \mathbb{L}_{X/\operatorname{Spec} k}^\bullet$ verifies the condition quoted above and we define the **intrinsic normal sheaf** of X as

$$\mathfrak{N}_X = \mathcal{H}^1/\mathcal{H}^0\left((\mathbb{L}\nu_{\mathrm{fl}}^*\mathbb{L}_X^\bullet)^{\mathbb{R}\vee}\right) \tag{1.9}$$

(if X is quasi-smooth it is a vector bundle stack over X). The intrinsic normal sheaf admits a simpler étale-local description. We call a **local embedding** of X a local immersion $f\colon U \to M$ of an affine k-scheme U of finite type in a smooth affine k-scheme of finite type M, with an étale morphism $i\colon U \to X$. Such a local embedding of X induces an isomorphism $[\mathcal{N}_{U/M}/f^*\mathcal{T}_M] \xrightarrow{\simeq} i^*\mathfrak{N}_X$ of cone stacks over U (where $\mathcal{N}_{U/M}$ is the normal sheaf of U in M and $\mathcal{T}_M$ the tangent bundle of M), so we can understand $\mathfrak{N}_X$ as being étale-locally presented by $\mathcal{N}_{U/X}$.

In the previous local description, we can replace $\mathcal{N}_{U/M}$ by the normal cone $C_{U/M}$ to obtain a closed subcone $[C_{U/M}/f^*\mathcal{T}_M]$. Then by [BF97, Corollary 3.9], a morphism $j\colon (U', M') \to (U, M)$ of local embeddings of X induces an isomorphism $[C_{U'/M'}/f'^*\mathcal{T}_{M'}] \simeq j^*[C_{U/M}/f^*\mathcal{T}_M]$ of closed subcones of $j^*[\mathcal{N}_{U/M}/f^*\mathcal{T}_M]$. It follows that these subcones glue to a closed subcone $\mathfrak{C}_X \subset \mathfrak{N}_X$, called the **intrinsic normal cone**, which is uniquely determined by the property that, for any local embedding (U, M) of X the square

$$\begin{tikzcd} C_{U/M} \arrow[r] \arrow[d] & \mathcal{N}_{U/M} \arrow[d] \\ \mathfrak{C}_X \arrow[r] & \mathfrak{N}_X \end{tikzcd} \tag{1.10}$$

is cartesian, so $\mathfrak{C}_X|_U \simeq [C_{U/M}/f^*\mathcal{T}_M]$. In addition, [BF97, Theorem 3.11] ensures that $\mathfrak{N}_X$ is the abelian hull of $\mathfrak{C}_X$ as a cone stack.

1.1.2.2 Virtual structure sheaf

Let $\phi\colon \mathcal{E}^\bullet \to \mathbb{L}_X^\bullet$ a perfect obstruction theory; then $\mathcal{H}^1/\mathcal{H}^0\left((\mathbb{L}\nu_{\mathrm{fl}}^*\mathcal{E}^\bullet)^{\mathbb{R}\vee}\right) =: \mathfrak{E}$ is an abelian cone stack over X and ϕ induces $\phi^{\mathbb{R}\vee}\colon \mathfrak{N}_X \to \mathfrak{E}$, which is a closed immersion (this is equivalent to ϕ being an obstruction theory by [BF97, Proposition 2.6, Theorem 4.5]). Equivalently, $\mathfrak{C}_X \to \mathfrak{E}$ is a closed immersion of cone stacks.

We write $\mathfrak{E}$ in the presentation $[\mathcal{E}_1/\mathcal{E}_0]$, where $\mathcal{E}_0 \to \mathcal{E}_1$ is the dual complex of

$\mathcal{E}^{-1} \to \mathcal{E}^0$. The subcone stack $\mathfrak{C}_X \hookrightarrow \mathfrak{E}$ induces a subcone $C_1 \subset \mathcal{E}_1$. We then set

$$\begin{aligned}\left[\mathcal{O}^{\mathrm{vir}}_{X,\phi}\right] &\coloneqq \left[\mathcal{O}_{C_1} \overset{\mathbb{L}}{\underset{\mathcal{O}_{\mathcal{E}_1}}{\otimes}} \mathcal{O}_X\right] \qquad \in K_\circ(X) \\ &= \sum_{i \geq 0} (-1)^i \left[\mathcal{T}or_i^{\mathcal{O}_{\mathcal{E}_1}}(\mathcal{O}_{C_1}, \mathcal{O}_X)\right],\end{aligned} \tag{1.11}$$

called the **virtual structure sheaf** of X relative to the perfect obstruction theory ϕ. In this definition the K-theoretic (derived) tensor product is to be interpreted as a K-theoretic (derived) pullback along the zero section of the local vector bundle $\mathcal{E}_1 \to X$.

Construction 1.1.2.2.1 (Reminders on K-theory)**.** Let X be an algebraic stack. Its K-groups $K^\circ(X)$ and $K_\circ(X)$ are defined as the Grothendieck groups of the additive categories respectively of vector bundles on X and of coherent sheaves on X, that is (*cf.* section 1.2.2.3) the quotient of the free abelian group on isomorphism classes of objects modulo the relations $[E] - [E'] - [E'']$ for all exact sequences $0 \to E' \to E \to E'' \to 0$. The tensor product of locally free sheaves gives a ring structure to $K^\circ(X)$ and a $K^\circ(X)$-module structure on $K_\circ(X)$.

Let $[\mathcal{F}] \in K_\circ(X)$. The functor $- \otimes \mathcal{F}$ is only right exact, so to obtain an exact sequence from an exact sequence of coherent sheaves we must use its left-derived functor $- \otimes^{\mathbb{L}} \mathcal{F}$. This induces a K-theoretic operation $[\mathcal{G}] \mapsto [\mathcal{G} \otimes^{\mathbb{L}} \mathcal{F}] = \sum_{i \geq 0} (-1)^i [\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})]$. Similarly, for any proper $f\colon X \to Y$, the direct image functor f_* is only left exact so its right derived functor induces $K_\circ(X) \to K_\circ(Y)$, $[\mathcal{F}] \mapsto [\mathbb{R}f_*\mathcal{F}] = \sum_{i \geq 0} (-1)^i [\mathbb{R}^i f_* \mathcal{F}]$.

Remark 1.1.2.2.2 (Virtual fundamental class)*.* Suppose X is a type of algebraic stack for which we have an intersection theory $A_\bullet X$. We make one of the following two assumptions:

(a) There exists a “Chern character” $\mathfrak{c}\colon K_\circ(X) \to A_\bullet X$.

(b) $\mathcal{E}^\bullet$ admits a global resolution, that is an isomorphism (in $\mathbb{D}(\mathfrak{Mod}_{\mathcal{O}_{X_{\text{ét}}}})$, so a quasi-isomorphism of complexes) with a two-term complex of vector bundles $[\mathcal{F}^{-1} \to \mathcal{F}^0]$ concentrated in degrees $[-1, 0]$.

Then we can define a **virtual fundamental class** $[X]^{\mathrm{vir}}_\phi \in A_{\mathrm{vdim}_{\mathcal{E}^\bullet} X} X$ (where $\mathrm{vdim}_{\mathcal{E}^\bullet} X = \operatorname{rk} \mathcal{E}^0 - \operatorname{rk} \mathcal{E}^{-1}$).

(a) We put $[X]^{\mathrm{vir}}_\phi = \mathrm{td}_{\mathfrak{c}}(\mathcal{E}^\bullet) \frown \mathfrak{c}\left(\left[\mathcal{O}^{\mathrm{vir}}_{X,\phi}\right]\right)$ where $\mathrm{td}_{\mathfrak{c}}$ is the associated Todd genus.

(b) We directly mimic the construction of the virtual sheaf in te setting of Chow homology by defining $[X]^{\mathrm{vir}}_\phi$ as the intersection of the closed subcone $C(\mathcal{F}^\bullet) \coloneqq \mathfrak{C}_X \times_{\mathfrak{E}} \mathcal{F}_1 \subset \mathcal{F}_1$ with the zero section of the vector bundle $\mathcal{F}_1$, that is $[X]^{\mathrm{vir}}_\phi = 0^![C(\mathcal{F}^\bullet)]$.

From [BF97, Remark 5.4] these two constructions agree when they are both possible.

We now describe two examples relevant for moduli spaces of (stable) maps from curves.

Example 1.1.2.2.3 (Canonical obstruction theory of a mapping space). Let C be a projective curve with dualising complex $\mathcal{D}_C$ (which is $(C \to \operatorname{Spec} k)^!(k)$) and X a smooth projective scheme. Let M denote the scheme of morphisms from C to X, with functor of points $U \rightsquigarrow \hom_U(C \times U, X \times U) = \hom(C \times U, X)$. Let $f\colon C \times M \to X$ be the universal morphism, and let $\pi\colon C \times M \to M$ and $p\colon C \times M \to M$ be the canonical projections. Note that since C is projective, the morphisms π and $C \to \operatorname{Spec} k$ are proper (and in particular we will use $\mathbb{R}\pi_* = \mathbb{R}\pi_!$). We have a cartesian square

$$\begin{tikzcd} C \times M \arrow[r, "\pi"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d] \\ C \arrow[r] & \operatorname{Spec} k \end{tikzcd} \tag{1.12}$$

which by the functoriality property 2 induces a (quasi)-isomorphism $\pi^*\mathbb{L}^\bullet_M \xrightarrow{\sim} \mathbb{L}^\bullet_{C\times M/C}$. Concomitantly the two commutative (although not cartesian) squares

$$\begin{tikzcd} C \times M \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k \arrow[r, equal] & \operatorname{Spec} k \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \times M \arrow[r, equal] \arrow[d] & C \times M \arrow[d] \\ C \arrow[r] & \operatorname{Spec} k \end{tikzcd} \tag{1.13}$$

give morphisms $f^*\mathbb{L}^\bullet_X \to \mathbb{L}^\bullet_{C\times M}$ and $\mathbb{L}^\bullet_{C\times M} \to \mathbb{L}^\bullet_{C\times M/C}$. Altogether these maps compose to $e\colon f^*\mathbb{L}^\bullet_X \to \pi^*\mathbb{L}^\bullet_M$.

By Verdier duality we have $\mathbb{R}\mathcal{H}om(\pi^*\mathbb{L}^\bullet_M, p^!\mathcal{D}_C) = \pi^!\mathbb{R}\mathcal{H}om(\mathbb{L}^\bullet_M, \mathbb{R}\pi_*p^!\mathcal{D}_C)$ where $\mathbb{R}\pi_*p^!\mathcal{D}_C =: \mathcal{D}_M$ is the dualising complex for M (by proper base change along eq. (1.12) and adjunctions). The morphism $\mathbb{R}\mathcal{H}om(e, p^!\mathcal{D}_C)\colon \pi^!\mathbb{R}\mathcal{H}om(\mathbb{L}^\bullet_M, \mathcal{D}_M) \to \mathbb{R}\mathcal{H}om(f^*\mathbb{L}^\bullet_X, p^!\mathcal{D}_C)$ induces by applying $\mathbb{R}\pi_!$ then again the Verdier duality functor

$$\begin{aligned} &\mathbb{R}\mathcal{H}om(\mathbb{R}\pi_!\mathbb{R}\mathcal{H}om(e, p^!\mathcal{D}_C), \mathcal{D}_M)\colon \\ &\quad \mathbb{R}\mathcal{H}om(\mathbb{R}\pi_!\mathbb{R}\mathcal{H}om(f^*\mathbb{L}^\bullet_X, p^!\mathcal{D}_C), \mathcal{D}_M) \to \mathbb{R}\mathcal{H}om(\mathbb{R}\pi_!\pi^!\mathbb{R}\mathcal{H}om(\mathbb{L}^\bullet_M, \mathcal{D}_M), \mathcal{D}_M), \end{aligned} \tag{1.14}$$

whose target can be expressed by Grothendieck duality as

$$\begin{aligned} \mathbb{R}\mathcal{H}om(\mathbb{R}\pi_!\pi^!\mathbb{R}\mathcal{H}om(\mathbb{L}^\bullet_M, \mathcal{D}_M), \mathcal{D}_M) &= \mathbb{R}\pi_!\mathbb{R}\mathcal{H}om(\pi^!\mathbb{R}\mathcal{H}om(\mathbb{L}^\bullet_M, \mathcal{D}_M), \pi^!\mathcal{D}_M) \\ &= \mathbb{R}\pi_!\mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(\pi^*\mathbb{L}^\bullet_M, p^!\mathcal{D}_C), \pi^!\mathcal{D}_M) \\ &= \mathbb{R}\pi_*(\pi^*\mathbb{L}^\bullet_M). \end{aligned} \tag{1.15}$$

Then postcomposition with the adjunction counit $\mathbb{R}\pi_*\pi^* \Rightarrow \mathbb{1}$ induces

$$\mathbb{R}\mathcal{H}om(\mathbb{R}\pi_!\mathbb{R}\mathcal{H}om(f^*\mathbb{L}^\bullet_X, p^!\mathcal{D}_C), \mathcal{D}_M) \to \mathbb{R}\mathbb{L}^\bullet_M. \tag{1.16}$$

But by further Grothendieck duality we have

$$\begin{aligned} \mathbb{R}\mathcal{H}om(\mathbb{R}\pi_*\mathbb{R}\mathcal{H}om(f^*\mathbb{L}^\bullet_X, p^!\mathcal{D}_C), \mathcal{D}_M) &\cong \mathbb{R}\pi_*\mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(f^*\mathbb{L}^\bullet_X, p^!\mathcal{D}_C), \pi^!\mathcal{D}_M) \\ &\cong \mathbb{R}\pi_*\mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(f^*\mathbb{L}^\bullet_X, p^!\mathcal{D}_C), p^!\mathcal{D}_C) \\ &\cong \mathbb{R}\pi_* f^*\mathbb{L}^\bullet_X \end{aligned} \tag{1.17}$$

Since π_* commutes with taking duals because π is proper and f^* does too by property of the inverse image, this eventually gives a morphism

$$\phi\colon \mathcal{E}^\bullet := (\mathbb{R}\pi_* f^* \mathbb{T}_X^\bullet)^{\mathbb{R}\vee} \to \mathbb{L}_M^\bullet. \tag{1.18}$$

By [BF97, Proposition 6.2] this is a perfect obstruction theory.

Remark 1.1.2.2.4. The scheme of morphisms from X to Y is typically constructed as an open subscheme of a Hilbert scheme of $X \times Y$, parameterising the immersions of the graphs of such morphisms, with ideal sheaves $\mathcal{I}_{\Gamma_g} \subset \mathcal{O}_{X\times Y}$. The cotangent space at a k-point x_g classifying $g\colon X \to Y$ is then $\mathcal{I}_{\Gamma_g}/\mathcal{I}_{\Gamma_g}^2 \cong g^*\Omega_Y^1$ by [Kol96, Theorem I.2.16].

Example 1.1.2.2.5 (Canonical obstruction theory of a universal curve). Let $\pi\colon \mathfrak{C} \to \mathfrak{M}$ be a smooth morphism of relative dimension 1 between Deligne–Mumford stacks which is a universal curve over (an open subset of) a moduli stack and such that the relative dualising complex $\mathcal{D}_{\mathfrak{C}/\mathfrak{M}}$ is a line bundle in degree -1 with inverse $\mathcal{D}_{\mathfrak{C}/\mathfrak{M}}^\vee$ such that $\mathbb{R}\mathcal{H}om(-, \mathcal{D}_{\mathfrak{C}/\mathfrak{M}}) = (- \otimes^{\mathbb{L}} \mathcal{D}_{\mathfrak{C}/\mathfrak{M}}^\vee)^{\mathbb{R}\vee}$ (note also that $\mathcal{D}_{\mathfrak{M}/\mathfrak{M}} = \mathcal{O}_{\mathfrak{M}}$ for $\mathbb{1}_{\mathfrak{M}}$). By the functoriality property 1 of the cotangent complex, the distinguished triangle $\pi^*\mathbb{L}_{\mathfrak{M}}^\bullet \to \mathbb{L}_{\mathfrak{C}}^\bullet \to \mathbb{L}_{\mathfrak{C}/\mathfrak{M}}^\bullet \to \pi^*\mathbb{L}_{\mathfrak{M}}^\bullet[1]$ furnishes the so-called Kodaira–Spencer map $\mathbb{L}_{\mathfrak{C}/\mathfrak{M}}^\bullet \to \pi^*\mathbb{L}_{\mathfrak{M}}^\bullet[1]$.

By Verdier duality we have a natural isomorphism $\mathbb{R}\mathcal{H}om(\pi^*-, \mathcal{D}_{\mathfrak{C}/\mathfrak{M}}) \cong \pi^!\mathbb{R}\mathcal{H}om(-, \mathcal{O}_{\mathfrak{M}})$, which induces for any $\mathcal{F}^\bullet \in \mathbb{D}^{\leq 0}\mathfrak{Mod}_{\mathcal{O}_{\mathfrak{C}}}$, $\mathcal{G}^\bullet \in \mathbb{D}^{\geq 0}\mathfrak{Mod}_{\mathcal{O}_{\mathfrak{M}}}$ a (quasi-)isomorphism of Ext complexes

$$\begin{aligned}\mathbb{R}\hom_{\mathcal{O}_{\mathfrak{C}}}(\mathcal{F}^\bullet, \pi^*\mathcal{G}^\bullet) &\simeq \mathbb{R}\hom_{\mathcal{O}_{\mathfrak{M}}}\left(\mathbb{R}\mathcal{H}om(\pi^*\mathcal{G}^\bullet, \mathcal{D}_{\mathfrak{C}/\mathfrak{M}}), \mathbb{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{D}_{\mathfrak{C}/\mathfrak{M}})\right)\\ &\simeq \mathbb{R}\hom_{\mathcal{O}_{\mathfrak{M}}}\left(\pi^!\mathbb{R}\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{O}_{\mathfrak{M}}), \mathbb{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{D}_{\mathfrak{C}/\mathfrak{M}})\right)\\ &\simeq \mathbb{R}\hom_{\mathcal{O}_{\mathfrak{M}}}\left((\mathcal{G}^\bullet)^{\mathbb{R}\vee}, \mathbb{R}\pi_*\mathbb{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{D}_{\mathfrak{C}/\mathfrak{M}})\right).\end{aligned} \tag{1.19}$$

In particular, setting

$$\begin{aligned}\mathcal{E}^\bullet[1] &= (\mathbb{R}\pi_*\mathbb{R}\mathcal{H}om(\mathbb{L}_{\mathfrak{C}/\mathfrak{M}}^\bullet, \mathcal{D}_{\mathfrak{C}/\mathfrak{M}}))^{\mathbb{R}\vee}\\ &= (\mathbb{R}\pi_*(\mathbb{L}_{\mathfrak{C}/\mathfrak{M}}^\bullet \otimes^{\mathbb{L}} \mathcal{D}_{\mathfrak{C}/\mathfrak{M}}^\vee)^{\mathbb{R}\vee})^{\mathbb{R}\vee} = \mathbb{R}\pi_*(\mathbb{L}_{\mathfrak{C}/\mathfrak{M}}^\bullet \otimes^{\mathbb{L}} \mathcal{D}_{\mathfrak{C}/\mathfrak{M}}^\vee)\end{aligned} \tag{1.20}$$

we obtain from the Kodaira–Spencer map an arrow in the derived category $\mathcal{E}^\bullet \to \mathbb{L}_{\mathfrak{M}}^\bullet$. By [BF97, Proposition 6.1 and remark], it is a perfect obstruction theory.

Remark 1.1.2.2.6 (Relative version). Let Y be smooth Artin k-stack of pure dimension, and $\pi\colon X \to Y$ a relative DM stack over Y. Then we can adapt the construction of the virtual structure sheaf to this relative context by replacing the absolute cotangent complex $\mathbb{L}_X^\bullet$ by the relative $\mathbb{L}_{X/Y}^\bullet$, which satisfies the condition so that the relative intrinsic normal cone $\mathfrak{N}_{X/Y} = \mathcal{H}^1/\mathcal{H}^0\left((\mathbb{L}\nu_{\mathrm{fl}}^*\mathbb{L}_{X/Y}^\bullet)^{\mathbb{R}\vee}\right)$ is an abelian cone stack. We define likewise relative obstruction theories as certain (absolutely) perfect complexes above $\mathbb{L}_{X/Y}^\bullet$.

The previous examples generalise to the case of a relative cotangent complex (and for example, in example 1.1.2.2.3, C is a relative curve over the base stack).

Suppose X is absolutely DM of finite type and Y is locally of finite type. Let $\phi\colon \mathcal{E}^\bullet \to \mathbb{L}_{X/Y}^\bullet$ be a relative perfect obstruction theory, and let η be the composite $\mathcal{E}^\bullet \to \mathbb{L}_{X/Y}^\bullet \to \pi^*\mathbb{L}_Y^\bullet[1]$. Then by [KKP03, Proposition 3], the induced $\psi\colon\ \mathrm{cone}(\eta)[-1] \to \mathbb{L}_X^\bullet$ is a perfect obstruction theory, and moreover the two virtual structure sheaves obtained from ϕ and ψ coincide in $K_\circ(X)$.

1.2 Derived spaces

1.2.1 The ∞-category of derived rings

1.2.1.1 Modelling ∞-categories

Although derived algebraic geometry requires the theory of ∞-categories in an essential way, it is model-independent, so it is possible to take the point of view presented in [GR17] that one does not actually need to know the details of the constructions of higher categories, but only what can be *done* with them. Nonetheless, we present some elements of the language of a model for $(\infty, 1)$-categories which will be useful. We mainly follow [Lur09, chapter 1] and the shorter [Gro10], [Ant13] and [GR17, chapter I.1]. We also refer to appendix A for further details.

It is well accepted that the theory of ∞-groupoids, or $(\infty, 0)$-categories, should be equivalent to that of topological spaces up to weak homotopy equivalence. Indeed, the homotopy type of a topological space is completely determined by its fundamental ∞-groupoid, and the homotopy hypothesis, which is taken as a guiding principle for the definition of higher categories, states that every **∞-groupoid** should arise as the fundamental ∞-groupoid of a topological space. It is then natural to define an $(\infty, 1)$-category as a category enriched in ∞-groupoids, that is a topologically enriched category. Since we only care about the morphism spaces *up to weak homotopy*, thanks to the Quillen equivalence $\mathrm{Sing}\colon \mathfrak{Top} \leftrightarrows \mathfrak{sSet}\colon |\cdot|$, we would obtain an equivalent theory by using categories enriched in simplicial sets. However the enrichment, in both cases, is strict, and thus ill-suited both conceptually and for the development of further aspects of the theory. A satisfying definition of $(\infty, 1)$-categories would realise them as categories with a *lax* enrichment in ∞-groupoids, with an infinite tower of coherences ruling the compositions. It is not possible to give such a definition, so we must instead seek a model which will contain those coherences.

Definition 1.2.1.1.1. An **∞-category** (or **quasicategory**) is a simplicial set $\mathfrak{C}$ respecting the weak Kan condition: for any $n \in \mathbb{N}$, for any inner horn $\Lambda^n_i \to \mathfrak{C}$, $1 \leq i \leq n-1$, there exists a extension $\Delta^n \to \mathfrak{C}$ to an n-simplex of $\mathfrak{C}$ along the inclusion $\Lambda^n_i \hookrightarrow \Delta^n$.

Let $\mathfrak{C}$ be an ∞-category. A 0-simplex of $\mathfrak{C}$ is simply called an object. If $x, y \in \mathfrak{C}_0$ are two objects of $\mathfrak{C}$, a 1-morphism from x to y is a 1-simplex $f \in \mathfrak{C}_1$ such that $d^0(f) = x$ and $d^1(f) = y$. If f, g are 1-morphisms with $d^0(g) = d^1(f)$, the datum defines a horn $\Lambda^2_1 \to \mathfrak{C}$, which then lifts to a 2-simplex θ, which we view as expressing a way that the 1-morphism $d^1(\theta)$ is homotopic to the composite $g \circ f$ of f by g.

Given any pair of objects (x, y), there exists a **mapping space** $\mathrm{Map}(x, y) \in \mathfrak{sSet}$, obtained for example by considering the corresponding hom in the associated simplicial (or topological) category. We can then associate to any ∞-category $\mathfrak{C}$ its **homotopy category** $\mathrm{Ho}(\mathfrak{C})$, which is a 1-category with objects those of $\mathfrak{C}$ and morphism sets $\hom_{\mathrm{Ho}(\mathfrak{C})}(x, y) = \pi_0 \mathrm{Map}_{\mathfrak{C}}(x, y)$. Given a simplicial set $\mathfrak{I}$ and an ∞-category $\mathfrak{C}$, the simplicial set $\mathrm{Map}(\mathfrak{I}, \mathfrak{C})_\bullet$ is an ∞-category. If $\mathfrak{I}$ is also a quasicategory, we call it the **∞-category of ∞-functors**, and denote it $\mathfrak{Fun}(\mathfrak{I}, \mathfrak{C})$. Its vertices are maps of simplical sets, referred to here as ∞-functors. We then see that the ∞-categories form an ∞-category $\mathfrak{Cat}_\infty$ themselves, and in fact even an $(\infty, 2)$-category.

Construction 1.2.1.1.2 (Dwyer–Kan localisation)**.** Let $\mathfrak{C}$ be an ∞-category with a selected class of morphisms W. A derived **localisation** of $\mathfrak{C}$ with respect to W is an ∞-category $\mathfrak{C}[W^{-1}]_\infty$ endowed with an ∞-functor $\mathcal{L}\colon \mathfrak{C} \to \mathfrak{C}[W^{-1}]_\infty$ such that for any ∞-category $\mathfrak{D}$ the induced map

$$\mathfrak{Fun}(\mathfrak{C}[W^{-1}]_\infty, \mathfrak{D}) \to \mathfrak{Fun}(\mathfrak{C}, \mathfrak{D}) \tag{1.21}$$

induces an equivalence with the full subcategory of ∞-functors sending all arrows in W to equivalences in $\mathfrak{D}$.

In the case where $\mathfrak{M}$ is a simplicial model category with class of weak equivalences W, the localisation of the corresponding ∞-category $N_\Delta(\mathfrak{M})$ can be constructed as the simplicial nerve $N_\Delta(\mathfrak{M}_{cf})$ of the full subcategory of fibrant–cofibrant objects in $\mathfrak{M}$.

In order to work with sheaves, we shall also need the notion of cartesian and cocartesian fibrations of ∞-category, for which there exists a higher version of the Grothendieck construction.

Definition 1.2.1.1.3 (Cartesian fibration)**.**

- Let $\mathcal{P}\colon \mathfrak{F} \to \mathfrak{C}$ be an ∞-functor. A morphism $\phi\colon \xi \to \psi$ in $\mathfrak{F}$, lifting $\mathcal{P}\xi = X \xrightarrow{\mathcal{P}\phi = f} Y = \mathcal{P}\psi$ in $\mathfrak{C}$, is $\mathcal{P}$**-cartesian** if the canonical map $\mathfrak{F}_{/\xi} \to \mathfrak{F}_{/\psi} \times_{\mathfrak{C}_{/Y}} \mathfrak{C}_{/X}$ it induces by postcomposition is an equivalence. We also call (ξ, ϕ) an inverse image of ψ by f, written $f^*\psi \xrightarrow{f_*} \psi$.
- The ∞-functor $\mathcal{P}$ is a **cartesian fibration** if every morphism of $\mathfrak{C}$ admits an inverse image for every object of $\mathfrak{F}$ lifting its target.
- Dually, $\mathcal{P}$ is a **cocartesian fibration** if $\mathcal{P}^{op}\colon \mathfrak{F}^{op} \to \mathfrak{C}^{op}$ is a cartesian fibration.

Theorem 1.2.1.1.4 (Grothendieck construction)**.** *Let $\mathfrak{C}$ be an ∞-category. There are equivalences of categories $\int\colon \mathfrak{Fun}(\mathfrak{C}^{op}, \mathfrak{S}) \simeq \mathfrak{Cart}^{cart}_{\infty,/\mathfrak{C}}$ and $\mathfrak{Fun}(\mathfrak{C}, \mathfrak{S}) \simeq \mathfrak{Cart}^{cocart}_{\infty,/\mathfrak{C}}$.*

Remark 1.2.1.1.5*.* Having established that there is a model for ∞-categories, we will whenever possible work in a model-independent manner. In particular, a category $\mathfrak{C}$ will implicitly be considered as a trivial example of ∞-category without taking its nerve.

1.2.1.2 Some commutative algebra for derived rings

We let Γ denote the category whose objects are pointed finite sets $(\langle n \rangle = \{0, \dots, n\}, 0)$ and whose morphisms are maps of sets preserving the marked point. For any n and any $0 \le i \le n$, let $\rho^n_i\colon \langle n \rangle \to \langle 1 \rangle$ be the arrow defined by $\rho^n_i(j) = \delta_{i,j}$ for all $j \in \langle n \rangle$.

Definition 1.2.1.2.1 (Monoidal ∞-category)**.** A symmetric monoidal ∞-category is a cocartesian fibration $\mathfrak{V}^\otimes \to N(\Gamma)$ (equivalently, an ∞-functor $N(\Gamma) \to \mathfrak{Cat}_\infty$) such that $\mathfrak{V}^\otimes_{\langle 0 \rangle}$ is contractible and, for every n, the ∞-functor $((\rho^n_1)_!, \dots, (\rho^n_n)_!)\colon \mathfrak{V}^\otimes_{\langle n \rangle} \to (\mathfrak{V}^\otimes_{\langle 1 \rangle})^n$ is a categorical equivalence.

Let k be a field of characteristic 0. The basic objects in derived algebraic geometry are the so-called *derived rings*, which can be modelled as simplicial k-algebras. Since we work over a base field of characteristic 0, we may as well apply the Dold–Kan correspondence and use dg-k-algebras[1]. The category $\mathfrak{dMod}_k^{\leq 0}$ of derived k-modules is defined as the derived localisation of the category of connective differential graded k-modules at the class of quasi-isomorphisms. The tensor product endows $\mathfrak{dMod}_k^{\leq 0}$ with a symmetric monoidal ∞-category structure, and we let $\mathfrak{dAlg}_k^{\leq 0}$ be the category of commutative monoids in it. If $A \in \mathfrak{dAlg}_k^{\leq 0}$, we write $\pi_n A := H^{-n}(A)$ for any $n \geq 0$. The (discrete) ring $\pi_0 A$ is called the classical part of A.

For any derived k-algebra $A \in \mathfrak{dAlg}_k^{\leq 0}$, we also have an ∞-category of A-modules $\mathfrak{dMod}_A$ (not necessarily bounded), with a tensor product, and its ∞-category of monoids $\mathfrak{dAlg}_A \simeq \mathfrak{dAlg}_k^{\leq 0}/A$.

Definition 1.2.1.2.2 (Space of derivations)**.** The space of A-derivations of B with coefficients in M is the space of sections of the projection $\pi_B \colon B \oplus M \to B$.

In other words, it is:

- the mapping space $\mathrm{Map}_{\mathfrak{dAlg}_A/B}(B, B \oplus M)$;
- the homotopy fiber of $\pi_{B,*} \colon \mathrm{Map}(B, B \oplus M) \to \mathrm{Map}(B, B)$ at $\mathbb{1}_B$.

Lemma 1.2.1.2.3. *The ∞-functor $M \rightsquigarrow \mathrm{Der}_A(B, M)$ is representable by an object $\mathbb{L}^{\bullet}_{B/A} \in \mathfrak{dMod}_B$, i.e. there is a functorial equivalence $\mathrm{Der}_A(B, -) \xrightarrow{\sim} \mathrm{Map}_{\mathfrak{dMod}_B}(\mathbb{L}^{\bullet}_{B/A}, -)$. The representing object is obtained as $\mathbb{L}^{\bullet}_{B/A} \colon \Omega_{R(B)/A} \otimes_{Q(B)} B$, with $Q(B)$ a cofibrant replacement of B.*

Proposition 1.2.1.2.4. *A morphism $A \to B$ in $\mathfrak{dAlg}_k^{\leq 0}$ is an equivalence if and only if it induces an isomorphism $H^0(A) \to H^0(B)$ in $\mathfrak{Alg}_k$ and $\mathbb{L}^{\bullet}_{B/A} \simeq 0$.*

Definition 1.2.1.2.5 (Properties of morphisms)**.** Let $f \colon A \to B$ be a morphism in $\mathfrak{dAlg}_k^{\leq 0}$. Then f is said to be

finitely presentated if the ∞-functor $\mathrm{Map}_{\mathfrak{dAlg}_A}(B, -)$ commutes with filtered colimits;

flat if $- \otimes_A B$ preserves finite colimits;

formally smooth if for any $M \in \mathfrak{dMod}_B$ such that $H^0(M) = 0$, we have $\mathrm{Map}(\mathbb{L}^{\bullet}_{B/A}, M) = 0$;

formally étale if $\mathbb{L}^{\bullet}_{B/A} \simeq 0$;

smooth if it is formally smooth and finitely presented;

étale if it is formally étale and finitely presented;

[1] In positive characteristic, the category of dg-algebras does not admit a model structure whose weak equivalences are quasi-isomorphisms and whose fibrations are degree-wise surjections.

a Zariski open immersion if it is flat, finitely presented, and the product $B \otimes_A B \xrightarrow{\times} B$ is an equivalence.

Definition 1.2.1.2.6 (Strong morphism)**.** A morphism $A \to B$ in $\mathfrak{dAlg}_k^{\leq 0}$ is **strong** if the induced map $\pi_0 B \otimes_{\pi_0 A} \pi_i A \to \pi_i B$ is an equivalence for all $i \geq 0$.

Theorem 1.2.1.2.7. *A morphism $A \to B$ in $\mathfrak{dAlg}_k^{\leq 0}$ is flat (resp. smooth, étale, a Zariski open immersion) if and only if it is strong and the induced map $\pi_0 A \to \pi_0 B$ of discrete rings is classically flat (resp. smooth, étale, a Zariski open immersion).*

1.2.2 Atlases for stacks

1.2.2.1 Sheaves of spaces

Definition 1.2.2.1.1 (Presheaves)**.** Let $\mathfrak{A}$ be an ∞-category. A **presheaf with values in** $\mathfrak{A}$ on an ∞-category $\mathfrak{C}$ is an ∞-functor $\mathfrak{C}^{\mathrm{op}} \to \mathfrak{A}$.

In particular, when $\mathfrak{A} = \mathfrak{S}$, a presheaf of spaces is also simply called a presheaf, or a **prestack**.

Categories of presheaves are determined by their exactness properties, so we must introduce the notion of (co)limits. Recall that in a (co)complete (1-)category $\mathfrak{C}$, the (co)limit functor for shape category $\mathfrak{I}$ is the functor of right (left) Kan extension along $\mathfrak{I} \to *$.

Construction 1.2.2.1.2 (Adjunction and Kan extensions)**.** An ∞-functor $\mathfrak{C} \to \mathfrak{D}$ is a diagram of ∞-categories of shape the interval category $[1]$, that is an ∞-functor $\mathcal{F}\colon [1] \to \mathfrak{Cat}_\infty$, which we can also see as a cocartesian fibration $\int \mathcal{F} \to [1]$. Its **partially defined right adjoint** is the full subcategory of $\int \mathcal{F}$ consisting of objects which are the source of a cartesian morphism over $0 \to 1$. It gives a cartesian fibration over $[1]$, so an ∞-functor $[1]^{\mathrm{op}} \to \mathfrak{Cat}_\infty$, determining an ∞-functor $\mathfrak{D} \to \mathfrak{C}$.

Let $\mathcal{K}\colon \mathfrak{J} \to \mathfrak{L}$ be an ∞-functor, and let $\mathfrak{C}$ be an ∞-category. The partially defined left and right adjoints to $\mathcal{K}^*\colon \mathfrak{Fun}(\mathfrak{L}, \mathfrak{C}) \to \mathfrak{Fun}(\mathfrak{J}, \mathfrak{C})$ are called respectively left and right (partially defined) Kan extensions.

Definition 1.2.2.1.3 (Limits)**.** Let $\mathcal{D}\colon \mathfrak{I} \to \mathfrak{C}$ be an ∞-functor. The colimit (respectively limit) of $\mathcal{D}$ is the right (resp. left) Kan extension along $\mathcal{P}\colon \mathfrak{I} \to *$:

$$\varinjlim_{\mathfrak{I}} \mathcal{D} = \mathrm{Ran}_{\mathcal{P}} \mathcal{D} \qquad \text{and} \qquad \varprojlim_{\mathfrak{I}} \mathcal{D} = \mathrm{Lan}_{\mathcal{P}} \mathcal{D}. \tag{1.22}$$

The colimit (resp. limit) is the initial (reps. final) object in the ∞-category of cocones under $\mathcal{F}$ (resp. cones over $\mathcal{D}$), determined up to a contractible space of choices. If $\mathfrak{C}$ is presented as the derived localisation of a simplicial model category $\mathfrak{M}$, the (co)limits in $\mathfrak{C}$ coincide with homotopy (co)limits in $\mathfrak{M}$. For this reason, we shall often adorn (co)limits in ∞-categories with an "h", especially when they are limits of objects of a 1-category embedded in an ∞-category.

An ∞-category is said to be (co)complete if it has all small (co)limits, and an ∞-functor is (co)continuous if it commutes with small (co)limits.

Proposition 1.2.2.1.4. *[Lur09, Theorem 5.1.5.6] Let $\mathfrak{C}$ and $\mathfrak{D}$ be two ∞-categories. There is an equivalence of ∞-categories $\mathfrak{Fun}^{\mathrm{colim}}(\mathfrak{PSh}(C), \mathfrak{D}) \xrightarrow{\simeq} \mathfrak{Fun}(\mathfrak{C}, \mathfrak{D})$ (induced by the Yoneda embedding of $\mathfrak{C}$), where $\mathfrak{Fun}^{\mathrm{colim}}$ indicates the colimit-preserving ∞-functors: the Yoneda embedding into the presheaf ∞-category is a cocompletion.*

We can now turn our attention to descent conditions. An ∞-site is simply a small ∞-category $\mathfrak{C}$ equipped with a Grothendieck topology τ on its homotopy category.

Definition 1.2.2.1.5 (Hypercovering)**.** Let $(\mathfrak{C}, \tau)$ be an ∞-site. A **τ-hypercovering** of an object $C \in \mathfrak{C}$ is an augmented simplicial presheaf $\mathcal{F}_\bullet \colon {\Delta_+}^{\mathrm{op}} \to \mathfrak{PSh}(\mathfrak{C})$ such that $\mathcal{F}_{-1}$ is the Yoneda reprenstable presheaf of C and for every $[n] \in \Delta$ the map $\mathcal{F}_n \to (\operatorname{cosk}_{n-1} \mathcal{F}_\bullet)_n$ is τ-covering (where $\operatorname{cosk}_{n-1}$ is the $(n-1)$-coskeleton, right adjoint to the restriction $(\Delta_{\leq n} \hookrightarrow \Delta)^*$).

A hypercovering is effective if its totalisation (colimit) is the augmentation.

Definition 1.2.2.1.6 (Derived stack)**.** A **derived stack** on an ∞-site $(\mathfrak{C}, \tau)$ is an ∞-functor $\mathcal{X} \colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{S}$ such that for every object C and every effective hypercover $\mathcal{F}_\bullet$ of C, the map

$$\mathcal{X}(C) = \operatorname{Map}(C, \mathcal{X}) = \operatorname{Map}\left(\varinjlim_n \mathcal{F}_n, \mathcal{X}\right) \to \varprojlim_n \operatorname{Map}(\mathcal{F}_n, \mathcal{X}) \tag{1.23}$$

is an equivalence in $\mathfrak{S}$.

We write $\mathfrak{dSt}(\mathfrak{C})$ the ∞-category of derived stacks, and simply $\mathfrak{dSt}$ when the site is unambiguous.

A **stack ∞-topos** is an ∞-category equivalent to $\mathfrak{dSt}(\mathfrak{C})$ for some ∞-site $(\mathfrak{C}, \tau)$.

Example 1.2.2.1.7 (Affine schemes)*.* Let $\mathfrak{C} = \mathfrak{dAlg}_k^{\leq 0}$ with the étale topology. Any derived ring A defines a representable derived stack, denoted $\operatorname{Spec} A$. If A is a discrete ring, we have $t_0(\operatorname{Spec} A) = \operatorname{Spec}^{\mathrm{cl}} A$ for its classical truncation, where $\operatorname{Spec}^{\mathrm{cl}}$ denotes the classical (underived) spectrum of a ring, the Yoneda embedding on $\mathfrak{Alg}_k$.

1.2.2.2 Derived algebraic stacks

Definition 1.2.2.2.1 (Geometric context)**.** Let $\mathfrak{C}$ be a monoidal ∞-category. We let $\mathfrak{Aff}_{\mathfrak{C}} = \mathfrak{CAlg}(\mathfrak{C})^{\mathrm{op}}$, endowed with a quasi-compact topology such that for any family $\{C_i\}_i$ of objects the family $\{C_i \to \coprod_j C_j\}_i$ is a covering family. A geometric context is a class $\mathbf{P}$ of morphisms in $\mathfrak{Aff}_{\mathfrak{C}}$ stable by composition, equivalences and (homotopy) fibred products such that

- any morphism in a covering family is in $\mathbf{P}$,
- being in $\mathbf{P}$ is a local property (for any $f \colon X \to Y$ in $\mathbf{P}$, if there exists a covering family $\{\rho_i \colon U_i \to X\}_i$ of X such that all the composites $f\rho_i$ are in $\mathbf{P}$, then f is in $\mathbf{P}$), and

- for any $X, Y \in \mathfrak{Aff}_{\mathfrak{C}}$, the natural morphisms $X, Y \rightrightarrows X \amalg^h Y$ are in **P**.

We will simply say of a morphism belonging to **P** that it is **P**.

Lemma 1.2.2.2.2. *[TV08, Lemma 1.3.2.12] Let* $\{C_i \to C\}$ *be a family of* **P** *morphisms. Then* $\coprod_i^h C_i \to C$ *is also* **P**.

Definition 1.2.2.2.3 (Geometric stack)**.** Let $\mathfrak{C}$ be a monoidal ∞-category, and let **P** be a geometric context on $\mathfrak{Aff}_{\mathfrak{C}}$. We define recursively (on $n \in \mathbb{N} \cup \{-1\}$) the notion of n-geometric stack for the context **P**. We will then say that a derived stack on $\mathfrak{Aff}_{\mathfrak{C}}$ is **geometric** if it is n-geometric for some n.

Base case:

- A derived stack is (-1)-geometric if it is affine, that is representable.
- A morphism of derived stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is (-1)-representable if for any representable stack $\mathcal{Z} \to \mathcal{Y}$ over $\mathcal{Y}$, the homotopy fibred product $\mathcal{X} \times^h_{\mathcal{Y}} \mathcal{Z}$ is (-1)-geometric.
- A morphism of derived stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is (-1)-**P** if it is (-1)-representable and for any $\mathcal{Z} \to \mathcal{Y}$ in $\mathfrak{dSt}_{/\mathcal{Y}}$ the induced morphism $\mathcal{X} \times^h_{\mathcal{Y}} \mathcal{Z} \to \mathcal{X}$ of representable stacks is **P**.

Fix $n \geq 0$:

- An n-**atlas** for a derived stack $\mathcal{X}$ is a family $\{\mathcal{U}_i \to \mathcal{X}\}_{i \in I}$ such that each $\mathcal{U}_i$ is (-1)-geometric (*i.e.* representable), each $\mathcal{U}_i \to \mathcal{X}$ is $(n-1)$-**P**, and $\coprod_i^h \mathcal{U}_i \to \mathcal{X}$ is an epimorphism.
- A derived stack $\mathcal{X}$ is n-**geometric** if it admits an n-atlas its diagonal morphism is $(n-1)$-representable.
- A morphism of derived stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is (n)-representable if for any representable stack $\mathcal{Z} \to \mathcal{Y}$ over $\mathcal{Y}$, the homotopy fibred product $\mathcal{X} \times^h_{\mathcal{Y}} \mathcal{Z}$ is (n)-geometric.
- A morphism of derived stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is (n)-**P** if it is (n)-representable and for any $\mathcal{Z} \to \mathcal{Y}$ in $\mathfrak{dSt}_{/\mathcal{Y}}$ the derived stack $\mathcal{X} \times^h_{\mathcal{Y}} \mathcal{Z}$ admits an n-atlas $\{\mathcal{U}_i \to \mathcal{X} \times^h_{\mathcal{Y}} \mathcal{Z}\}$ such that each composite $\mathcal{U}_i \to \mathcal{X}$ between representables is **P**.

As for classical geometric stacks, there is an alternative characterisation of geometricity in terms of "quotients" of internal ∞-groupoids.

Definition 1.2.2.2.4 (Segal groupoid objects)**.**

- Let $\mathfrak{C}$ be an ∞-category. A **Segal groupoid object** in $\mathfrak{C}$ is a simplicial object $C_\bullet \in \mathfrak{Fun}(\Delta^{op}, \mathfrak{C})$ such that the morphism $d_0 \times d_1\colon C_2 \to C_1 \times^h_{C_0} C_1$ is an equivalence, and for any $n > 0$, the natural morphism $\prod_i \sigma_i\colon C_n \to C_1 \times^h_{C_0} \cdots \times^h_{C_0} C_1$ is an equivalence.
- A Segal groupoid $\mathcal{F}_\bullet$ in $\mathfrak{dSt}$ is an n-**P** Segal groupoid if the stacks $\mathcal{F}_0$ and $\mathcal{F}_1$ are disjoint unions of n-geometric stacks and the morphism $d_0\colon \mathcal{F}_1 \to \mathcal{F}_0$ is n-**P**.

Note that (by the discussion following [TV08, Definition 1.3.4.1]) if $\mathcal{F}_\bullet$ is n-**P** Segal, then all its faces $\mathcal{F}_i \to \mathcal{F}_{i-1}$.

Theorem 1.2.2.2.5. *[TV08, Proposition 1.3.4.2] Let $\mathcal{X}$ be a derived stack, and let $n \geq 0$. Then $\mathcal{X}$ is n-geometric if and only if there is an $(n-1)$-**P** Segal groupoid object $\mathcal{U}_\bullet$ in $\mathfrak{dSt}$ with an isomorphism $\mathcal{X} \simeq |\mathcal{U}_\bullet| \coloneqq \mathbb{R}\varinjlim_{n\in\Delta} \mathcal{U}_n$.*

Definition 1.2.2.2.6 (Derived algebraic geometry)**.** For derived algebraic geometry we will work in the monoidal ∞-category $\mathfrak{C} = \mathfrak{dMod}_k^{\leq 0}$ of dg-vector spaces over the field of characteristic 0 k. We write $\mathfrak{dAff} \coloneqq \mathfrak{Aff}_{\mathfrak{C}} = \mathfrak{dAlg}_k^{\leq 0\,\mathrm{op}}$.

We will usually endow $\mathfrak{Aff}$ with the **étale topology**. A geometric derived stack for the context consisting of étale morphisms is called a derived Deligne–Mumford stack. A geometric derived stack in the context of smooth morphisms is called a **derived Artin stack**, or a **derived algebraic stack**.

The inclusion $\mathfrak{Alg}_k \hookrightarrow \mathfrak{dAlg}_k^{\leq 0}$ induces by composition a functor $\mathfrak{Fun}(\mathfrak{dAff}^{\mathrm{op}} = \mathfrak{dAlg}_k^{\leq 0}, \mathfrak{S}) \to \mathfrak{Fun}(\mathfrak{Aff}^{\mathrm{op}} = \mathfrak{Alg}_k, \mathfrak{S})$ restricting to the **truncation** functor $t_0\colon \mathfrak{dSt}(\mathfrak{dAlg}_k^{\leq 0}) \to \mathfrak{dSt}(\mathfrak{Alg}_k)$. By [TV08, Lemma 2.2.4.1], it admits a fully faithful left adjoint ι. The truncation functor commutes with homotopy limits and colimits, while the extension functor ι commutes with homotopy colimits, but not homotopy limits in general.

We will usually omit writing down ι, especially when considering a truncation $t_0(\mathcal{X})$ again as a derived stack.

Definition 1.2.2.2.7 (Monos and epis)**.**

- A morphism of derived stacks $a\colon \mathcal{F} \to \mathcal{G}$ is an **epimorphism** if the induced morphism $\pi_0(t_0(\mathcal{F})) \to \pi_0(t_0(\mathcal{G}))$ of sheaves of sets on $(\mathfrak{Aff}_k, \tau_{\text{ét}})$ is an epimorphism.
- A morphism of derived stacks $a\colon \mathcal{F} \to \mathcal{G}$ is a **monomorphism** if its diagonal morphism $\Delta_a\colon \mathcal{F} \to \mathcal{F} \times_{\mathcal{G}} \mathcal{F}$ is a equivalence.

Property 1.2.2.2.8. *[TV08, Proposition 2.2.4.7] The components of the counit $j\colon \iota t_0 \Rightarrow \mathbb{1}_{\mathfrak{dSt}}$ are closed immersions $j_{\mathcal{X}}\colon t_0\mathcal{X} \hookrightarrow \mathcal{X}$.*

Definition 1.2.2.2.9 (Zariski open immersion)**.** Let $\mathcal{X}$ be a derived stack over $\mathfrak{dAff}$.

- A morphism $u\colon \mathcal{X} \to \operatorname{Spec} A$ to an affine derived stack is a Zariski open immersion if it is a monomorphism and there is a family $\{j_\alpha\colon \operatorname{Spec} A_\alpha \to \operatorname{Spec} A\}$ of Zariski open immersions of derived rings such that each j_α factors through u as $u \circ p_\alpha$, $p_\alpha\colon \operatorname{Spec} A_\alpha \to \mathcal{X}$, where $\coprod_\alpha p_\alpha\colon \coprod_\alpha \operatorname{Spec} A_\alpha \to \mathcal{X}$ is an epimorphism.
- Let $\mathcal{Y}$ be a derived stack. A morphism $\mathcal{X} \to \mathcal{Y}$ is a **Zariski open immersion** if for any $\mathcal{S} \to \mathcal{Y}$ with $\mathcal{S}$ affine the base change $\mathcal{X} \times_{\mathcal{Y}} \mathcal{S} \to \mathcal{S}$ is a Zariski open immersion.

Property 1.2.2.2.10. *Let $\mathcal{X}$ be a 0-truncated algebraic stack. If $i\colon \mathcal{U} \hookrightarrow \mathcal{X}$ is a Zariski open immersion, then $\mathcal{U}$ is also 0-truncated.*

Proof. We may assume that $\mathcal{X}$ is affine, $\mathcal{X} = \operatorname{Spec} A$ with $A \in \mathfrak{Alg}_k$ a discrete ring. Fix an epimorphism $\coprod_\alpha A_\alpha \xrightarrow{\coprod_\alpha p_\alpha} \mathcal{U}$ such that each $ip_\alpha\colon \operatorname{Spec} A_\alpha \to \operatorname{Spec} A$ is a Zariski open immersion. In particular $A \to A_\alpha$ is strong, so A_α is a classical ring. The proof is then finished by induction on the geometricity of $\mathcal{U}$ as in [TV08, Proposition 2.2.4.4 (4)]. $\square$

Proposition 1.2.2.2.11. *[TV08, Corollary 2.2.2.10] Let X be a derived affine scheme. There is an equivalence of ∞-categories from the ∞-category of Zariski open immersions into X to that of Zariski open immersions into $t_0(X)$.*

Corollary 1.2.2.2.12. *[STV15, Proposition 2.1] Let $\mathcal{X}$ be a derived algebraic stack. There is a bijection $\phi_{\mathcal{X}}$ from the set of (equivalence classes of) Zariski open substacks of $t_0(\mathcal{X})$ to that of Zariski open derived substacks of $\mathcal{X}$. For any Zariski open substack $\mathcal{U} \subset t_0(\mathcal{X})$, the diagram of derived stacks*

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook] \arrow[d] & t_0(\mathcal{X}) \arrow[d, "j_{\mathcal{X}}"] \\ \phi_{\mathcal{X}}(\mathcal{U}) \arrow[r, hook] & \mathcal{X} \end{tikzcd} \tag{1.24}$$

is homotopy cartesian.

Proof. We construct $\phi_{\mathcal{X}}$ by defining its action on a Zariski open substack $\mathcal{U}$ of $t_0(\mathcal{X})$ as

$$\phi_{\mathcal{X}}(\mathcal{U})\colon \mathfrak{dAlg}_k^{\leq 0} \in A \rightsquigarrow \mathcal{U}(\pi_0 A) \times^h_{t_0(\mathcal{X})(\pi_0 A)} \mathcal{X}(A) \tag{1.25}$$

where the map $\mathcal{X}(A) \to t_0(\mathcal{X})(\pi_0 A)$ is induced by the map

$$\mathrm{Map}_{\mathfrak{dSt}}(\operatorname{Spec} A, \mathcal{X}) \to \mathrm{Map}_{\mathfrak{St}}(t_0 \operatorname{Spec} A = \operatorname{Spec}^{\mathrm{cl}} \pi_0 A, t_0 \mathcal{X}) \tag{1.26}$$

of the ∞-functor $t_0\colon \mathfrak{dSt} \to \mathfrak{St}$. If A is a discrete ring then, by definition of the truncation, $t_0(\mathcal{X})(A) = \mathcal{X}(A)$ and we recover $\phi_{\mathcal{X}}(\mathcal{U})(A) = \mathcal{U}(A)$, that is $t_0(\phi_{\mathcal{X}}(\mathcal{U})) = \mathcal{U}$, or $\phi_{\mathcal{X}}(\mathcal{U})$ is a derived enhancement of $\mathcal{U}$ (and t_0 is left inverse to $\phi_{\mathcal{X}}$). We see from (1.25) that t_0 is also right inverse to $\phi_{\mathcal{X}}$ because a Zariski open immersion is locally given by a strong morphism of derived rings. Since the truncation functor t_0 commutes with homotopy pullbacks, we find $t_0(\phi_{\mathcal{X}}(\mathcal{U}) \times^h_{\mathcal{X}} t_0(\mathcal{X})) = t_0(\phi_{\mathcal{X}}(\mathcal{U})) \times_{t_0(\mathcal{X})} t_0(t_0\mathcal{X}) = \mathcal{U} \times_{t_0\mathcal{X}} t_0\mathcal{X} = \mathcal{U}$. By the above equivalence of categories, we can then identify the homotopy fibre product $\phi_{\mathcal{X}}(\mathcal{U}) \times^h_{\mathcal{X}} t_0(\mathcal{X})$ with $\mathcal{U}$. □

1.2.2.3 Stable ∞-categories and Grothendieck K-groups

Recall that a zero object is an object that is both initial and final. An ∞-category with a zero object is said to be **pointed**. In this case the zero object will always be denoted 0.

Definition 1.2.2.3.1 (Stable ∞-category)**.** An ∞-category $\mathfrak{A}$ is **stable** if it is pointed, has finite limits and finite colimits, and any square in it is cartesian (a pullback square) if and only if it is cocartesian (a pushout square).

Example 1.2.2.3.2. Let $A \in \mathfrak{dAlg}_k^{\leq 0}$ be a derived ring. The ∞-category $\mathfrak{dMod}_A$ is stable. If A is a discrete ring, then $\mathrm{Ho}(\mathfrak{dMod}_A) = \mathbb{D}^{\leq 0}\mathfrak{Mod}_A$.

A coherent square in a finitely complete and finitely cocomplete pointed ∞-category of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & C \end{tikzcd} \tag{1.27}$$

is called a **triangle**. A triangle is said to be **exact** if it is a cartesian square, and **coexact** if it is cocartesian.

Proposition 1.2.2.3.3. *For a finitely complete and finitely cocomplete pointed ∞-category to be stable, it is sufficient (and necessary!) that a triangle be exact if and only if it is coexact.*

A 2-simplex $A \to B \to C$ in a a finitely complete and finitely cocomplete pointed ∞-category $\mathfrak{A}$ is called a fibre sequence if there is an exact triangle as in eq. (1.27), and a cofibre sequence if there is such a coexact triangle. It follows that, if $\mathfrak{A}$ is stable, then a sequence is exact if and only if it is coexact.

Construction 1.2.2.3.4 (Suspension and desuspension)**.** Let $A \in \mathfrak{A}$ be an object in a finitely complete and finitely cocomplete pointed ∞-category. We define the **loop space object** ΩX of X by the fibre sequence $\Omega X \to 0 \to X$. The **suspension object** ΣX of X is dually defined by the cofibre sequence $X \to 0 \to \Sigma X$.

These assignments give rise to adjoint ∞-functors $\Sigma\colon \mathfrak{A} \rightleftarrows \mathfrak{A}\colon \Omega$.

Proposition 1.2.2.3.5. *A finitely complete and finitely cocomplete pointed ∞-category is stable if and only if the suspension and loop space functors are mutually quasi-inverse.*

If $A \xrightarrow{f} B \xrightarrow{g} C$ is a fibre sequence, we also say that f is a fibre of g, written $f = \mathrm{fib}(g)$. If it is a cofibre sequence, we say that g is a cofibre of f, written $g = \mathrm{cofib}(f)$. This defines adjoint ∞-functors $\mathrm{cofib}\colon \mathfrak{Arr}(\mathfrak{A}) \rightleftarrows \mathfrak{Arr}(\mathfrak{A})\colon\ \mathrm{fib}$, where the arrows ∞-category $\mathfrak{Arr}(\mathfrak{A})$ is $\mathfrak{Fun}(\Delta^1, \mathfrak{A})$.

The fibres and cofibres should be seen as homotopy kernels and cokernels. Note that we will often abuse notation and write $C = \mathrm{cofib}(f)$ if g is, or $A = \mathrm{fib}(g)$ if f is.

Proposition 1.2.2.3.6. *A finitely complete and finitely cocomplete pointed ∞-category is stable if and only if the fibre and cofibre functors are mutually quasi-inverse.*

Suppose $A \to B \to C$ is a (co)fibre sequence in a stable ∞-category. There is an induced morphism $C \to \Sigma A$.

Theorem 1.2.2.3.7. *The homotopy category of a stable category has a triangulated structure induced by the suspension functor and the class of distinguished triangles isomorphic to the triangles of fibre sequences.*

Example 1.2.2.3.8 (Quasicoherent sheaves on a stack). Let $X = \operatorname{Spec} A$ be an affine derived stack. We set $\mathfrak{QCoh}(X) = \mathfrak{dMod}_A^{\leq 0}$. We also define the structure sheaf $\mathcal{O}_X \in \mathfrak{QCoh}(X)$ as $A \in \mathfrak{dMod}_A^{\leq 0}$.

Let now $\mathcal{X}$ be a derived algebraic stack. We define its ∞-category of quasicoherent sheaves as

$$\mathfrak{QCoh}(\mathcal{X}) = \varprojlim_{\operatorname{Spec} A \to \mathcal{X}} \mathfrak{QCoh}(\operatorname{Spec} A), \tag{1.28}$$

which thus comes equipped with ∞-functors $x^*\colon \mathfrak{QCoh}(\mathcal{X}) \to \mathfrak{QCoh}(\operatorname{Spec} A)$ for any $x\colon \operatorname{Spec} A \to \mathcal{X}$. The **structure sheaf** of $\mathcal{X}$ is the unique (up to equivalence) object $\mathcal{O}_\mathcal{X} \in \mathfrak{QCoh}(\mathcal{X})$ such that for any A-point x of $\mathcal{X}$ we have $x^*\mathcal{O}_\mathcal{X} = \mathcal{O}_{\operatorname{Spec} A}$.

Definition 1.2.2.3.9 (Cotangent complex of a stack)**.** Let $\mathcal{X} \in \mathfrak{dSt}(\mathfrak{dAff})$ be a derived stack. Let $x\colon \operatorname{Spec} A \to \mathcal{X}$ be a point, seen also as $x \in \mathcal{X}(A)$. We have an ∞-functor

$$\begin{aligned} \mathcal{D}er(\mathcal{X}_x)\colon \mathfrak{dMod}_A^{\leq 0} &\to \mathfrak{S}, \\ M &\rightsquigarrow \operatorname{hofib}\left(\mathcal{X}(A \oplus M) \to \mathcal{X}(A), x\right) \end{aligned} \tag{1.29}$$

We say that $\mathcal{X}$ admits a cotangent complex at x if there exists $\mathbb{L}^\bullet_{\mathcal{X},x} \in \mathfrak{dMod}_A^{\leq 0}$ corepresenting $\mathcal{D}er(\mathcal{X}_x)$.

We say that $\mathcal{X}$ has a cotangent complex if there is $\mathbb{L}^\bullet_{\mathcal{X}} \in \mathfrak{QCoh}(\mathcal{X})$ such that $x^*\mathbb{L}^\bullet_{\mathcal{X}} = \mathbb{L}^\bullet_{\mathcal{X},x}$ for any $x\colon \operatorname{Spec} A \to \mathcal{X}$.

Definition 1.2.2.3.10 (Grothendieck group)**.** Let $\mathfrak{A}$ be a stable ∞-category. We define its **Grothendieck group** $K_0(\mathfrak{A})$ as the abelian group freely generated on the set $\pi_0(\mathfrak{A})$ of equivalence classes of objects of $\mathfrak{A}$, modulo the relations $[B] - [A] - [C]$ whenever there is a fiber sequence $A \to B \to C$.

If $\mathcal{X}$ is a derived algebraic stack, we denote $K_0(\mathcal{X})$ the Grothendieck group $K_0(\mathfrak{QCoh}(\mathcal{X}))$, called the K-theory of $\mathcal{X}$, and we note $G_0(X)$ the Grothendieck group of the sub-∞-category of perfect complexes, called the G**-theory** of X.

1.3 The virtual sheaf as a shadow of the derived enrichment

1.3.1 Obstruction theories and derived enhancements

1.3.1.1 Obstruction theory induced by a derived structure

Definition 1.3.1.1.1 (Connectivity)**.** A morphism f in $\mathfrak{dAlg}_k^{\leq 0}$ is n**-connective** if $\pi_k(f) = 0$ for all $0 \leq k \leq n$ (where $\pi_k = H^{-k}$). An object A is n-connective if $A \to 0$ is.

Similarly, f is n-coconnective if $\pi_k(f) = 0$ for all $k > n$.

Lemma 1.3.1.1.2 (Connectivity estimate)**.** *[Lur12, Corollary 7.4.3.2] Let* $f\colon A \to B$ *be a morphism in* $\mathfrak{dAlg}_k^{\leq 0}$ *such that* $\operatorname{cofib}(f)$ *is* n*-connective for some* $n \geq 0$*. Then* $\mathbb{L}^\bullet_f$ *is* n*-connective.*

Then from the fact that for any $A \in \mathfrak{dAlg}_k^{\leq 0}$ the canonical morphism $A \twoheadrightarrow \pi_0(A)$ has mapping cone $[A^0/d_A^{-1}(A^{-1}) \twoheadleftarrow A^0 \leftarrow A^{-1} \leftarrow \cdots]$ whose ith homotopy groups $\pi_i = H^{-i}$ vanish for $i < 2$ (and then $\pi_i(\text{cone}) = \pi_{i-1}(A)$ for $A \geq 2$), we obtain the following:

Proposition 1.3.1.1.3. *[STV15, Corollary 1.3] Let* $\mathbb{R}\mathcal{M}$ *be a quasi-smooth derived DM enhancement of a DM stack* $\mathcal{M}$*, and let* $j = j_{\mathbb{R}\mathcal{M}}\colon \mathcal{M} \to \mathbb{R}\mathcal{M}$ *be the associated closed immersion. The induced morphism* $j^*\mathbb{L}^\bullet_{\mathbb{R}\mathcal{M}} \to \mathbb{L}^\bullet_{\mathcal{M}}$ *is a perfect obstruction theory.*

Proof. By the functoriality property of the cotangent complex we have a cofibre sequence $j^*\mathbb{L}^\bullet_{\mathbb{R}\mathcal{M}} \to \mathbb{L}^\bullet_{\mathcal{M}} \to \mathbb{L}^\bullet_{\mathcal{M}/\mathbb{R}\mathcal{M}}$ whose cofibre $\mathbb{L}^\bullet_{\mathcal{M}/\mathbb{R}\mathcal{M}}$ has been shown to be 2-connective. Quasi-smoothness of $\mathbb{R}\mathcal{M}$ means that $\mathbb{L}^\bullet_{\mathbb{R}\mathcal{M}}$ is 1-coconnective, which implies the proposition. □

Hence, by direct application of the construction of section 1.1.2.2, a quasi-smooth derived enhancement of $\mathcal{M}$ induces a virtual structure sheaf in $G_0(\mathcal{M})$.

We now describe another process by which to obtain a virtual sheaf on $\mathcal{M}$ from a derived enhancement.

The structure sheaf $\mathcal{O}_{\mathbb{R}\mathcal{M}}$ induces a family of quasi-coherent sheaves $\pi_i(j^*\mathcal{O}_{\mathbb{R}\mathcal{M}}), i \geq 0$ on $\mathcal{M}$, which by abuse of notation we shall write $\pi_i(\mathcal{O}_{\mathbb{R}\mathcal{M}})$.

Proposition 1.3.1.1.4. *[Toë12] The quasi-coherent $\mathcal{O}_{\mathcal{M}}$-modules $\pi_i(\mathcal{O}_{\mathbb{R}\mathcal{M}})$ are coherent on $\pi_0(\mathcal{O}_{\mathbb{R}\mathcal{M}}) = \mathcal{O}_{\mathcal{M}}$, and only a finite number of them are non-vanishing.*

Lemma 1.3.1.1.5. *[Bar15, Proposition 9.2] Let $j\colon \mathcal{M} \hookrightarrow \mathbb{R}\mathcal{M}$ be a locally nœtherian derived DM enhancement of a DM stack $\mathcal{M}$. Then $j_*\colon G_0(\mathcal{M}) \xrightarrow{\simeq} G_0(\mathbb{R}\mathcal{M})$.*

In fact we have $(j_*)^{-1}[\mathcal{O}_{\mathbb{R}\mathcal{M}}] = \sum_{i\geq 0}(-1)^i[\pi_i(\mathcal{O}_{\mathbb{R}\mathcal{M}})]$.

1.3.1.2 Derived enhancement determined by an obstruction theory

Let $\mathcal{M}$ be a classical Artin stack. Let $\phi\colon \mathcal{E}^\bullet \to \mathbb{L}^\bullet_{\mathcal{M}}$ be a perfect obstruction theory. Recall from section 1.1.2.2 that by [BF97, Proposition 4.5], ϕ induces a closed immersion of cone stacks $\phi^\vee\colon \mathfrak{C}_{\mathcal{M}} \hookrightarrow \mathcal{H}^1/\mathcal{H}^0((\mathcal{E}^\bullet)^{\mathbb{R}\vee}) =: \mathfrak{E}$.

Definition 1.3.1.2.1 (Induced enhancement)**.** The derived Artin stack $\mathbb{R}\mathcal{O}bs(\phi)$ is the homotopy fibre product

$$\begin{tikzcd} \mathbb{R}\mathcal{O}bs(\phi) \arrow[r] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner_h"] & \mathfrak{C}_{\mathcal{M}} \arrow[d, hook, "\phi^\vee"] \\ \mathcal{M} \arrow[r, hook, "\zeta"'] & \mathfrak{E} \end{tikzcd} \tag{1.30}$$

in the ∞-category of derived stacks, where $\zeta\colon \mathcal{M} \to \mathfrak{E}$ is the zero section of the vector bundle stack.

Proposition 1.3.1.2.2. *The derived Artin stack $\mathbb{R}\mathcal{O}bs(\phi)$ is a (non-trivial) derived enhancement of $\mathcal{M}$.*

Proof. Note first that $\mathcal{M}$ also admits a closed embedding into its intrinsic normal cone, so that the fibre product of (classical) Artin stacks $\mathcal{M} \times_{\mathfrak{E}} \mathfrak{C}_{\mathcal{M}}$ is again $\mathcal{M}$. However the inclusion ι of classical stacks into derived stacks does not commute with products, which explains why the homotopy fibre product can be non-trivial.

However, the truncation functor t_0 does commute with limits, so $t_0(\mathbb{R}\mathcal{O}bs(\phi)) = t_0(\mathcal{M}) \times_{t_0(\mathfrak{E})} t_0(\mathfrak{C}_{\mathcal{M}}) = \mathcal{M} \times_{\mathfrak{E}} \mathfrak{C}_{\mathcal{M}} = \mathcal{M}$. □

Let $i\colon \mathcal{M} \hookrightarrow \mathbb{R}\mathcal{O}bs(\phi)$ denote the canonical closed immersion.

Recall that the virtual structure sheaf was defined as $\left[\mathcal{O}^{\mathrm{vir}}_{\mathcal{M},\phi}\right] = \left[\mathcal{O}_{\mathfrak{C}_{\mathcal{M}}|_{\mathcal{E}_1}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{E}_1}} \mathcal{O}_{\mathcal{M}}\right] = \sum_{n\geq 0} \pi_n \left(\left[\mathcal{O}_{\mathfrak{C}_{\mathcal{M}}|_{\mathcal{E}_1}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{E}_1}} \mathcal{O}_{\mathcal{M}}\right]\right)$. This can clearly be identified with $(i_*)^{-1}[\mathcal{O}_{\mathbb{R}\mathcal{O}bs(\phi)}]$ constructed above.

Remark 1.3.1.2.3 (Splitting). Since $\mathcal{M}$, $\mathfrak{C}_{\mathcal{M}}$ and $\mathfrak{E}$ are classical stacks, we may also consider the classical fibre product $\mathcal{M}$ in the category of stacks in groupoids over $\mathfrak{Aff}_k$. It canonically provides a (commutative, hence in particular coherent) cone over the diagram of (1.30), inducing a canonical morphism $\mathcal{M} \to \mathbb{R}\mathcal{O}bs(\phi)$, which is the closed immersion $i\colon \mathcal{M} \hookrightarrow \mathbb{R}\mathcal{O}bs(\phi)$ and makes the total diagram coherent:

$$\begin{tikzcd}
\mathcal{M} \arrow[dr, hook, "i"] \arrow[drr, hook, bend left] \arrow[ddr, equal, bend right] & & \\
& \mathbb{R}\mathcal{O}bs(\phi) \arrow[r] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner h"] & \mathfrak{C}_{\mathcal{M}} \arrow[d, hook, "\phi^\vee"] \\
& \mathcal{M} \arrow[r, hook, "\zeta"'] & \mathfrak{E}
\end{tikzcd} \quad . \tag{1.31}$$

In particular, we see that $r \circ i$ is homotopic to $\mathbb{1}_{\mathcal{M}}$, making r a retract of i. The exact triangle of cotangent complexes associated to i is $i^*\mathbb{L}^\bullet_{\mathbb{R}\mathcal{O}bs(\phi)} \to \mathbb{L}^\bullet_{\mathcal{M}} \to \mathbb{L}^\bullet_i \to i^*\mathbb{L}^\bullet_{\mathbb{R}\mathcal{O}bs(\phi)}[1]$, and the one associated to r is $r^*\mathbb{L}^\bullet_{\mathcal{M}} \to \mathbb{L}^\bullet_{\mathbb{R}\mathcal{O}bs(\phi)} \to \mathbb{L}^\bullet_r \to r^*\mathbb{L}^\bullet_{\mathcal{M}}[1]$. Applying the exact ∞-functor i^* to the latter, we obtain $\mathbb{L}^\bullet_{\mathcal{M}} \to i^*\mathbb{L}^\bullet_{\mathbb{R}\mathcal{O}bs(\phi)} \to i^*\mathbb{L}^\bullet_r \to \mathbb{L}^\bullet_{\mathcal{M}}[1]$. This shows

$$\mathbb{L}^\bullet_{\mathcal{M}} = \mathrm{fib}(i^*\mathbb{L}^\bullet_{\mathbb{R}\mathcal{O}bs(\phi)} \to i^*\mathbb{L}^\bullet_r) \text{ while } \mathbb{L}^\bullet_i[-1] = \mathrm{fib}(i^*\mathbb{L}^\bullet_{\mathbb{R}\mathcal{O}bs(\phi)} \to \mathbb{L}^\bullet_{\mathcal{M}}), \tag{1.32}$$

providing a splitting $i^*\mathbb{L}^\bullet_{\mathbb{R}\mathcal{O}bs(\phi)} \simeq \mathbb{L}^\bullet_i[-1] \oplus \mathbb{L}^\bullet_{\mathcal{M}}$.

Remark 1.3.1.2.4 (Comparison). Let $j\colon \mathcal{M} \to \mathbb{R}\mathcal{M}$ be a quasi-smooth derived enhancement, inducing the perfect obstruction theory $j^\sharp\colon j^*\mathbb{L}^\bullet_{\mathbb{R}\mathcal{M}} \to \mathbb{L}^\bullet_{\mathcal{M}}$. We have the span of closed immersions $\mathbb{R}\mathcal{O}bs(j^\sharp) \overset{i}{\hookleftarrow} \mathcal{M} \overset{j}{\hookrightarrow} \mathbb{R}\mathcal{M}$. As i splits (has a retract) while j need not, the two derived enhancements cannot in general be identified. However we wish to show an equality of the virtual structure sheaves induced in G-theory. By [MR18a, Proposition 4.3.2], this is the case.

1.3.2 Comparison of obstruction theories

Let $\mathcal{X}, \mathcal{Y}$ be classical Artin stacks, with $\mathcal{X}$ a (relative) curve, and let $\mathcal{M}$ be the stack of morphisms from $\mathcal{X}$ to $\mathcal{Y}$. Then we have:

- the canonical obstruction theory as constructed in example 1.1.2.2.3;
- the derived enhancement to the derived mapping stack $\mathbb{R}\mathcal{M}ap(\mathcal{X}, \mathcal{Y})$, defined by right adjointness to the homotopy product.

We wish to compare the two induced virtual sheaves.

Construction 1.3.2.0.1 (Tangent stack)**.** Let $\mathcal{X}$ be a derived Artin stack. We define its **tangent stack** as $T\mathcal{X} = \mathbb{R}\mathcal{M}ap(\operatorname{Spec}(k[\epsilon]), \mathcal{X})$. The natural inclusion $\operatorname{Spec} k \hookrightarrow \operatorname{Spec}(k[\epsilon])$ induces a map $T\mathcal{X} \to \mathcal{X}$ allowing us to view $T\mathcal{X}$ as a derived stack over $\mathcal{X}$.

By [TV08, Proposition 1.4.1.6], if $x\colon \operatorname{Spec} A \to \mathcal{X}$ is an A-point of $\mathcal{X}$ we have

$$\operatorname{Map}_{\mathfrak{dSt}_{/\mathcal{X}}}(\operatorname{Spec} A, T\mathcal{X}) \simeq \operatorname{Map}_{\mathfrak{dMod}_A}(\mathbb{L}^\bullet_{\mathcal{X},x}, A) \simeq \operatorname{Map}_{\mathfrak{dMod}_A}(A, \mathbb{T}^\bullet_{\mathcal{X},x}) \tag{1.33}$$

(which we can of course identify with $\operatorname{Map}_{\mathfrak{QCoh}(\operatorname{Spec} A)}(\mathcal{O}_{\operatorname{Spec} A}, \mathbb{T}^\bullet_{\mathcal{X},x})$). This shows that the A-points of $T\mathcal{X}$ over $\mathcal{X}$ are completely determined by the (co)tangent complex.

Lemma 1.3.2.0.2. *[MR18a, eq. 4.3.4] Let $\mathcal{X}, \mathcal{Y}$ be derived Artin stacks. Consider the universal derived mapping stack diagram*

$$\begin{tikzcd} & \mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y}) \times \mathcal{X} \arrow[dl, "\pi"'] \arrow[dr, "\mathrm{ev}"] & \\ \mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y}) & & \mathcal{Y} \end{tikzcd}, \tag{1.34}$$

where π is the projection and ev *the evaluation map.*

There is a canonical equivalence $\mathbb{T}^\bullet_{\mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y})} \simeq \pi_* \operatorname{ev}^* \mathbb{T}^\bullet_{\mathcal{Y}}$ in $\mathfrak{QCoh}(\mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y}))$.

Proof. We will show that the formula holds at the level of stalks. Let $A \in \mathfrak{dAlg}_k^{\leq 0}$ and $x_f\colon \operatorname{Spec} A \to \mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y})$ be an A-point of $\mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y})$, classifying $f\colon \operatorname{Spec} A \times \mathcal{X} \to \mathcal{Y}$. We must show that $\mathbb{T}^\bullet_{\mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y}),x_f} = (\pi_* \operatorname{ev}^* \mathbb{T}^\bullet_{\mathcal{Y}})_{x_f} = x_f^* \pi_* \operatorname{ev}^* \mathbb{T}^\bullet_{\mathcal{Y}}$.

We apply construction 1.3.2.0.1 to the derived mapping stack. Notice first that

$$\begin{aligned} T\,\mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y}) &:= \mathbb{R}\mathcal{M}ap(\operatorname{Spec} k[\epsilon], \mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y})) \\ &= \mathbb{R}\mathcal{M}ap(X, \mathbb{R}\mathcal{M}ap(\operatorname{Spec} k[\epsilon], \mathcal{Y})) = \mathbb{R}\mathcal{M}ap(\mathcal{X}, T\mathcal{Y}). \end{aligned} \tag{1.35}$$

Hence the space of A-points of $T\,\mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y})$ above $\mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y})$, that is of liftings for the left diagram in eq. (1.36), is by adjunction equivalent to that of liftings in the right diagram:

$$\begin{tikzcd} & T\,\mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y}) \arrow[d] \\ \operatorname{Spec} A \arrow[r, "x_f"'] \arrow[ur, dashed] & \mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y}) \end{tikzcd} \iff \begin{tikzcd} & T\mathcal{Y} \arrow[d] \\ \operatorname{Spec} A \times \mathcal{X} \arrow[r, "f"'] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}. \tag{1.36}$$

By eq. (1.33), this means that, assuming X affine (since the tangent complex satisfies descent)

$$\begin{aligned} \operatorname{Map}_{\mathfrak{dMod}_A}(A, \mathbb{T}^\bullet_{\mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y}),x_f}) &= \operatorname{Map}_{\mathfrak{dSt}_{/\mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y})}}(\operatorname{Spec} A, T\,\mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y})) \\ &= \operatorname{Map}_{\mathfrak{dSt}_{/\mathcal{Y}}}(\operatorname{Spec} A \times \mathcal{X}, T\mathcal{Y}) \\ &= \operatorname{Map}_{\mathfrak{QCoh}(\operatorname{Spec} A \times \mathcal{X})}(\mathcal{O}_{\operatorname{Spec} A \times \mathcal{X}}, \mathbb{T}^\bullet_{\mathcal{Y},f}). \end{aligned} \tag{1.37}$$

Writing $p\colon X \times \operatorname{Spec} A \to \operatorname{Spec} A$ for the projection, we have $p^*A = \mathcal{O}_{\operatorname{Spec} A \times X}$ and thus (by adjunction)

$$\operatorname{Map}_{\mathfrak{dMod}_A}\left(A, \mathbb{T}^\bullet_{\mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y}),x_f}\right) = \operatorname{Map}_{\mathfrak{dMod}_A}\left(A, p_*\mathbb{T}^\bullet_{\mathcal{Y},f}\right) \tag{1.38}$$

where we remind that $\mathbb{T}^\bullet_{\mathcal{Y},f} = f^*\mathbb{T}^\bullet_{\mathcal{Y}}$.

The diagram

$$\begin{tikzcd}
\operatorname{Spec} A \times \mathcal{X} \arrow[r, "x_f \times \mathbb{1}_{\mathcal{X}}"'] \arrow[d, "p"'] \arrow[rr, bend left, "f"] & \mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y}) \times \mathcal{X} \arrow[r, "ev"] \arrow[d, "\pi"] & \mathcal{Y} \\
\operatorname{Spec} A \arrow[r, "x_f"] & \mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y}) &
\end{tikzcd} \tag{1.39}$$

is coherent, so we have $\mathbb{T}^\bullet_{\mathcal{Y},f} = (x_f \times \mathbb{1}_{\mathcal{X}})^* \operatorname{ev}^* \mathbb{T}^\bullet_{\mathcal{Y}}$ and by base change in the left square we obtain

$$p_*\mathbb{T}^\bullet_{\mathcal{Y},f} = p_*(x_f \times \mathbb{1}_{\mathcal{X}})^* \operatorname{ev}^* \mathbb{T}^\bullet_{\mathcal{Y}} = x_f^*\pi_* \operatorname{ev}^* \mathbb{T}^\bullet_{\mathcal{Y}}. \tag{1.40}$$

Finally, we have shown that

$$\operatorname{Map}_{\mathfrak{dMod}_A}\left(A, \mathbb{T}^\bullet_{\mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y}),x_f}\right) = \operatorname{Map}_{\mathfrak{dMod}_A}\left(A, (\pi_* \operatorname{ev}^* \mathbb{T}^\bullet_{\mathcal{Y}})_{x_f}\right). \tag{1.41}$$

By the universal property of the colimit, the cotangent complex satisfies the following descent property: for any A-point $\alpha\colon \operatorname{Spec} A \to \mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y})$ and B-point $\beta\colon \operatorname{Spec} B \to \mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y})$, and any morphism of derived rings $A \to B$ over $\mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y})$, the induced morphism $\mathbb{L}^\bullet_{\mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y}),\alpha} \otimes^{\mathbb{L}}_A B \to \mathbb{L}^\bullet_{\mathbb{R}\mathcal{M}ap(\mathcal{X},\mathcal{Y}),\beta}$ is an equivalence in $\mathfrak{dMod}_B = \mathfrak{QCoh}(\operatorname{Spec} B)$. Hence we can proceed by reduction to the affine case and conclude from eq. (1.41). □

Similarly to the case of remark 1.1.2.2.6, this formula can be recast into the relative context. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are defined above a base derived Artin stack $\mathcal{B}$, so that $\mathbb{R}\mathcal{M}ap_{\mathcal{B}}(\mathcal{X},\mathcal{Y})$ also is. We replace the previous diagrams with

$$\begin{tikzcd}
\mathcal{X} \times_{\mathcal{B}} \mathbb{R}\mathcal{M}ap_{\mathcal{B}}(\mathcal{X},\mathcal{Y}) \arrow[r, "ev"] \arrow[d, "\pi"'] & \mathcal{Y} \arrow[d] \\
\mathbb{R}\mathcal{M}ap_{\mathcal{B}}(\mathcal{X},\mathcal{Y}) \arrow[r] & \mathcal{B}
\end{tikzcd} \tag{1.42}$$

to obtain an equivalence $\mathbb{T}^\bullet_{\mathbb{R}\mathcal{M}ap_{\mathcal{B}}(\mathcal{X},\mathcal{Y})/\mathcal{B}} \simeq \pi_* \operatorname{ev}^* \mathbb{T}^\bullet_{\mathcal{Y}/\mathcal{B}}$. In particular, in the case $\mathcal{Y} = \mathcal{W} \times \mathcal{B}$, where the structure map is the canonical projection to $\mathcal{B}$, we have $\mathbb{T}^\bullet_{\mathbb{R}\mathcal{M}ap_{\mathcal{B}}(\mathcal{X},\mathcal{Y})/\mathcal{B}} \simeq \pi_* \operatorname{ev}^* \mathbb{T}^\bullet_{\mathcal{W}}$.

Corollary 1.3.2.0.3. *Let $\mathcal{C}$ be a (discrete) curve over a discrete algebraic stack $\mathcal{B}$, and let $\mathcal{Y}$ be a discrete algebraic stack over $\mathcal{B}$. The relative perfect obstruction theory*

$$j^*\mathbb{L}^\bullet_{\mathbb{R}\mathcal{M}ap_{\mathcal{B}}(\mathcal{C},\mathcal{Y})/\mathcal{B}} \to \mathbb{L}^\bullet_{\mathcal{M}ap_{\mathcal{B}}(\mathcal{C},\mathcal{Y})/\mathcal{B}} \tag{1.43}$$

coincides with that (for a mapping stack from a relative curve) constructed in example 1.1.2.2.3 and remark 1.1.2.2.6.

Proof. The diagram (1.42) and its truncation (whose maps are denoted with a "0") fit into the commutative diagram

$$
\begin{tikzcd}
\mathcal{C} \times_{\mathcal{B}} \mathcal{M}ap_{\mathcal{B}}(\mathcal{C}, \mathcal{Y}) \arrow[rr, hook, "\hat{\jmath}"] \arrow[dr, "\mathrm{ev}_0"] \arrow[dd, "\pi_0"'] & & \mathcal{C} \times_{\mathcal{B}} \mathbb{R}\mathcal{M}ap_{\mathcal{B}}(\mathcal{C}, \mathcal{Y}) \arrow[dl, "\mathrm{ev}"'] \arrow[dd, "\pi"] \\
& \mathcal{Y} \arrow[dd] & \\
\mathcal{M}ap_{\mathcal{B}}(\mathcal{C}, \mathcal{Y}) \arrow[rr, hook, "j"] \arrow[dr] & & \mathbb{R}\mathcal{M}ap_{\mathcal{B}}(\mathcal{C}, \mathcal{Y}) \arrow[dl] \\
& \mathcal{B} &
\end{tikzcd} . \tag{1.44}
$$

By the relative version of lemma 1.3.2.0.2, we have $j^* \mathbb{T}^{\bullet}_{\mathbb{R}\mathcal{M}ap_{\mathcal{B}}(\mathcal{C},\mathcal{Y})/\mathcal{B}} = j^* \pi_* \,\mathrm{ev}^* \,\mathbb{T}^{\bullet}_{\mathcal{Y}/\mathcal{B}}$ (note that j^* commutes with taking duals, by definition of the structure sheaves in example 1.2.2.3.8). The back square is cartesian (as $\mathcal{C}$ is 0-truncated), so by the base-change formula this gives $\pi_{0,*} \hat{\jmath}^* \,\mathrm{ev}^* \,\mathbb{T}^{\bullet}_{\mathcal{Y}/\mathcal{B}}$. By the commutativity of the upper triangle, we finally obtain $\pi_{0,*} \hat{\jmath}^* \,\mathrm{ev}^* \,\mathbb{T}^{\bullet}_{\mathcal{Y}/\mathcal{B}} = \pi_{0,*} (\mathrm{ev} \circ \hat{\jmath})^* \mathbb{T}^{\bullet}_{\mathcal{Y}/\mathcal{B}} = \pi_{0,*} \,\mathrm{ev}_0^* \,\mathbb{T}^{\bullet}_{\mathcal{Y}/\mathcal{B}}$, which is exactly the construction (1.18). ☐

Remark 1.3.2.0.4. Finally, note also that for any derived enhancement $j \colon \mathcal{M} \hookrightarrow \mathbb{R}\mathcal{M}$ of a discrete algebraic stack, since the Euler characteristic $\chi_{\mathcal{M}} \colon G_0(\mathcal{M}) \to G_0(\operatorname{Spec} k) = \mathbb{Z}$ in G-theory is defined as the direct image along the projection to the point, for any $[\mathcal{F}] \in G_0(\mathbb{R}\mathcal{M})$ the diagram

$$
\begin{tikzcd}
\mathcal{M} \arrow[rr, hook, "j"] \arrow[dr, "a"'] & & \mathbb{R}\mathcal{M} \arrow[dl, "a^{\mathbb{R}}"] \\
& \operatorname{Spec} k &
\end{tikzcd} \tag{1.45}
$$

gives $\chi_{\mathcal{M}} \left((j_*)^{-1}[\mathcal{F}] \right) \coloneqq a_* (j_*)^{-1}[\mathcal{F}] = (a^{\mathbb{R}})_* j_* (j_*)^{-1}[\mathcal{F}] = (a^{\mathbb{R}})_*[\mathcal{F}] \eqqcolon \chi_{\mathbb{R}\mathcal{M}}([\mathcal{F}])$.

Chapter 2

∞-operads and brane actions

2.1 Higher operads

2.1.1 Models for ∞-operads

There are (at least) two ways of thinking of classical (coloured) operads:

- as a "multicategory" $\mathfrak{O}$ with a set of **colours** and "multilinear arrows" from any given sequence of colours to another colour;
- as its **category of operators** $\mathfrak{O}^{\otimes}$, an actual category whose objects are finite sequences of colours of $\mathfrak{O}$.

Definition 2.1.1.0.1 (Coloured operad)**.** A **coloured operad** (or symmetric multicategory) $\mathfrak{O}$ in $\mathfrak{Set}$ is the data of:

- a set $C = C(\mathfrak{O})$ of **colours**,
- for any $n \in \mathbb{F}$ and any collection of colours $c_1, \ldots, c_n, d \in C$, a set $\hom_{\mathfrak{O}}(c_1, \ldots, c_n; d)$ of **multimorphisms**,
- for any colour $c \in C$, a distinguished **identity** morphism $\mathbb{1}_c \in \hom(c; c)$,
- for any collection of colours $(c_1, \ldots, c_n), (d_{1,1}, \ldots, d_{1,k_1}), \ldots, (d_{n,1}, \ldots, d_{n,k_n}), d$, a **composition** map

$$\begin{aligned}&\hom(c_1, \ldots, c_n; d)\times \\ &\qquad \hom(d_{1,1}, \ldots, d_{1,k_1}; c_1) \times \cdots \times \hom(d_{n,1}, \ldots, d_{n,k_n}; c_n) \\ &\to \hom(d_{1,1}, \ldots, d_{n,k_n}; d),\end{aligned} \tag{2.1}$$

- for any n, any collection $c_1, \ldots, c_n, d$ of colours and any permutation $\sigma \in \mathbb{S}_n$, a morphism $\sigma^*\colon \hom(c_1, \ldots, c_n; d) \to \hom(c_{\sigma(1)}, \ldots, c_{\sigma(n)}; d)$

such that the σs form a representation of $\mathbb{S}_n$ and the composition law is associative, unital and $\mathbb{S}_n$-equivariant. The commutative diagrams expressing these conditions are spelled out in[EM06, Definition 2.1].

Example 2.1.1.0.2. Let $(\mathfrak{V}, \otimes)$ be a monoidal category. We can then define a coloured operads whose colours are the objects of $\mathfrak{V}$ with $\mathrm{hom}(v_1, \dots, v_n; w) := \mathrm{hom}_{\mathfrak{V}}(v_1 \otimes \cdots \otimes v_n, w)$. If $(\mathfrak{V}, \otimes) = (\mathfrak{Set}, \times)$ is the category of sets with its cartesian product, we also write $\mathfrak{Set}$ for the resulting coloured operad.

A **multifunctor** $\mathcal{F}\colon \mathfrak{O} \to \mathfrak{P}$ between two coloured operads consists of a function $C(\mathfrak{O}) \to C(\mathfrak{P})$ and for any sequence of colors $c_1, \dots, c_n, d \in C(\mathfrak{O})$, a map $\mathrm{hom}_{\mathfrak{O}}(c_1, \dots, c_n; d) \to \mathrm{hom}_{\mathfrak{P}}(\mathcal{F}c_1, \dots, \mathcal{D}c_n; \mathcal{F}d)$ respecting the compositions, identities, and $\mathbb{S}_n$-actions ([Wei07, Definition 1.1.5]).

Construction 2.1.1.0.3 (Category of operators)**.** To a coloured operad $\mathfrak{O}$, we associate its **category of operators** $\mathfrak{O}^{\otimes}$, endowed with a canonical projection to Γ. An object of $\mathfrak{O}^{\otimes}$ is a family $(O_1, \dots, O_n)$ of n (not necessarily different) colours of $\mathfrak{O}$, for any $\langle n \rangle \in \Gamma$. If $(O_1, \dots, O_m)$ and $(P_1, \dots, P_n)$ are two objects of $\mathfrak{O}^{\otimes}$, a morphism from $(O_1, \dots, O_m)$ to $(P_1, \dots, P_n)$ consists of a map $\alpha\colon \langle m \rangle \to \langle n \rangle$ (to be thought of as selecting the sources of multimorphisms) and for every $i \in \langle n \rangle^{\circ}$ a multimorphism from $(O_j)_{j \in \alpha^{-1}(i)}$ to P_i. Composition is given in the obvious way by composition of the selection maps α and using the composition operation for multimorphisms.

There is a canonical functor $\mathfrak{O}^{\otimes} \to \Gamma$, which at the level of objects maps $(O_1, \dots, O_n) \mapsto \langle n \rangle$ and at the level of morphisms forgets the multimorphisms and remembers only the selection map $\langle n \rangle \mapsto \langle m \rangle$.

2.1.1.1 ∞-operads

Definition 2.1.1.1.1 (Inert and active morphisms)**.**
- A morphism $f\colon \langle m \rangle \to \langle n \rangle$ in Γ is **inert** if for every $i \in \langle n \rangle^{\circ}$, the preimage $f^{-1}(i) \subset \langle m \rangle$ has a single element.
- A morphism $f\colon \langle m \rangle \to \langle n \rangle$ in Γ is **semi-inert** if for every $i \in \langle n \rangle^{\circ}$, the preimage $f^{-1}(i) \subset \langle m \rangle$ has *at most* one element.
- A morphism $f\colon \langle m \rangle \to \langle n \rangle$ in Γ is **active** if $f^{-1}(0) = \{0\}$.

Example 2.1.1.1.2. The morphisms $\rho^n_i\colon \langle n \rangle \to \langle 1 \rangle$ are inert.

Lemma 2.1.1.1.3. *Let $\mathfrak{Cat}^{\mathrm{mult}}_{/\Gamma}$ be the (non-full) subcategory of the slice category $\mathfrak{Cat}_{/\Gamma}$ whose objects are the functors $\mathcal{P}\colon \mathfrak{C} \to \Gamma$ which satisfy the following conditions:*

1. *for any inert morphism $\phi\colon \langle m \rangle \to \langle n \rangle$ in Γ and every object C in the fibre of $\mathcal{P}$ above $\langle m \rangle$, there is an object X^{ϕ} above $\langle n \rangle$ and a $\mathcal{P}$-cocartesian lift $\widetilde{\phi}\colon X \to X^{\phi}$ of ϕ;*
2. *for any $\langle n \rangle \in \Gamma$ the functor $\mathfrak{C}_{\langle n \rangle} \to \mathfrak{C}^n_{\langle 1 \rangle}$ induced by the cocartesian lifts of the ρ^n_i is essentially surjective;*
3. *for every morphism $\phi\colon \langle m \rangle \to \langle n \rangle$ in Γ and any $X \in \mathfrak{C}_{\langle m \rangle}, Y \in \mathfrak{C}_{\langle n \rangle}$, writing $\mathrm{hom}^{\phi}(X, Y)$ for the set of arrows $X \to Y$ lifting ϕ, composition with cocartesian lifts $Y \to Y_i$ of the ρ^n_i induces an isomorphism $\mathrm{hom}^{\phi}(X, Y) \xrightarrow{\sim} \prod_i \mathrm{hom}^{\rho^n_i \circ \phi}(X, Y_i)$;*

and whose morphisms are the functors above $\mathbb{\Gamma}$ which send cocartesian lifts of inert morphisms to cocartesian morphisms. Then the functor $\mathfrak{O} \rightsquigarrow \mathfrak{O}^{\otimes}\colon \mathfrak{Op} \to \mathfrak{Cat}_{/\mathbb{\Gamma}}$ induces an equivalence of categories with $\mathfrak{Cat}^{\mathrm{mult}}_{/\mathbb{\Gamma}}$.

Proof. Here we simply transpose the proof of [GH15] to the symmetric context.

Let us first describe some consequences of the axioms for $\mathfrak{Cat}^{\mathrm{mult}}_{/\mathbb{\Gamma}}$. Fix an inert morphism $\phi\colon \langle m\rangle \to \langle n\rangle$. Consider a morphism $g\colon X \to Y$ in the fiber over $\langle m\rangle$. Since $X \to X^{\phi}$ is cocartesian, the morphism $X \xrightarrow{g} Y \to Y^{\phi}$ uniquely determines a morphism $g^{\phi}\colon X^{\phi} \to Y^{\phi}$ lifting $\mathbb{1}_{\langle n\rangle}$, in a functorial manner, hence ϕ induces a functor $\phi_!\colon \mathfrak{C}_{\langle m\rangle} \to \mathfrak{C}_{\langle n\rangle}$.

It is easy to see that $(-)^{\otimes}$ factors through $\mathfrak{Cat}^{\mathrm{mult}}_{/\mathbb{\Gamma}}$ on objects, *i.e.* that every category of operators of a coloured operad satisfy the required conditions (for $\phi\colon \langle m\rangle \to \langle n\rangle$ inert and $X = (X_1, \dots, X_n) \in \mathfrak{O}^{\otimes}_{\langle n\rangle}$, the object $(X^{\phi}_1, \dots, X^{\phi}_m)$ is determined by $X^{\phi}_i = X_{\phi^{-1}(i)}$ and the morphism $(X_1, \dots, X_n) \to (X^{\phi}_1, \dots, X^{\phi}_m)$ is induced by the projections).

We show that it is essentially surjective. Let $\mathcal{P}\colon \mathfrak{C} \to \mathbb{\Gamma}$ be an object of $\mathfrak{Cat}^{\mathrm{mult}}_{/\mathbb{\Gamma}}$. We construct a coloured operad $\mathfrak{O}_{\mathcal{F}}$. The set of colours of $\mathfrak{O}_{\mathcal{F}}$ is defined as the set of objects of the fibre $\mathfrak{C}_{\langle 1\rangle}$ of $\mathcal{F}$ over $\langle 1\rangle$. Let $O_1, \dots, O_n, P$ be colours of $\mathfrak{O}_{\mathcal{F}}$. Since $\mathfrak{C}_{\langle n\rangle} \to \mathfrak{C}^{n}_{\langle 1\rangle}$ is essentially surjective, the family $(O_1, \dots, O_n)$ is induced by an object $C \in \mathfrak{C}_{\langle n\rangle}$. Let $\alpha_n\colon \langle n\rangle \to \langle 1\rangle$ be the map sending 0 to 0 and everything else to 1 (the unique active map $\langle n\rangle \to \langle 1\rangle$); we set $\hom_{\mathfrak{O}_{\mathcal{F}}}(O_1, \dots, O_n; P) \coloneqq \hom^{\alpha_n}_{\mathfrak{C}}(C, P)$.

Consider now a collection of colours $O_{1,1}, \dots, O_{1,n_1}, \dots, O_{k,n_k}, P_1, \dots, P_k, Q$ and set $m = \sum_k n_k$. Let $\beta^k\colon \langle m\rangle \to \langle k\rangle$ be the map of $\mathbb{\Gamma}$ sending any j such that $\sum_{\ell \le i-1} n_\ell < j \le \sum_{\ell \le i} n_\ell$ to i and 0 to 0; notice that it is active. Using composition with the lifts of the ρ^n_i we can identify (using implicitly the equivalence $\mathfrak{C}^n_{\langle 1\rangle} \simeq \mathfrak{C}_{\langle n\rangle}$):

$$\begin{aligned}&\hom^{\alpha_{n_1}}((O_{1,1}, \dots, O_{1,n_1}); P_1) \times \cdots \times \hom^{\alpha_{n_k}}((O_{k,1}, \dots, O_{k,n_k}); P_k) \times \hom^{\alpha_k}((P_1, \dots, P_k); Q)\\ &\simeq \hom^{\beta_k}((O_{1,1}, \dots, O_{k,n_k}); (P_1, \dots, P_k)) \times \hom^{\alpha_k}((P_1, \dots, P_k); Q)\end{aligned} \tag{2.2}$$

and it follows that composition of morphisms in $\mathfrak{C}$ defines the composition operation $\hom(O_{1,1}, \dots, O_{1,n_1}; P_1) \times \cdots \times \hom(O_{k,1}, \dots, O_{k,n_k}; P_k) \times \hom(P_1, \dots, P_k; Q)$ in $\mathfrak{O}_{\mathcal{F}}$, whence we also get the associativity and unitality of composition.

Let $\sigma\colon \langle n\rangle \to \langle n\rangle$ be induced by a permutation in $\mathbb{S}_n$; in particular it is both active and inert. An object $X \simeq (X_1, \dots, X_n) \in \mathfrak{C}_{\langle n\rangle}$ will be sent by $\sigma_!$ to the object $X^{\sigma} = (X^{\sigma}_1, \dots, X^{\sigma}_n) \in \mathfrak{C}_{\langle n\rangle}$ such that $X^{\sigma}_i = \rho^n_{i,!}X^{\sigma} = \rho^n_{i,!}\sigma_! X = (\rho^n_i\sigma)_! X$ (it is clear that $(-)_!$ is functorial), from which it follows that $X^{\sigma}_i = X_{\sigma(i)}$, hence the $\mathbb{S}_n$ actions on the sets of multimorphisms from X.

We have shown that $\mathfrak{O}_{\mathcal{F}}$ is a coloured operad, and it is easy to verify that its category of operators is indeed isomorphic (over $\mathbb{\Gamma}$) to $\mathfrak{C}$.

It now remains to check that the functor $(-)^{\otimes}$ is fully faithful. A multifunctor $\mathcal{G}\colon \mathfrak{O} \to \mathfrak{P}$ induces a functor $\mathcal{G}^{\otimes}\colon \mathfrak{O}^{\otimes} \to \mathfrak{P}^{\otimes}$ on the categories of operators, clearly defined over $\mathbb{\Gamma}$, and by the above description of the cocartesians lifts of inert morphisms, the condition of preserving them for $\mathcal{G}^{\otimes}$ is equivalent to the condition that $\mathcal{G}$ respect the sources of multimorpihsms, which is the definition of a multifunctor. Thus $(-)^{\otimes}$ also does factor through $\mathfrak{Cat}^{\mathrm{mult}}_{/\Gamma}$. Conversely, if we have a morphism $\mathcal{F}\colon \mathfrak{C} \to \mathfrak{D}$ in $\mathfrak{Cat}^{\mathrm{mult}}_{/\Gamma}$, the decompositions of objects allow us to see that $\mathcal{F}$ defines a multifunctor, which shows fullness of $(-)^{\otimes}$ and in fact determines an inverse operation for its action on morphisms. Thus $(-)^{\otimes}$ is an equivalence of categories. □

We can now simply adapt the categorical redefinition of coloured operads to the ∞-categorical setting.

Definition 2.1.1.1.4 (∞-operad)**.** An ∞**-operad** is an ∞-functor $\mathcal{P}\colon \mathfrak{O}^{\otimes} \to \Gamma$ such that:

1. For every inert morphism $f\colon \langle m\rangle \to \langle n\rangle$ in Γ and every object $X \in \mathfrak{O}^{\otimes}_{\langle m\rangle}$ there exists a $\mathcal{P}$-cocartesian lift $\widetilde{f}\colon X \to X'$ of f, so that f induces an ∞-functor $f_!\colon \mathfrak{O}^{\otimes}_{\langle m\rangle} \to \mathfrak{O}^{\otimes}_{\langle n\rangle}$.
2. For $X \in \mathfrak{O}^{\otimes}_{\langle m\rangle}$, $Y \in \mathfrak{O}^{\otimes}_{\langle n\rangle}$, and $f\colon \langle m\rangle \to \langle n\rangle$ a morphism in Γ, let $\mathrm{Map}^{f}_{\mathfrak{O}^{\otimes}}(X, Y)$ be the union of the connected components of $\mathrm{Map}_{\mathfrak{O}^{\otimes}}(X, Y)$ mapped to f by $\mathcal{P}$. For any $1 \leq i \leq n$, choose a $\mathcal{P}$-cocartesian lift $Y \to Y_i$ of $\rho^n_i\colon \langle n\rangle \to \langle 1\rangle$. Then the induced map $\mathrm{Map}^{f}_{\mathfrak{O}^{\otimes}}(X, Y) \to \prod_{1\leq i\leq n} \mathrm{Map}^{\rho^n_i\circ f}_{\mathfrak{O}^{\otimes}}(X, Y_i)$ is a homotopy equivalence.
3. For every $C_1, \ldots, C_n \in \mathfrak{O}^{\otimes}_{\langle 1\rangle}$, there exists $X \in \mathfrak{O}^{\otimes}_{\langle n\rangle}$ and a collection of $\mathcal{P}$-cocartesian lifts $X \to C_i$ of the ρ^n_i.

We say that an ∞-operad is **monochromatic** if it is equipped with an essentially surjective ∞-functor $[0] \to \mathfrak{O}^{\otimes}_{\langle 1\rangle}$.

Remark 2.1.1.1.5 (Interpretation). Let $\mathfrak{O}^{\otimes} \to \Gamma$ be an ∞-operad. By [Lur12, Remark 2.1.1.15], the functors $\rho^n_{i,!}$ induce an equivalence $\mathfrak{O}^{\otimes}_{\langle n\rangle} \simeq \mathfrak{O}^{\otimes}_{\langle 1\rangle}$. We call ${\mathfrak{O}^{\otimes}_{\langle 1\rangle}}^{n}$ the **underlying ∞-category** of $\mathfrak{O}^{\otimes}$, written $\mathfrak{O}$. Thus for any $X \in \mathfrak{O}^{\otimes}_{\langle n\rangle}$ we have $X_i = \rho^n_{i,!}(X) \in \mathfrak{O}$, and we shall write $X =: X_1 \oplus \cdots \oplus X_n$. The union of the connected components of the space $\mathrm{Map}_{\mathfrak{O}^{\otimes}}(X_1 \oplus \cdots \oplus X_n, Y)$ which are mapped to the unique active morphism $\alpha_n\colon \langle n\rangle \to \langle 1\rangle$ is called the space of multimorphisms from $X_1, \ldots, X_n$ to Y. The composition operation is obtained as in the proof of lemma 2.1.1.1.3.

If $\mathfrak{O}^{\otimes}$ is monochromatic, it has essentially one single colour $C \in \mathfrak{O}$. We then denote $\mathfrak{O}(n)$ the space of multimorphisms from n copies of C to C. This allows us to treat monochromatic ∞-operads similarly to topological operads.

Example 2.1.1.1.6. By [Lur12, Remark 2.1.2.19], a symmetric monoidal ∞-category is an ∞-operad.

Let $\mathfrak{O}$ be a classical (coloured) operad. Then the nerve of its category of operators, with its natural projection, is an ∞-operad. We obtain the same result by taking the homotopy coherent nerve of a topological or simplicial operad.

Example 2.1.1.1.7 (The little k-disks operads $\mathcal{E}^{\otimes}_k$). We define a monochromatic topological operad $\mathcal{E}_k$ in the following way. For any $n \in \mathbb{N}$, the topological space $\mathcal{E}_k(n)$ is the configuration space of n disjoint k-dimensional disks inside the unit k-sphere. Composition is given by insertion of disks and forgetting the surrounding one. The little k-disks ∞-operad is the homotopy coherent nerve of its category of operators, which we call $\mathcal{E}_k^{\otimes}$.

A direct (and rigorous) construction of a topological category equivalent to $\mathcal{E}_k^{\otimes}$, the little k-cubes operad, is given as [Lur12, Definition 5.1.0.2].

Remark 2.1.1.1.8. By [Lur12, Corollary 5.1.1.5], the ∞-operad $\mathcal{E}_\infty^{\otimes} \coloneqq \varinjlim(\mathcal{E}_0^{\otimes} \to \mathcal{E}_1^{\otimes} \to \mathcal{E}_2^{\otimes} \to \cdots)$ is equivalent to the commutative ∞-operad Γ (and similarly the associative ∞-operad and the $\mathcal{A}_\infty$-operad coincide). This is an example of the difference between ∞-categorical constructions and their presentations: we do not have to differentiate between "strict" commutativity and lax commutativity (or associativity). In other words, since we are only working with fibrant objects, there is not need to resolve our algebraic operads.

Definition 2.1.1.1.9. If $\mathfrak{O}^{\otimes} \to \Gamma$ is an ∞-operad, we will say that a morphism f in $\mathfrak{O}^{\otimes}$ is **inert** if it is cocartesian and its projection is inert in Γ.

A morphism f in $\mathfrak{O}^{\otimes}$ is **active** if its projection is an active morphism in Γ.

A morphism f in $\mathfrak{O}^{\otimes}$ is **semi-inert** if its projection is semi-inert in Γ and for any inert morphism g composable with f in $\mathfrak{O}^{\otimes}$, the composite $g \circ f$ is inert in $\mathfrak{O}^{\otimes}$ when its projection is inert in Γ.

Definition 2.1.1.1.10 (Map of ∞-operads)**.**

- Let $\mathfrak{O}^{\otimes} \to \Gamma$ and $\mathfrak{P}^{\otimes} \to \Gamma$ be two ∞-operads. An ∞**-operad map** from $\mathfrak{O}^{\otimes}$ to $\mathfrak{P}^{\otimes}$ is an ∞-functor above Γ carrying inert morphisms in $\mathfrak{O}^{\otimes}$ to inert morphisms in $\mathfrak{P}^{\otimes}$.

  The ∞-category of ∞-operad maps, denoted $\mathfrak{Alg}_{\mathfrak{O}}(\mathfrak{P})$, is the full sub-$\infty$-category of $\mathfrak{Fun}_{\Gamma}(\mathfrak{O}^{\otimes}, \mathfrak{P}^{\otimes})$ spanned by the ∞-functors which are maps of ∞-operads.

  We let $\mathfrak{Op}_\infty$ denote the ∞-category of ∞-operads (though, from the above discussion, it should be extended to an ∞-bicategory).

- Let $\mathfrak{V}^{\otimes}$ be a symmetric monoidal ∞-category. An $\mathfrak{O}^{\otimes}$-algebra in $\mathfrak{V}^{\otimes}$ is a map of ∞-operads from $\mathfrak{O}^{\otimes}$ to $\mathfrak{V}^{\otimes}$.

Example 2.1.1.1.11. In the spirit of remark 2.1.1.1.5, let us give an explicit description of a map of monochromatic ∞-operads. Let $\mathfrak{O}^{\otimes} \to \Gamma$ and $\mathfrak{P}^{\otimes} \to \Gamma$ be monochromatic ∞-operads with respective colours O and P, and let $\mathcal{F} \colon \mathfrak{O}^{\otimes} \to \mathfrak{P}^{\otimes}$ be a map of ∞-operads between them. Preservation of inert morphisms means that $\mathcal{F}$ preserves the colour decompositions and the $\mathbb{S}_n$-actions. Since it is defined over Γ, the functor $\mathcal{F}$ sends the active morphisms of $\mathfrak{O}^{\otimes}$ to active morphisms in $\mathfrak{P}^{\otimes}$, hence it determines maps of spaces $\mathfrak{O}(n) \to \mathfrak{P}(n)$ for all n.

Recall the composition map $\mathfrak{O}(n_1) \times \cdots \times \mathfrak{O}(n_k) \times \mathfrak{O}(k) \to \mathfrak{O}(n_1 + \cdots + n_k)$, induced by $\beta_k \colon \langle n_1 + \cdots + n_k \rangle \to \langle k \rangle$, that is by composition $\hom^{\beta_k}(O^{\oplus n_1 + \cdots + n_k}, O^{\oplus k}) \times \hom^{\alpha_k}(O^{\oplus k}, O) \xrightarrow{\circ} \hom^{\alpha_k \beta_k}(O^{\oplus n_1 + \cdots + n_k}, O)$, where α_k and β_k are active. The functor $\mathcal{F}$ will therefore produce coherent squares

$$\begin{tikzcd} \mathfrak{O}(n_1) \times \cdots \times \mathfrak{O}(n_k) \times \mathfrak{O}(k) \arrow[r, "\mathcal{F}"] \arrow[d, "\circ"] & \mathfrak{P}(n_1) \times \cdots \times \mathfrak{P}(n_k) \times \mathfrak{P}(k) \arrow[d, "\circ"] \\ \mathfrak{O}(n_1 + \cdots + n_k) \arrow[r, "\mathcal{F}"] & \mathfrak{P}(n_1 + \cdots + n_k) \end{tikzcd} . \tag{2.3}$$

2.1.1.2 Quasi-operads

Construction 2.1.1.2.1 (Category of rooted trees)**.** We shall call **tree** a loop-free non-empty connected finite (non-planar) graph. A **rooted** tree is a tree with a distinguished outer vertex called the output and a (possibly empty) set of outer vertices called the inputs, where a vertex is called outer if it is only attached to one edge.

Any rooted tree T determines a symmetric coloured operad $\Omega(T)$, whose colours are the edges of T. A vertex with output edge d and input edges $c_1, \dots, c_n$ defines an element in $\hom(c_1, \dots, c_n; d)$ and composition is given by grafting of trees (while the symmetry action is obviously given by permutations of the edges).

The category of trees $\mathbf{\Omega}$ is the category whose objects are trees with morphisms between two trees being the morphisms between the operads they generate. In other words, $\mathbf{\Omega}$ is the full subcategory of the category of (symmetric) coloured operads spanned by the operads generated by trees.

A **dendroidal object** in a category $\mathfrak{C}$ is a $\mathfrak{C}$-valued presheaf on $\mathbf{\Omega}$. The category of dendroidal objects is denoted $\mathfrak{d}\mathfrak{C} = \mathfrak{C}^{\mathbf{\Omega}^{\mathrm{op}}}$.

Remark 2.1.1.2.2 (Faces and degeneracies). Just as in the simplex category $\mathbf{\Delta}$, the morphisms in the tree category $\mathbf{\Omega}$ are generated by two classes of elementary morphisms, which we now describe.

Face maps Let T be a tree with an inner edge e from a vertex v to a vertex w. Write T/e for the tree obtained by contracting e (and identifying v and w as a unique vertex u). There is a natural morphism $T/e \to T$, which is the identity on the unaffected components and sends u to the operadic partial composition $w \circ_e v$. This type of morphism is called an **inner face map**. Let T_1 denote the subtree with e as output edge and T_2 the complementary subtree with e as an input edge; the face map can be interpreted as factoring through the gluing of T_1 and T_2 along e, which we may write informally as “$T_1 \amalg T_2$” (since the trees are required to be connected such “coproducts” will not actually exist in $\mathbf{\Omega}$, so this form is nothing but a useful abuse of notation, which can be given meaning by completing to the $(\infty$-)category of presheaves, especially with the Segal condition below).

Let T be a tree with a vertex v that has only one inner edge attached to it (and any number of outer edges, including possibly zero). Write T/v for the tree obtained by removing v and its outer edges. There is once again a natural map $T/v \to T$ which is the identity on all elements of T/v. These types of morphisms are called **outer face maps**.

There is another special case of outer face maps. Denote T_n, called the **corolla with n leaves**, for the tree with one vertex and $n+1$ outer edges, the last of which, representing the output, shall be forgotten. Then we may remove the unique vertex and obtain the tree I with one edge and no vertex; a face map $I \to T_n$ corresponds to the choice of an edge in T_n.

Degeneracy maps Let T be a tree containing a vertex v with one ingoing edge e_i and outgoing edge e_f. Write $T \setminus v$ for tree obtained by removing v and joining e_i and

e_f to a single vertex e. There is then a map $T \to T \setminus v$ which is the identity on components unaffected by the construction, sends both colours (edges) e_i and e_f to e and sends the vertex v (seen as a multimorphism from e_i to e_f) to $\mathbb{1}_e$. Such a morphism is called a **degeneracy map**.

By [MT10], any morphism in Ω factors as a composition of degeneracy maps followed by an isomorphism followed by a composition of face maps.

Definition 2.1.1.2.3 (Boundaries and horns)**.** Let $T \in \Omega$ be a tree and $\phi \colon S \to T$ be a face map. The **ϕ-face** of the representable $\Omega[T]$ is the sub-dendroidal set of $\Omega[T]$ generated by the image of the induced natural transformation $\Omega[\phi]$. It is denoted $\partial_\phi \Omega[T]$.

The **boundary** $\partial\Omega[T]$ of the representable $\Omega[T]$ is the union of all the faces of $\Omega[T]$.

The **ϕ-horn** of $\Omega[T]$ is the sub-dendroidal set of $\partial\Omega[T]$ given by the union of all the faces not equal to $\partial_\phi \Omega[T]$. The horn is said to be an **inner horn** if ϕ is an inner face map.

Definition 2.1.1.2.4 (Quasi-operad)**.** A **quasi-operad** is a dendroidal set having the right lifting property for all inner horn inclusions.

Construction 2.1.1.2.5 (Segal conditions for operads)**.** Since any Reedy categroy (see definition A.1.2.1.1.1) must be skeletal with no non-trivial automorphism, the category Ω cannot be Reedy. For this reason, [MT10, Part I, Definition 5.3.1] introduces a notion of generalised Reedy category, relaxing the requirements of a Reedy category to allow for isomorphisms. It is shown in [MT10, Part I, Theorem 5.4.5] that there is still a generalised Reedy model structure on the category $\mathfrak{M}^{\mathfrak{R}}$ of functors from a generalised Reedy category $\mathfrak{R}$ to a cofibrantly generated model category $\mathfrak{M}$. In particular[MT10, Example 5.3.3 (v)], the category Ω is generalised Reedy, so $\mathfrak{d}\mathfrak{M}$ has an induced model structure.

Since the category $\mathfrak{M}$ has all colimits, it is tensored over $\mathfrak{Set}$, with the tensor or copower of an object M by a set S given by a coproduct of copies of M indexed by S. This induces a functor $\mathfrak{Set} \to \mathfrak{M}$, taking copowers of the final object $*$, which is strong monoidal for the cartesian monoidal structures (more generally, we could endow $\mathfrak{M}$ with any monoidal structure and replace $*$ by the unit). This in turn allows us to view objects of $\mathfrak{d}\mathfrak{Set}$ as objects of $\mathfrak{d}\mathfrak{M}$, which we will do implicitly. By [MT10, Part I, § 4.2], there is a closed monoidal structure on $\mathfrak{d}\mathfrak{Set}$, which the strong monoidal fuctor $\mathfrak{Set} \to \mathfrak{M}$ also extends to a closed monoidal structure on $\mathfrak{d}\mathfrak{M}$, with internal homs denoted $\underline{\mathrm{hom}}$.

Let $T \in \Omega$ be a tree with at least one vertex. The **Segal core** $S[T]$ of the representable dendroidal set $\Omega[T]$ is the subobject given by the union of all the corollas in T (found at the vertices and given by their inputs); it is a coproduct of representables. If T is the tree with no vertex, we set $S[T] = \Omega[T]$. We also define the dendroidal set J as obtained from the category $[0 \xrightarrow{\simeq} 1]$.

Let $\mathfrak{M}$ be a cofibrantly generated monoidal model category. A dendroidal object $X \in \mathfrak{d}\mathfrak{M}$ is said to be a **dendroidal Segal object** if for any tree T the map $\underline{\mathrm{hom}}(\Omega[T], X) \to$

$\underline{\mathrm{hom}}(S[T], X)$ is a generalised Reedy weak equivalence in $\mathfrak{dM}$. We say furthermore that X is a **complete dendroidal Segal object** if it is Segal and in addition the map $\underline{\mathrm{hom}}(J, X) \to \underline{\mathrm{hom}}(\{0\}, X) = X$ is a weak equivalence.

When $\mathfrak{M}$ is the category of simplicial sets with the standard Kan–Quillen model structure (and cartesian closed monoidal structure), we call **Segal operad** a complete dendroidal Segal space.

Lemma 2.1.1.2.6. *[MT10]*

- *[MT10, Part I, Proposition 8.4.2] There is a model structure on the category of dendroidal sets whose fibrant objects are exactly the quasi-operads.*
- *[MT10, Part I, Theorem 5.6.3] There is a model structure on the category of dendroidal spaces whose fibrant objects are exactly the Segal operads.*
- *[MT10, Part I, Theorem 5.6.4] These two model categories are Quillen equivalent.*

Theorem 2.1.1.2.7. *[HHM16, Corollary 2.5.4] The theories of quasi-operads with no nullary dendrices and of unital ∞-operads (with no nullary operation, see definition 2.2.1.4.1) are equivalent in the following sense: there exist simplicial model categories encoding quasi-operads and ∞-operads, and a zigzag of Quillen equivalences between them.*

We also give a result allowing us to speak of higher operads in the more natural language of simplicial operads.

Proposition 2.1.1.2.8. *[CM13, Theorem 8.15] There is a simplicial model structure on the category of simplicial operads which is Quillen equivalent to the model structure for quasi-operads on $\mathfrak{dSet}$.*

2.1.2 Variants

2.1.2.1 Operads in a stack ∞-topos

Let $\mathfrak{T} = \mathfrak{Sh}_\tau(\mathfrak{C})$ be a stack ∞-topos. We wish to study operads in $\mathfrak{T}$. Note that if $\mathfrak{C}$ is the point category with τ its unique topology, so that $\mathfrak{T} = \mathfrak{S}$ is the category of spaces, then an operad in $\mathfrak{T}$ will be an ∞-operad as described above. Further, by theorem 2.1.1.2.7 and lemma 2.1.1.2.6, ∞-operads can be modelled as Segal operads, which are functors $\Omega^{\mathrm{op}} \to \mathfrak{S} = \mathfrak{T}$ satisfying the Segal condition.

Definition 2.1.2.1.1 ($\mathfrak{T}$-operad)**.** The ∞-category of **operads in the stack ∞-topos** $\mathfrak{T}$ is the ∞-category

$$\mathfrak{Op}_\infty(\mathfrak{T}) \coloneqq \mathfrak{Fun}^{\mathrm{Segal}}(\Omega^{\mathrm{op}}, \mathfrak{T}) \tag{2.4}$$

of ∞-functors from (the nerve of) the category of trees to $\mathfrak{T}$ satisfying the Segal condition.

Proposition 2.1.2.1.2. *The datum of an operad in $\mathfrak{T} = \mathfrak{Sh}_\tau(\mathfrak{C})$ is equivalent to that of an $\mathfrak{Op}_\infty$-valued sheaf on $(\mathfrak{C}, \tau)$.*

Proof. We have the chain of equivalences

$$\begin{aligned}\mathfrak{Fun}^{\mathrm{Segal}}(\mathbf{\Omega}^{\mathrm{op}}, \mathfrak{T}) &\simeq \mathfrak{Fun}^{\mathrm{Segal}}(\mathbf{\Omega}^{\mathrm{op}}, \mathfrak{Fun}^{\tau}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{G})) \\ &\simeq \mathfrak{Fun}^{\mathrm{Segal},\tau}(\mathbf{\Omega}^{\mathrm{op}} \times \mathfrak{C}^{\mathrm{op}}, \mathfrak{G}) \\ &\simeq \mathfrak{Fun}^{\tau}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Fun}^{\mathrm{Segal}}(\mathbf{\Omega}^{\mathrm{op}}, \mathfrak{G})) =: \mathfrak{Sh}_{\tau}(\mathfrak{C}, \mathfrak{Op}_{\infty}).\end{aligned} \tag{2.5}$$

□

Corollary 2.1.2.1.3. *The ∞-category of operads in $\mathfrak{T}$ is equivalent to that of limit-preserving ∞-functors from $\mathfrak{T}^{\mathrm{op}}$ to $\mathfrak{Op}_{\infty}$.*

Proof. By the exactness properties of ∞-categories of presheaves[Lur09, Theorem 5.1.5.6] extended to sheaves by [Lur09, Proposition 5.5.4.20] we have

$$\mathfrak{Fun}^{\tau}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Op}_{\infty}) \simeq \mathfrak{Fun}^{\mathrm{colim}}(\mathfrak{Sh}_{\tau}(\mathfrak{C}^{\mathrm{op}}), \mathfrak{Op}_{\infty}) \simeq \mathfrak{Fun}^{\mathrm{lim}}(\mathfrak{T}^{\mathrm{op}}, \mathfrak{Op}_{\infty}), \tag{2.6}$$

where the last equivalence is by [Lur09, Proposition 5.2.6.2]. □

We now restrict our attention exclusively to sheaves of ∞-operads on $\mathfrak{C}$ taking values in unital monochromatic ∞-operads, *i.e.* sheaves of ∞-operads $\mathcal{O}^{\otimes}$ on $(\mathfrak{C}, \tau)$ such that for every $Z \in \mathfrak{C}$ the ∞-operad $\mathcal{O}^{\otimes}(C)$ is monochromatic and unital, with colour C_Z.

Let us give an explicit description of such operads. Let $\mathcal{O}^{\otimes} \in \mathfrak{Sh}_{\tau}(\mathfrak{C}, \mathfrak{Op}_{\infty})$ be a sheaf of unital monochromatic ∞-operads, and write $\mathcal{O}^{\otimes}_{\mathbf{\Omega}} \in \mathfrak{Fun}^{\mathrm{Segal}}(\mathbf{\Omega}^{\mathrm{op}}, \mathfrak{T})$ for the corresponding complete dendroidal Segal object of $\mathfrak{T}$. For every tree $T \in \mathbf{\Omega}$, we obtain a sheaf of ∞-groupoids $\mathcal{O}^{\otimes}_{\mathbf{\Omega}}(T)$ on $\mathfrak{C}$.

In particular, the corolla with n leaves T_n gives the sheaf $\mathcal{O}_n := \mathcal{O}^{\otimes}_{\mathbf{\Omega}}(T_n)$ which, by the Segal condition, sends $Z \in \mathfrak{C}$ to $\mathrm{Map}_{\mathcal{O}^{\otimes}(Z)_{\mathrm{act}}}(C_Z^{\oplus n}, C_Z) = \mathcal{O}^{\otimes}(Z)(n)$. By the Yoneda lemma after embedding $\mathfrak{T}$ in the larger ∞-category $\mathfrak{PSh}(\mathfrak{C})$, this space is also identified with $\mathrm{Map}_{\mathfrak{PSh}(\mathfrak{C})}(Z, \mathcal{O}_n)$. Using this functorial identification and the fact $\mathfrak{PSh}(\mathfrak{C})$ is generated by representables under colimits, the composition operations $\mathcal{O}^{\otimes}(-)(k) \times \mathcal{O}^{\otimes}(-)(i_1) \times \cdots \times \mathcal{O}^{\otimes}(-)(i_k) \to \mathcal{O}^{\otimes}(-)(i_1 + \cdots + i_k)$ furnish maps $\mathcal{O}_k \times \mathcal{O}_{i_1} \times \cdots \times \mathcal{O}_{i_k} \to \mathcal{O}_{i_1 + \cdots + i_k}$ between the underlying presheaves (which, since $\mathfrak{T} \hookrightarrow \mathfrak{PSh}(\mathfrak{C})$ is fully faithful, are maps of sheaves), showing that we can think of the ∞-operad $\mathcal{O}^{\otimes}$ in $\mathfrak{T}$ using the classical language of operads enriched in the ∞-topos $\mathfrak{T}$.

2.1.2.2 Graded ∞-operads

Let B be a monoid (in sets) with indecomposable zero (that is, if $\beta_1 + \beta_2 = 0$ then $\beta_1 = \beta_2 = 0$).

Lemma 2.1.2.2.1. *[MR18a, Proposition 2.3.2] Let Γ^B be the category whose*

objects *are those of Γ,*

morphisms *from $\langle m \rangle$ to $\langle n \rangle$ are pairs (f, β) of an arrow $f\colon \langle m \rangle \to \langle n \rangle$ in Γ and a function $\beta\colon \langle n \rangle^{\circ} \to B$,*

composition *of* $(f,\beta)\colon \langle m\rangle \to \langle n\rangle$ *and* $(g,\gamma)\colon \langle n\rangle \to \langle p\rangle$ *is* $(g\circ f, \gamma\circ\beta)$ *where* $\gamma\circ\beta\colon \langle p\rangle^\circ \to B$ *is given by*

$$(\gamma\circ\beta)(i) = \begin{cases} \lambda(i) & g^{-1}(i)=\emptyset \\ \lambda(i) + \sum_{j\in g^{-1}(i)}\beta(j) & \textit{else.} \end{cases} \tag{2.7}$$

The ∞*-functor* $N(\Gamma^B) \to N(\Gamma)$ *induced by forgetting the* B*-grading is an* ∞*-operad.*

Definition 2.1.2.2.2 (Graded ∞-operad)**.** A B-graded ∞-operad is a map of ∞-operads $\mathfrak{O}^\otimes \to \Gamma^B$.

2.2 Brane actions

2.2.1 Algebras in correspondances

2.2.1.1 Cobordisms in the operad $\mathcal{E}_2^\otimes$

As motivation to understand the brane action, we treat the example of the little 2-disks ∞-operad $\mathcal{E}_2^\otimes$, whose space of n-ary operations is the configuration space of n disjoint disks in the unit disk. In particular, observe the space $\mathcal{E}_2(2)$ is homotopy equivalent to the circle S^1.

For simplicity, we will consider $\mathcal{E}_2$ as an operad in the category $\mathfrak{S}$ of spaces, whose monoidal structure is given by the cartesian product. Hence an $\mathcal{E}_2$-algebra X in $\mathfrak{S}^\times$ is given by the data of, for each $\sigma\in\mathcal{E}_2(n)$, a continuous map $X^n \to X$.

Recall the following:

Definition 2.2.1.1.1 (Cobordism)**.** Let X, Y be smooth oriented $(k-1)$-manifolds without boundary. A **cobordism** from X to Y is an oriented k-manifold Σ with boundary such that $\partial\Sigma = \overline{X}\sqcup Y$, where $\overline{X}$ means X with the opposite orientation.

Let $\sigma\in\mathcal{E}_2(n)$ be a configuration of n disks. Then σ defines a cobordism from $\coprod_n S^1$ to S^1, as a "pair of pants with n legs". Indeed, we join the n copies of S^1 to the boundaries of the n little disks determined by σ, and we join the target S^1 to the boundary of the unit disk, to obtain the required cobordism.

Since the disk is contractible, the space of insertions of an additional disk into the configuration σ is homotopy equivalent to σ itself (that is, to the unit disk with the little disks removed). This is equivalently the space of configurations σ' of $n+1$ disks such that forgetting the last disk gives back σ, denoted $\mathrm{Ext}(\Sigma)$. Finally, we note that this space is also homotopy equivalent to a wedge $\bigvee^n S^1$ of n circles.

Let X be a topological space. Applying the functor $\mathrm{Map}(-,X)$ (which turns coproducts to products) to the cobordism constructed above we obtain a structure of $\mathcal{E}_2^\otimes$-algebra on the loop space $\mathrm{Map}(S^1,X)$.

2.2.1.2 Categories of correspondences

Let $\mathfrak{C}$ be a category with finite limits. We can form a 2-category $\mathfrak{Span}(\mathfrak{C})$ of **correspondences** or **spans** in $\mathfrak{C}$. Its objects are the objects of $\mathfrak{C}$. A morphism for X to Y, written $X \dashrightarrow Y$, is a span with extremities X and Y, that is a diagram

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}, \tag{2.8}$$

with composition given by pullbacks of the respective vertices. A 2-morphism of spans is a morphism of their vertices above both the source and the target of the span. There is an obvious functor $\mathfrak{C} \to \mathfrak{Span}(\mathfrak{C})$ inducing the identity on object and sending a morphism $X \to Y$ to the span $X = X \to Y$ with vertex X, and there is another obvious functor $\mathfrak{C}^{op} \to \mathfrak{Span}(\mathfrak{C})$ sending $X \leftarrow Y$ to the span $X \leftarrow Y = Y$ with vertex Y.

Property 2.2.1.2.1. *For any functor $\mathcal{F}\colon \mathfrak{C}^{op} \to \mathfrak{D}$ to a 2-category $\mathfrak{D}$ such that*

- *for any $f\colon X \to Y$ in $\mathfrak{C}$, the 1-morphism $\mathcal{F}f\colon \mathcal{F}Y \to \mathcal{F}X$ has a right adjoint $f_*\colon \mathcal{F}X \to \mathcal{F}Y$ in $\mathfrak{D}$, and*
- *for every cartesian square*

$$\begin{tikzcd} X \arrow[r, "u'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "u"'] & Y' \end{tikzcd}, \tag{2.9}$$

the canonical base-change 2-morphism $\mathcal{F}(u) \circ f_* \Rightarrow (f')_* \circ \mathcal{F}(u')$ is invertible,

then $\mathcal{F}$ extends to a functor of 2-categories $\widehat{\mathcal{F}}\colon \mathfrak{Span}(\mathfrak{C}) \to \mathfrak{D}$ in a unique way.

Proof. First let us notice that $\mathfrak{C}^{op} \to \mathfrak{Span}(\mathfrak{C})$ verifies these properties, since spans define morphisms in either direction, and by associativity of pullbacks.

We define $\widehat{\mathcal{F}}$ to act as $\mathcal{F}$ on objects, and on morphisms to send a span $X \xleftarrow{f} Z \xrightarrow{g} Y$ to $g_* \circ \mathcal{F}(f)$. By the base-change property of $\mathcal{F}$ this is well-defined with regard to composition.

Let $\mathcal{G}\colon \mathfrak{Span}(\mathfrak{C}) \to \mathfrak{D}$ be another extension of $\mathcal{F}$ through $\mathfrak{Span}(\mathfrak{C})$. By definition, $\mathcal{G}$ and $\widehat{\mathcal{F}}$ have the same effect on objects. Let now $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a morphism $X \dashrightarrow Y$ in $\mathfrak{Span}(\mathfrak{C})$. We now observe that the span factors as

$$\begin{tikzcd} & & Z \times_Z Z = Z \arrow[dl, "\mathbb{1}_Z"'] \arrow[dr, "\mathbb{1}_Z"] & & \\ & Z \arrow[dl, "f"'] \arrow[dr, "\mathbb{1}_Z"'] & & Z \arrow[dl, "\mathbb{1}_Z"] \arrow[dr, "g"] & \\ X & & Z & & Y \end{tikzcd} \tag{2.10}$$

Since $\mathcal{G}$ extends $\mathcal{F}$, we have that $\mathcal{G}(X \xleftarrow{f} Z = Z) = \mathcal{F}(f)$, and we noticed above that $Z = Z \xrightarrow{g} Y$ is adjoint to $Y \xleftarrow{g} Z = Z$, mapped by $\mathcal{G}$ to $\mathcal{F}(g)$. We have thus recovered the behaviour of $\widehat{F}$ on morphisms. $\square$

An ∞-bicategorical version of the bicategory of spans is described in [GR17, Chapter V.1], with a categorification of the previous property proved as [GR17, V.1, Theorem 3.2.2]. We will not need it, as we are not interested in ∞-functors from the ∞-bicategory of spans but in ∞-functors *to* its maximal ∞-category.

The decategorification (*i.e.* the maximal category) of $\mathfrak{Span}(\mathfrak{C})$ is denoted $(\mathfrak{C})^{\mathrm{corr}}$.

We also recall (see example A.1.1.2.7 for the ∞-categorical version) the **twisted arrows category** of a category $\mathfrak{D}$, whose objects are arrows of $\mathfrak{D}$ and whose morphisms from f to g are factorisations of f through g, that is commutative squares

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot & \cdot \arrow[l] \end{tikzcd} . \tag{2.11}$$

Property 2.2.1.2.2. *For any category $\mathfrak{D}$, there is an equivalence between the category of functors $\mathfrak{D} \to (\mathfrak{C})^{\mathrm{corr}}$ and the category of functors $\mathcal{F}\colon \mathfrak{Tw}(\mathfrak{D}) \to \mathfrak{C}$ such that for any composable morphisms $f\colon X \to Y$, $g\colon Y \to Z$ in $\mathfrak{D}$, the object $\mathcal{F}(gf)$ is isomorphic to $\mathcal{F}(g) \times_{\mathcal{F}(\mathbb{1}_Y)} \mathcal{F}(f)$.*

Proof. Let $\mathcal{F}\colon \mathfrak{Tw}(\mathfrak{D}) \to \mathfrak{C}$ be as above. For any $X \in \mathfrak{D}$, we let $\mathcal{F}^t(X) = \mathcal{F}(\mathbb{1}_X)$. Note that a twisted morphism from $\mathbb{1}_X$ to $\mathbb{1}_Y$ is an isomorphism $X \xrightarrow{\simeq} Y$. For any $f\colon X \to Y$ in $\mathfrak{D}$, set $\mathcal{F}^t(f)\colon \mathcal{F}(\mathbb{1}_X) \dashrightarrow \mathcal{F}(\mathbb{1}_Y)$ to be the span with vertex $\mathcal{F}(f)$ whose arrows are given by the tautological factorisations of f through the identities:

$$\begin{tikzcd} & \mathcal{F}(f) \arrow[dl, "\mathcal{F}(*)"'] \arrow[dr, "\mathcal{F}(**)"] & \\ \mathcal{F}(\mathbb{1}_X) & & \mathcal{F}(\mathbb{1}_Y) \end{tikzcd} \quad \text{with } * = \begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"] & X \arrow[d, equal, "\mathbb{1}_X"'] \\ Y & X \arrow[l, "f"] \end{tikzcd} \text{ and } ** = \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[d, equal, "\mathbb{1}_Y"'] \\ Y & Y \arrow[l, equal] \end{tikzcd} . \tag{2.12}$$

This is functorial by the property required of $\mathcal{F}$, and the operation $(-)^t$ is clearly an equivalence of categories. $\square$

An ∞-categorical generalisation of this construction is exposed in [Ras14]. We have a categorification of the previous property.

Proposition 2.2.1.2.3. *[Ras14, § 20.9] Let $\mathfrak{C}$ be an ∞-category with fiber products and $\mathfrak{D}$ an ∞-category. The ∞-category of ∞-functors $\mathfrak{D} \to (\mathfrak{C})^{\mathrm{corr}}$ is canonically equivalent to the ∞-category of ∞-functors $\mathcal{F}\colon \mathfrak{Tw}(\mathfrak{D}) \to \mathfrak{C}$ such that for every 2-simplex $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathfrak{D}$, the morphism $\mathcal{F}(gf) \to \mathcal{F}(f) \times_{\mathcal{F}(\mathbb{1}_Y)} \mathcal{F}(g)$ is an equivalence in $\mathfrak{C}$.*

This can in fact be taken as a definition of $(\mathfrak{C})^{\mathrm{corr}}$: the ∞-functor $(\bullet)^{\mathrm{corr}}$ is a sub-∞-functor of the right adjoint to the ∞-functor $\mathfrak{Tw}(-)$.

Finally, we show that there is a monoidal version of the above. Since $\mathfrak{C}$ has finite limits, it admits the cartesian monoidal structure $\mathfrak{C}^{\times}$. This induces a symmetric monoidal structure on $(\mathfrak{C})^{\mathrm{corr}}$, which we denote $(\mathfrak{C}^{\times})^{\mathrm{corr}}$ (it is not given by the cartesian products in $(\mathfrak{C})^{\mathrm{corr}}$). Now let $\mathfrak{D}^{\otimes}$ be a symmetric monoidal ∞-category, that is a commutative monoid in $\mathfrak{Cat}_{\infty}^{\times}$. Then by[Lur12, Example 5.2.2.23], the morphism $\mathfrak{Tw}(\mathfrak{D}) \to \mathfrak{D} \times \mathfrak{D}^{\mathrm{op}}$ admits a structure of commutative monoid in the ∞-category of pairings exhibiting a symmetric monoidal structure on $\mathfrak{Tw}(\mathfrak{D})$, which we denote $\mathfrak{Tw}(\mathfrak{D})^{\otimes}$.

Corollary 2.2.1.2.4. *[MR18a, Corollary 2.1.3] For any ∞-category $\mathfrak{C}$ admitting finite limits and any symmetric monoidal ∞-category $\mathfrak{D}^{\otimes}$, the ∞-groupoid of monoidal ∞-functors $\mathfrak{D}^{\otimes} \to (\mathfrak{C}^{\times})^{\mathrm{corr}}$ is canonically equivalent to the ∞-groupoid of monoidal ∞-functors $\mathcal{F}\colon \mathfrak{Tw}(\mathfrak{D})^{\otimes} \to \mathfrak{C}^{\times}$ such that for any composable sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathfrak{D}$, $\mathcal{F}(gf) \simeq \mathcal{F}(g) \times_{\mathcal{F}(\mathbb{1}_Y)} \mathcal{F}(f)$.*

Proof. The ∞-functor $(\bullet)^{\mathrm{corr}} : \mathfrak{Cat}_{\infty}^{\lim,\times} \to \mathfrak{Cat}_{\infty}^{\times}$ commutes with limits and thus sends commutative monoids to commutative monoids, so the (sub-)adjunction $\mathfrak{Tw}(-) \dashv (-)^{\mathrm{corr}}$ passes to commutative monoid objects. □

Remark 2.2.1.2.5 (Cospans). We can similarly define ∞-bicategories of cospans and ∞-categories of cocorrespondences in an ∞-category $\mathfrak{C}$, which coincide with respectively $\mathfrak{Span}(\mathfrak{C}^{\mathrm{op}})$ and $(\mathfrak{C}^{\mathrm{op}})^{\mathrm{corr}}$. The symmetric monoidal structure $\left(\mathfrak{C}^{\amalg}\right)^{\mathrm{cocorr}} = \left(\mathfrak{C}^{\mathrm{op}\,\times^{\mathrm{op}}}\right)^{\mathrm{corr}}$ is now given by coproducts in $\mathfrak{C}$.

Example 2.2.1.2.6 (Endomorphisms operad). Let C be an object of $\mathfrak{C}$. A correspondence $C^n \dashrightarrow C$ in $\mathrm{Map}_{(\mathfrak{C})^{\mathrm{corr}}}(C^n, C)$ is equivalently given by an object of $\mathfrak{C}_{/C^{n+1}}$, an object of $\mathfrak{C}$ over C^{n+1} (and, trivially, over the final object). Write $\mathfrak{E}[C](n) \coloneqq \mathfrak{Iso}\left(\mathfrak{C}_{/C^{n+1}}\right)$.

If we have an object $A \to C^{n+1}$ and another $B \to C^{m+1}$, then $A \times_C B \to C^{n+m}$ (using the morphism from A to any factor C and the one from B to the final ($m+1$th) factor) gives an object of $\mathfrak{E}[C](n+m-1)$, providing a structure of monochromatic ∞-operad on $\mathfrak{E}[C]$. (Note that there ought to be in fact a structure of cyclic ∞-operad, *cf.* section 3.1.2.2.)

The ∞-operad $\mathfrak{E}[C]^{\otimes}$ is a sub-∞-operad of the symmetric monoidal ∞-category $(\mathfrak{C}^{\times})^{\mathrm{corr}}$. An algebra in $(\mathfrak{C}^{\times})^{\mathrm{corr}}$ over a monochromatic ∞-operad $\mathfrak{O}^{\otimes}$ with colour O such that the image of O is C can be seen equivalently as a map of ∞-operads $\mathfrak{O}^{\otimes} \to \mathfrak{E}[C]^{\otimes}$.

2.2.1.3 Lax morphisms and categorical operads

Construction 2.2.1.3.1 (First formulation of lax morphisms)**.** Suppose $\mathfrak{O}^{\otimes}$ and $\mathfrak{P}^{\otimes}$ are two monochromatic ∞-operads in an ∞-bicategory: for example we can take two ∞-operads and consider their spaces of operations as objects in the bicategory of spaces. We will not pursue in detail the language of $(\infty, 2)$-categories (though see construction A.1.2.3.7); however working with the ∞-bicategory of spans in an ∞-category $\mathfrak{C}$ will allow us to describe 2-morphisms in terms of morphisms of $\mathfrak{C}$.

We wish to consider a *lax* map of ∞-operads $\mathcal{F}\colon \mathfrak{O}^{\otimes} \rightsquigarrow \mathfrak{P}^{\otimes}$ between them. The coherent diagram (2.3) of example 2.1.1.1.11, written in terms of partial compositions to simplify, should then be reinterpreted as

$$\begin{tikzcd}
\mathfrak{O}(n_1) \times \mathfrak{O}(n_2) \arrow[r, dashed, "\mathcal{F}_{n_1} \times \mathcal{F}_{n_2}"] \arrow[d, dashed, "\circ"'] & \mathfrak{P}(n_1) \times \mathfrak{P}(n_2) \arrow[d, dashed, "\circ"] \arrow[dl, Rightarrow, "c_{n_1,n_2}"'] \\
\mathfrak{O}(n_1 + n_2 - 1) \arrow[r, dashed, "\mathcal{F}_{n_1+n_2-1}"'] & \mathfrak{P}(n_1 + n_2 - 1)
\end{tikzcd}, \tag{2.13}$$

where the 2-arrow between the composites is not necessarily invertible.

We will give a more rigorous description of a lax morphism in the case where $\mathfrak{P} = \mathfrak{E}[X]$ is the endomorphisms operad of a space X (seen this time in spans instead of correspondences), or more generally an object in a stack ∞-topos.

Definition 2.2.1.3.2 (Categorical operad)**.** [Toë13] A **categorical ∞-operad in spaces** is a presheaf of ∞-categories on Ω satisfying the Segal conditions.

Let $\mathfrak{T}$ be a stack ∞-topos. The ∞-category $\mathfrak{CatOp}(\mathfrak{T})$ of **categorical ∞-operads** in $\mathfrak{T}$ is $\mathfrak{Fun}^{\lim}(\mathfrak{T}^{op}, \mathfrak{Fun}^{\mathrm{Segal}}(\Omega^{op}, \mathfrak{Cat}_\infty))$, the ∞-category of limit-preserving ∞-functors from $\mathfrak{T}$ to the ∞-category of categorical operads.

By cartesian closure, the ∞-category of categorical ∞-operads in $\mathfrak{T}$ is equivalent to $\mathfrak{Fun}^{\lim,\mathrm{Segal}}(\mathfrak{T}^{op} \times \Omega^{op}, \mathfrak{Cat}_\infty)$. Applying the Grothendieck construction, an object of this ∞-category can also be seen as a category over $\mathfrak{T} \times \Omega^{op}$, whose structure morphism is a cocartesian fibration in the first variable (over $\mathfrak{T}$) and a cartesian fibration in the second (over Ω^{op}), that is a bifibration in the terminology of [Lur09, Definition 2.4.7.2].

Example 2.2.1.3.3. Let $\mathfrak{V}^\otimes$ be a symmetric monoidal ∞-bicategory (which we have not defined, but one may think of the cartesian monoidal structure on an ∞-bicategory of spans). This defines a categorical ∞-operad in the following fashion. To a corolla with n leaves, we associate the coproduct $\coprod_{(X_1,\dots,X_n,Y)\in\mathfrak{V}^{n+1}} \mathfrak{Map}(X_1 \otimes \cdots \otimes X_n, Y)$ of mapping ∞-categories. On a general tree T, we define the action of the categorical operad by decomposing T into the corollas of its Segal core, and impose the Segal condition.

In particular, the ∞-bicategory of spans in an ∞-category $\mathfrak{C}$ has a monoidal structure similar to that on $(\mathfrak{C}^\times)^{\mathrm{corr}}$, so we can present it more simply by a categorical ∞-operad.

Remark 2.2.1.3.4. Let $\mathcal{O}^\otimes \colon \mathfrak{T}^{op} \times \Omega^{op} \to \mathfrak{Cat}_\infty$ be a categorical ∞-operad in $\mathfrak{T}$. For the purposes of this paragraph, we shall omit all consideration of the fibration over $\mathfrak{T}$ in its Grothendieck construction $\int \mathcal{O}^\otimes$, and only consider the cocartesian fibration over Ω^{op}, which we recast in this paragraph as a cartesian fibration over Ω. Let $\phi \colon \Xi \to \Psi$ be a cartesian edge in the Grothendieck construction whose projection to Ω is homotopic to the inclusion of a corolla T_n of the Segal core of a tree τ. Let $g \colon \ \to T_n$ be a morphism in Ω and $\eta \colon \Upsilon \to \Psi$ a lift of the composite of g by $T_n \hookrightarrow \tau$; since ϕ is cartesian there exists a unique lift γ of g making the triangle commute, *i.e.* such that $\phi\gamma = \eta$. If we now consider the union of these morphisms over the set of all corollas of τ (formally corresponding to its Segal core), we obtain : the cartesian property expresses exactly the Segal condition.

Construction 2.2.1.3.5. Consider now a morphism of categorical ∞-operads $\mathcal{F}^{\otimes}\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$, and the associated morphism of bifibrations $\mathcal{F}\colon \int \mathcal{O}^{\otimes} \to \int \mathcal{P}^{\otimes}$ over $\mathfrak{T} \times \Omega^{\mathrm{op}}$. By definition it sends Ω^{op}-cocartesian morphisms in $\int \mathcal{O}^{\otimes}$ to Ω^{op}-cocartesian morphisms in $\int \mathcal{P}^{\otimes}$. Let $f\colon T_{n+m-1} \to$ "$T_n \amalg T_m$" be a face map in Ω, giving (by the Segal condition) a partial composition $f^*\colon \mathcal{O}^{\otimes}(-)(n) \times \mathcal{O}^{\otimes}(-)(m) \to \mathcal{O}^{\otimes}(-)(n+m-1)$ in the sheaf of categorical ∞-operads $\mathcal{O}^{\otimes}(-)$ (f^* is a natural transformation of sheaves of ∞-categories, whose components are thus ∞-functors). The property, for a lift ϕ of f^{op}, of being cocartesian, corresponds to the essential uniqueness of iterated compositions, that is the homotopy associativity. Since $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a morphism of ∞-operads, the corresponding $\int \mathcal{O}^{\otimes} \to \int \mathcal{P}^{\otimes}$ is a morphism of cocartesian fibrations over Ω^{op}, and sends ϕ to a cocartesian morphism in $\int \mathcal{P}^{\otimes}$.

We now fix an object of $\mathfrak{T}$ to work over, so as to only remember the fibration over Ω^{op}. Let us relax this condition, and suppose $\phi\colon \Xi \to \Psi$ (with Ξ over T_{n+m-1} and Ψ over the contraction "$T_n \amalg T_m$") is a cocartesian lift that is not sent to a cocartesian edge: we require that $\int \mathcal{O}^{\otimes} \to \int \mathcal{P}^{\otimes}$ be defined over Ω^{op} (so ϕ is still sent to a lift of f in $\int \mathcal{P}^{\otimes}$), but not a morphism of fibrations. There is a cocartesian lift $\widetilde{\phi}\colon \mathcal{F}(\Xi) \to \widetilde{\Psi}$ of f in $\int \mathcal{P}^{\otimes}$ (which differs from the image of ϕ by $\mathcal{F}$).

Let us now compare the two branches in the square defining a morphism of operads. Then operadic composition in $\mathcal{O}^{\otimes}$ (given for Ξ by direct image to Ψ) followed by application of $\mathcal{F}^{\otimes}$ sends Ξ to $\mathcal{F}(\Psi)$. Now $\mathcal{F}$ sends ϕ to $\mathcal{F}(\phi)\colon \mathcal{F}(\Xi) \to \mathcal{F}(\Psi)$. There is also the cocartesian $\widetilde{\phi}\colon \mathcal{F}(\Xi) \to \widetilde{\Psi}$, and both $\mathcal{F}(\phi)$ and $\widetilde{\phi}$ are lifts of f (or $\mathbb{1}_{\text{“}T_n \amalg T_m\text{”}} \circ f\circ$). It follows that $\mathcal{F}(\phi)$ factors (in an essentially unique way) as $\varpi_{\Xi} \circ \widetilde{\phi}$, with $\varpi_{\Xi}\colon \widetilde{\Psi} \to \mathcal{F}(\Psi)$ lifting $\mathbb{1}_{\text{“}T_n \amalg T_m\text{”}}$ (that is, by the Segal condition, a morphism in $\mathcal{P}(n) \times \mathcal{P}(m)$). This ϖ_{Ξ} is the component of the 2-morphism making the diagram of ∞-categories only 2-coherent:

$$\begin{tikzcd} \mathcal{O}(n) \times \mathcal{O}(m) \arrow[r, "\mathcal{F}_n \times \mathcal{F}_m"] \arrow[d, "f_*"'] & \mathcal{P}(n) \times \mathcal{P}(m) \arrow[d, "f_*"] \arrow[dl, Rightarrow, "\varpi"] \\ \mathcal{O}(n+m-1) \arrow[r, "\mathcal{F}_{n+m-1}"'] & \mathcal{P}(n+m-1) \end{tikzcd} \qquad \begin{tikzcd} \Xi \arrow[r, mapsto] \arrow[dd, mapsto] & \mathcal{F}(\Xi) \arrow[dr, mapsto] & \\ & & \widetilde{\Psi} \arrow[dl, "\varpi_{\Xi}"] \\ \Psi \arrow[r, mapsto] & \mathcal{F}(\Psi) & \end{tikzcd} . \tag{2.14}$$

We have now motivated the following definition.

Definition 2.2.1.3.6 (Lax morphism of categorical operads)**.** Let $\mathfrak{T} = \mathfrak{Sh}_{\tau}(\mathfrak{C})$, and let $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ be categorical ∞-operads in $\mathfrak{T}$. A **lax morphism** from $\mathcal{O}^{\otimes}$ to $\mathcal{P}^{\otimes}$ is an ∞-functor $\int \mathcal{O}^{\otimes} \to \int \mathcal{P}^{\otimes}$ over $\mathfrak{T} \times \Omega^{\mathrm{op}}$ sending $\mathfrak{T}$-cartesian arrows in $\int \mathcal{O}^{\otimes}$ to $\mathfrak{T}$-cartesian arrows in $\int \mathcal{P}^{\otimes}$.

In the terminology of [Toë13], this is called a very lax morphism of categorical ∞-operads.

Notice that for classical ∞-operads, the Grothendieck construction produces fibrations in spaces over Ω^{op}, where all morphisms are cocartesian, so the lax morphisms are exactly the morphisms of ∞-operads.

Example 2.2.1.3.7 (Endomorphisms in correspondences). In the ∞-bicategory of spans, the 2-coherent diagram (2.13) is written explicitly as the fully (homotopy) commutative diagram

$$\begin{tikzcd}
\mathfrak{O}(n_1) \times \mathfrak{O}(n_2) & & Z_N \arrow[ll] \arrow[r] & \mathfrak{P}(n_1) \times \mathfrak{P}(n_2) \\
& & Z_n \times_P Z_E \arrow[u] \arrow[r] \arrow[dl, "c_{n_1,n_2}"'] \arrow[ur, phantom, "\lrcorner^h"] & Z_E \arrow[u] \arrow[dd] \\
Z_W \arrow[uu] \arrow[d] & Z_W \times_O Z_S \arrow[l] \arrow[d] \arrow[dl, phantom, "{}^h\llcorner"] & & \\
\mathfrak{O}(n_1 + n_2 - 1) & Z_S \arrow[l] \arrow[rr] & & \mathfrak{P}(n_1 + n_2 - 1)
\end{tikzcd} \quad . \tag{2.15}$$

Suppose that $\mathfrak{O}^\otimes$ and $\mathfrak{P}^\otimes$ are both simply the symmetric monoidal ∞-bicategory $\mathfrak{Span}(\mathfrak{G}^\times)$ itself. Seeing them as categorical ∞-operads as in example 2.2.1.3.3, this diagram should be encoded in the datum of a lax morphism of categorical operads. The discussion preceding the definition 2.2.1.3.6 then provides the required morphism c_{n_1,n_2}.

2.2.1.4 Coherent ∞-operads

Definition 2.2.1.4.1.

- An ∞-operad $\mathfrak{O}^\otimes \to \Gamma$ is **unital** if $\mathfrak{O}^\otimes$ is pointed, if and only if for each object $X \in \mathfrak{O}$ of the underlying ∞-category the space $\mathrm{Map}(\emptyset, X)$ is contractible (here $\emptyset$ lies above $\langle 0 \rangle \in \Gamma$).
- An ∞-operad $\mathfrak{O}^\otimes \to \Gamma$ is **reduced** if it is unital and its underlying ∞-category $\mathfrak{O}$ is an essentially trivial ∞-groupoid (contractible Kan complex).

Remark 2.2.1.4.2.

- A monochromatic ∞-operad $\mathfrak{O}^\otimes$ is unital if and only if $\mathfrak{O}(0)$ is contractible.
- A reduced ∞-operad is necessarily monochromatic.

Definition 2.2.1.4.3 (Extensions of an active morphism)**.** Let $\mathcal{Q}\colon \mathfrak{O}^\otimes \to \Gamma$ be a unital ∞-operad. Let $\sigma\colon O \to P$ be an active morphism in $\mathfrak{O}^\otimes$, also seen as an edge $[\sigma]\colon [1] \to \mathfrak{O}^\otimes$. The ∞-category $\mathrm{Ext}(\sigma)$ of extensions of σ is the full subcategory of $\mathfrak{Fun}([1], \mathfrak{O}^\otimes)_{[\sigma]/}$ spanned by the diagrams

$$\begin{tikzcd}
O \arrow[r, "f"] \arrow[d, "\sigma"'] & O^+ \arrow[d, "\sigma^+"] \\
P \arrow[r, "g"'] & P^+
\end{tikzcd} \tag{2.16}$$

such that:

1. g is an equivalence;

2. f is semi-inert (see definition 2.1.1.1.9) and $\mathcal{Q}(f)$ is an inclusion $\langle m \rangle \coloneqq \mathcal{Q}(O) \to \langle m+1 \rangle$ which misses a single element i of $\langle m+1 \rangle$;

3. σ^+ is active.

This definition is generalised combinatorially for a composable sequence of active morphisms as [Lur12, Definition 3.3.1.4]. It expresses the following. Write $O = O_1 \oplus \cdots \oplus O_m$. Then an extension of the multimorphism σ is given by a colour $O \in \mathfrak{O}$ and a multimorphism σ^+ from $O_1 \oplus \cdots \oplus O_{i-1} \oplus O \oplus O_i \oplus \cdots \oplus O_m$ to $P^+ \simeq P$, extending σ (up to equivalence).

Remark 2.2.1.4.4 (Extensions of a monochromatic multimorphism). Let $\mathfrak{O}$ be a unital monochromatic ∞-operad with colour C. Let $\sigma \in \mathfrak{O}(n) = \mathrm{Map}_{\mathfrak{O}^{\otimes},\mathrm{act}}(C \oplus \cdots \oplus C, C)$ be an active morphism. We define the space of **extensions** of σ as the homotopy fibre product $\mathrm{Ext}(\sigma) = * \times^{h}_{\mathfrak{O}(n)} \mathfrak{O}(n+1)$ where the map $* \to \mathfrak{O}(n)$ selects σ and $\mathfrak{O}(n+1) \to \mathfrak{O}(n)$ is the map forgetting the last input (remember also that $\mathfrak{O}(0) \simeq *$).

If $\sigma \in \mathrm{Map}_{\mathfrak{O}^{\otimes},\mathrm{act}}(C^{\oplus n}, C^{\oplus m}) \simeq \mathfrak{O}(n)^m$ corresponds to the family $(\sigma_1, \dots, \sigma_m)$, with $\sigma_i \in \mathfrak{O}(n)$, then we have $\mathrm{Ext}(\sigma) \coloneqq \coprod_{i=1}^{m} \mathrm{Ext}(\sigma_i)$.

Remark 2.2.1.4.5. Suppose $\mathfrak{O}(1) \simeq \{\mathbb{1}_C\}$ is contractible. We then have

$$\begin{tikzcd}
\mathrm{Ext}(\mathbb{1}_C) \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\lrcorner^{h}"] & \mathfrak{O}(2) \arrow[d] \\
* \arrow[r, equal] & \mathfrak{O}(1) = \{\mathbb{1}_C\}
\end{tikzcd} . \tag{2.17}$$

Definition 2.2.1.4.6 (Coherent ∞-operad). Let $\mathfrak{O}^{\otimes}$ be a monochromatic ∞-operad with colour C. We say $\mathfrak{O}^{\otimes}$ is **coherent** if it is reduced and for every $\sigma \in \mathfrak{O}(n)^m, \tau \in \mathfrak{O}(m)$ composable, the square

$$\begin{tikzcd}
\mathrm{Ext}(\mathbb{1}_{C^{\oplus m}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "{}^{h}\ulcorner"] & \mathrm{Ext}(\tau) \arrow[d] \\
\mathrm{Ext}(\sigma) \arrow[r] & \mathrm{Ext}(\tau \circ \sigma)
\end{tikzcd} \tag{2.18}$$

is homotopy cocartesian.

2.2.2 Construction of the brane actions

2.2.2.1 Brane action in spaces

Lemma 2.2.2.1.1. *[Lur12, Proposition 2.2.4.9] The forgetful functor seeing a symmetric monoidal ∞-category as an ∞-operad admits a left adjoint given by $\mathfrak{O}^{\otimes} \rightsquigarrow \mathfrak{O}^{\otimes} \times_{\Gamma} \mathfrak{Act}(\Gamma)$ where $\mathfrak{Act}(\Gamma)$ denotes the full subcategory of the ∞-category $\mathfrak{Fun}([1], \Gamma)$ of mophisms in Γ spanned by the active morphisms (and Γ is identified with $\mathfrak{Fun}([0], \Gamma)$), and where the structure fibration is induced by evaluation at $\{1\} \subset [1]$. This symmetric monoidal ∞-category is called the **monoidal envelope** of $\mathfrak{O}^{\otimes}$ and denoted $\mathfrak{Env}^{\otimes}(\mathfrak{O})$.*

Remark 2.2.2.1.2. An object of $\mathfrak{Env}(\mathcal{O})^{\otimes}_{\langle m\rangle}$ can be seen as a pair (C, f) of an object $C = (C_1 \oplus \cdots \oplus C_n) \in \mathcal{O}^{\otimes}$ and an active morphism $f\colon \langle n\rangle \to \langle m\rangle$ in Γ. In particular, when $m = 1$, there is a single active morphism from $\langle n\rangle$ to $\langle 1\rangle$, so the underlying category $\mathfrak{Env}(\mathcal{O}) = \mathfrak{Env}(\mathcal{O})^{\otimes}_{\langle 1\rangle}$ has as objects those of $\mathcal{O}^{\otimes}$ and as morphisms the *active* morphisms of $\mathcal{O}^{\otimes}$. It follows that morphisms of the (symmetric monoidal) ∞-category $\mathfrak{Env}(\mathcal{O})^{\otimes}$ are given by families of multimorphisms of the ∞-operad $\mathcal{O}^{\otimes}$.

An object of the induced symmetric monoidal twisted arrows ∞-category $\mathfrak{Tw}(\mathfrak{Env}(\mathcal{O}))^{\otimes}$ is a family $(\langle n\rangle, (\sigma_i\colon X_i \to Y_i)_{i\in\langle n\rangle^\circ})$ of n active morphisms of $\mathcal{O}^{\otimes}$ (n indexing the fibre), which as above we can see as a family of multimorphisms. A morphism is a twisted arrow

$$\begin{tikzcd}
(X_i)_{i\in\langle n\rangle^\circ} \arrow[r, "(\bigoplus_i f_{i,j})_j"] \arrow[d, "(\sigma_i)_{i\in\langle n\rangle^\circ}"'] & (A_j)_{j\in\langle m\rangle^\circ} \arrow[d, "(\tau_j)_{j\in\langle n\rangle^\circ}"] \\
(Y_i)_{i\in\langle n\rangle^\circ} & (B_j)_{j\in\langle m\rangle^\circ} \arrow[l, "(\bigoplus_j g_{i,j})_i"]
\end{tikzcd} \tag{2.19}$$

where the $\oplus$ is to be interpreted as the concatenation of multimorphisms (considering the family $(Z_i)_i$ as the object $\bigoplus_i Z_i$), and where $\bigoplus_i f_{i,j}$ (resp. $\bigoplus_j g_{i,j}$) is a multimorphism from $\bigoplus_i X_i$ to A_j (resp. from $\bigoplus_j B_j$ to Y_i).

Theorem 2.2.2.1.3 (Lax brane action)**.** *[Toë13, Theorem 3.1] Let $\mathcal{O}^{\otimes}$ be a reduced (monochromatic) operad with colour O. The space of binary operations $\mathcal{O}(2)$ has a structure of lax $\mathcal{O}^{\otimes}$-algebra in cocorrespondences: there is a lax map of ∞-operads $\mathcal{B}\colon \mathcal{O}^{\otimes} \rightsquigarrow \mathfrak{C}[\mathcal{O}(2)]^{\otimes}$, that is $\mathcal{B}\colon \mathcal{O}^{\otimes} \rightsquigarrow \left(\mathfrak{S}^{\amalg}\right)^{\mathrm{cocorr}}$ sending O to $\mathcal{O}(2)$.*

Theorem 2.2.2.1.4 (Brane action)**.** *[Toë13, Proposition 3.5][MR18a, Theorem 2.1.7] The lax brane action is a map of ∞-operads if and only if $\mathcal{O}^{\otimes}$ is coherent.*

Strategy of the proof. Let $\mathcal{O}^{\otimes}$ be a coherent ∞-operad. We wish to construct a (non-lax, eventually) map of ∞-operads $\mathcal{O}^{\otimes} \to \left(\mathfrak{S}^{\amalg}\right)^{\mathrm{cocorr}}$ sending the unique colour of $\mathcal{O}^{\otimes}$ to the space $\mathcal{O}(2)$, which be the universal property of the free construction is equivalent to a symmetric monoidal functor $\mathfrak{Env}^{\otimes}(\mathcal{O}) \to \left(\mathfrak{S}^{\amalg}\right)^{\mathrm{cocorr}} = \left(\mathfrak{S}^{\mathrm{op}\times}\right)^{\mathrm{corr}}$.

But we also have the left adjoint to $(-^{\times})^{\mathrm{corr}}$ given informally by

$$\left\{\mathfrak{D}^{\otimes} \xrightarrow{\mathcal{F}} \left(\mathfrak{C}^{\times}\right)^{\mathrm{corr}}\right\} = \left\{\mathfrak{Tw}(\mathfrak{D})^{\otimes} \xrightarrow{\widetilde{\mathcal{F}}} \mathfrak{C}^{\times} \mid \widetilde{\mathcal{F}}(s's) = \widetilde{\mathcal{F}}(s') \times_{\mathbb{1}_{d'}} \widetilde{\mathcal{F}}(s)\right\} \tag{2.20}$$

where $\widetilde{\mathcal{F}}(s\colon d \to d') = \mathcal{F}(s) \in \operatorname{obj}\mathfrak{C}$.

So we are reduced to constructing a functor $\mathfrak{Tw}(\mathfrak{Env}(\mathcal{O}))^{\otimes} \to \mathfrak{S}^{\mathrm{op}}$ satisfying the above conditions and defining a monoidal ∞-functor to the cartesian structure on $\mathfrak{S}^{\mathrm{op}}$, which by the Grothendieck construction is equivalent to a fibred category in spaces over $\mathfrak{Tw}(\mathfrak{Env}(\mathcal{O}))^{\otimes}$ respecting conditions. $\square$

Construction 2.2.2.1.5 (Classifying fibration)**.** We shall now define an ∞-category over $\mathfrak{Tw}(\mathfrak{Env}(\mathcal{O}))^{\otimes}$ which will then be shown to be a cartesian fibration (in ∞-groupoids).

Consider the ∞-functor

$$s = \mathrm{ev}_0\colon \mathfrak{Fun}\left([1], \mathfrak{Tw}(\mathfrak{Env}^{\otimes}(\mathcal{O}))\right) \to \mathfrak{Tw}(\mathfrak{Env}(\mathcal{O}))^{\otimes}, \tag{2.21}$$

and let also

- $p_e \colon \mathfrak{Env}^{\otimes}(\mathcal{O}) \to \Gamma$ be the cocartesian fibration defining the symmetric monoidal structure for the monoidal envelope ;
- $p_t \colon \mathfrak{Tw}(\mathfrak{Env}(\mathcal{O}))^{\otimes} \to \Gamma$ be the cocartesian fibration defining the symmetric monoidal structure for its twisted arrows ∞-category;
- $p = p_e \operatorname{ev}_0 \colon \mathfrak{Tw}(\mathfrak{Env}(\mathcal{O}))^{\otimes} \to \Gamma$ (where ev_0 computes the source of an object of the twisted arrows category seen as a morphism of the base category; it is not the same ev_0 as s);
- $p_o \colon \mathcal{O}^{\otimes} \to \Gamma$ the ∞-functor defining the operad structure on $\mathcal{O}^{\otimes}$.

By [Lur09, Corollary 2.4.7.11] followed with [Lur09, Lemma 2.4.7.5], s is a cartesian fibration. We define the quasi-category $B\mathcal{O}$ as the 2-full (but not full) sub-∞-category of the quasi-category $\mathfrak{Fun}\left(\Delta^1, \mathfrak{Tw}(\mathfrak{Env}(\mathcal{O}))\right)^{\otimes}$ of twisted edges of $\mathfrak{Env}^{\otimes}(\mathcal{O})$ (recall the description of $\mathfrak{Tw}(\mathfrak{Env}(\mathcal{O}))^{\otimes}$ from remark 2.2.2.1.2) whose:

objects are twisted morphisms p_t-over the active map $\langle n\rangle \to \langle 1\rangle$:

$$\begin{tikzcd} (X_i)_{i\in\langle n\rangle^\circ} \arrow[r, "\bigoplus_i f_i"] \arrow[d, "(\sigma_i)_{i\in\langle n\rangle^\circ}"'] & A \arrow[d, "\delta"] \\ (Y_i)_{i\in\langle n\rangle^\circ} & B \arrow[l, "(g_i)_i"] \end{tikzcd} \tag{2.22}$$

such that (notice the similarity with definition 2.2.1.4.3):

1. the map $(g_i)_i$ is an equivalence;
2. the active map $\bigoplus_i f_i \colon \bigoplus_i X_i \to A$ is semi-inert in $\mathcal{O}^{\otimes}$ and lifts one of the maps $\langle m\rangle \coloneqq p_o\left(\bigoplus_i X_i\right) \to \langle m+1\rangle$ corresponding to an injection missing a single element of $\langle m+1\rangle$;

morphisms are given by: a morphism from the twisted arrow $\sigma = (\sigma_i)_i \xrightarrow{(\bigoplus_i f_i, (g_i)_i)} \delta$ to $\tau = (\tau_j)_j \xrightarrow{(\bigoplus_j a_j, (b_j)_j)} \epsilon$ mapped by s over $t \colon \sigma \to \tau$ is a commutative square of twisted arrows

$$\begin{tikzcd} \delta \arrow[r, "{(r,s)}"] & \epsilon \\ \sigma \arrow[u, "{(\bigoplus_i f_i, (g_i)_i)}"] \arrow[r, "{\left(\left(\bigoplus_i t_{i,j}\right)_j, \left(\bigoplus_j u_{i,j}\right)_i\right)}"'] & \tau \arrow[u, "{(\bigoplus_j a_j, (b_j)_j)}"'] \end{tikzcd} \quad p_t\text{-over} \quad \begin{tikzcd} \langle 1\rangle \arrow[r, "\mathbb{1}_{\langle 1\rangle}"] & \langle 1\rangle \\ \langle n\rangle \arrow[u] \arrow[r, "\phi"'] & \langle m\rangle \coloneqq p_t(\tau) \arrow[u] \end{tikzcd} \tag{2.23}$$

satisfying the following property: the square above induces a diagram

$$\begin{tikzcd} A \arrow[rr, "r"] & & U \\ (X_i)_{i\in\langle n\rangle^\circ} = \left((X_k)_{k\in\phi^{-1}(j)}\right)_{j\in\langle m\rangle^\circ} \arrow[u, "\bigoplus_i f_i"] \arrow[rr, "\left(\bigoplus_i t_{i,j}\right)_j"'] & & (S_j)_{j\in\langle m\rangle^\circ} \arrow[u, "\bigoplus_j a_j"'] \end{tikzcd} \tag{2.24}$$

with the naming as $\epsilon\colon U \to V$ and $\tau_j\colon S_j \to T_j, j \in \langle m \rangle^\circ$ (and notations for σ and δ exactly as in the description of objects), and as follows $t_{i,j}\colon X_i \to S_j$, $u_{i,j}\colon T_j \to Y_i$ and $r\colon A \to U$, $s\colon V \to B$. Remember that, by the above definition of the objects of $B\mathfrak{O}$, the map $\bigoplus_i f_i$ lifts the inclusion of all points but one of $p_\circ(A)$. Then we require that the map $p_\circ(r)$ sends this missing point to the unique point of $p_\circ(U)$ missed by $p_\circ(\bigoplus_i \alpha_i)$.

This makes it so that the fibre over $(\sigma_i)_i$ is its space of extensions $\coprod_i \mathrm{Ext}(\sigma_i)$.

Proof (of theorem 2.2.2.1.3). Let $\pi\colon B\mathfrak{O} \subset \mathfrak{Fun}([1], \mathfrak{Tw}(\mathfrak{Env}(\mathfrak{O}))^\otimes) \xrightarrow{s} \mathfrak{Tw}(\mathfrak{Env}(\mathfrak{O}))^\otimes$ be the restriction of the cartesian fibration s along the inclusion of $B\mathfrak{O}$. We must show that π is a cartesian fibration. It is enough (since the space of extensions of several morphisms is the coproduct of the individual spaces) to verify the lift property for a "family" in $\mathfrak{Tw}(\mathfrak{Env}(\mathfrak{O}))^\otimes$ consisting of a single multimorphism (or active morphism).

Let $\sigma = (\sigma_1\colon X_1 \to Y_1), \tau = (\tau_1\colon S_1 \to T_1)$ be objects of $\mathfrak{Tw}(\mathfrak{Env}(\mathfrak{O}))^\otimes$, that is active morphisms of $\mathfrak{O}^\otimes$, and let $t = (t_1\colon X_1 \to S_1, u_1\colon T_1 \to Y_1)\colon \sigma \to \tau$ be a twisted arrow between them. Recall that, by construction, the fibre of π over τ is the space of extensions $\mathrm{Ext}(\tau)$. To simplify notations, for all extensions will replace the equivalence by the identity and omit it.

Consider then an extension $\tau^+ = (\tau_1^\circ\colon S_1 \to S_1^+, \tau_1^+\colon S_1^+ \to T_1)$ of τ. Since s is a cartesian fibration, there must then exist a twisted arrow $\sigma^+ = (\sigma_1^\circ\colon X_1 \to X_1^+, \sigma_1^+\colon X_1^+ \to Y_1', \sigma_1'\colon Y_1' \to Y_1)$ factoring σ and an s-cartesian lift of t (by a coherent square of twisted arrows) between them:

$$\begin{tikzcd}
X_1^+ \arrow[rrr] \arrow[dr, "\sigma_1^+"] & & & S_1^+ \arrow[dl, "\tau_1^+"'] \\
& Y_1' \arrow[d, "\sigma_1'"] & T_1 \arrow[l] \arrow[d, "\mathbb{1}_{T_1}"] & \\
& Y_1 & T_1 \arrow[l, "u_1"] & \\
X_1 \arrow[uuu, "\sigma_1^\circ"] \arrow[ur, "\sigma_1"] \arrow[rrr, "t_1"'] & & & S_1 \arrow[uuu, "\tau_1^\circ"'] \arrow[ul, "\tau_1"']
\end{tikzcd} . \tag{2.25}$$

But [Lur09, Lemma 2.4.7.5] implies that the morphism of twisted arrows is sent by the target functor ev_1 to an equivalence (between σ^+ and τ^+) in $\mathfrak{Tw}(\mathfrak{Env}(\mathfrak{O}))^\otimes$. As we have taken X_1 (and Y_1) to consist of a single object of $\mathfrak{O}^\otimes$, we may restrict σ^+ to an extension of σ, which we denote $\sigma^{+0} = (\sigma_1^\circ\colon X_1 \to X_1^{+0}, \sigma_1' \circ \sigma_1^+|_{X_1^{+0}}\colon X_1^{+0} \to Y_1)$ (where X_1^{+0} is a subobject of X_1^+ lying $p_\circ$-above $p_\circ(X_1) + 1 \in \Gamma$).

We now check that the map from this restricted extension is π-cartesian in $B\mathfrak{O}$. Let $(a_1\colon U_1 \to X_1, b_1\colon Y_1 \to V_1)$ be a twisted map from another active morphism $\lambda = (\lambda_1\colon U_1 \to V_1)$ to σ. Let also $\lambda^+ = (\lambda_1^\circ\colon U_1 \to U_1^+, \lambda_1^+\colon U_1^+ \to V_1)$ be an extension of λ, and $r = (r_1\colon U_1^+ \to S_1^+)$ a morphism from λ^+ to τ^+ in $B\mathfrak{O}$ whose projection is homotopic (in $\mathfrak{Tw}(\mathfrak{Env}(\mathfrak{O}))^\otimes_{/\tau}$) to the composite $(t_1, u_1) \circ (a_1, b_1)$. Since $\sigma^+ \to \tau^+$ is

s-cartesian, there is an essentially unique filling $\lambda^+ \to \sigma^+$. Finally, we see that the space of submorphims factoring through σ^{+0} and satisfying the condition needed to define a morphism in $B\mathfrak{O}$ is contractible.

Since π has been shown to be a cartesian fibration in spaces, it defines an ∞-functor $\mathfrak{Tw}(\mathfrak{Env}(\mathfrak{O}))^{\otimes} \to \mathfrak{G}^{\mathrm{op}}$. We now need to ensure that it sends the monoidal structure of $\mathfrak{Tw}(\mathfrak{Env}(\mathfrak{O}))^{\otimes}$ to the cartesian monoidal structure of $\mathfrak{G}^{\mathrm{op}}$.

Definition 2.2.2.1.6. Let $\mathfrak{V}^{\otimes} \to \Gamma$ be an ∞-operad. A **lax cartesian structure** on $\mathfrak{V}^{\otimes}$ in an ∞-category $\mathfrak{D}$ is an ∞-functor $\mathcal{C}\colon \mathfrak{V}^{\otimes} \to \mathfrak{D}$ such that for any object X of $\mathfrak{V}^{\otimes}_{\langle n\rangle}$, written as $X_1 \oplus \cdots \oplus X_n$, the canonical maps $\mathcal{C}(X) \to \mathcal{C}(X_i)$ exhibit $\mathcal{C}(X)$ as a product of the $\mathcal{C}(X_i)$ in $\mathfrak{D}$.

We say furthermore that $\mathcal{C}$ is a **weak cartesian structure** if $\mathfrak{V}^{\otimes}$ is a symmetric monoidal ∞-category and, for any cocartesian active morphism $f\colon X \to Y$ covering the active morphism $\langle n\rangle \to \langle 1\rangle$, the morphism $\mathcal{C}(f)$ is an equivalence in $\mathfrak{D}$.

The ∞-category of weak cartesian structures on $\mathfrak{V}^{\otimes}$ in $\mathfrak{D}$ is the full sub-∞-category of $\mathfrak{Fun}(\mathfrak{V}^{\otimes}, \mathfrak{D})$ spanned by the weak cartesian structures.

A symmetric monoidal structure $\mathfrak{V}^{\otimes}$ on an ∞-category $\mathfrak{V}$ is said to be cartesian if its unit object is final in $\mathfrak{V}$ and for any pair of objects C, D the induced maps $C \otimes D \to C, D$ exhibit $C \otimes D$ as a product $C \times D$ in $\mathfrak{V}$. Such a structure is constructed and seen to be weak cartesian in [Lur12, Proposition 2.4.1.5], and it is shown in [Lur12, Corollary 2.1.4.8] to be unique up to monoidal equivalence whose restriction to $\mathfrak{V}$ is homotopic to the identity. We shall write this structure as $\mathfrak{V}^{\times}$.

Then, according to [Lur12, Proposition 2.4.1.6], if $\mathfrak{V}^{\otimes}$ is a symmetric monoidal ∞-category and $\mathfrak{D}$ is an ∞-category admitting finite products, there is an equivalence between the ∞-category of cartesian structures on $\mathfrak{V}^{\otimes}$ in $\mathfrak{D}$ and the ∞-category of monoidal ∞-functors from $\mathfrak{V}^{\otimes}$ to $\mathfrak{D}^{\times}$.

We thus need to check that the ∞-functor $\int^{-1}\pi$ associated to π is a weak cartesian structure. But remember that the fibre of π above an active morphism σ, that is the value of $\int^{-1}\pi$ at σ, is the space $\mathrm{Ext}(\sigma)$, and that for a family of active morphisms $(\sigma_i)_i$ we have $\mathrm{Ext}(\otimes_i \sigma_i) = \coprod_i \mathrm{Ext}(\sigma_i)$. The product in $\mathfrak{G}^{\mathrm{op}}$ is given by the coproduct in $\mathfrak{G}$, so $\int^{-1}\pi$ is a lax cartesian structure. The same property of extensions of families of active morphisms explains why a map of extension spaces covering the active map $\langle n\rangle \to \langle 1\rangle$ will be an equivalence of ∞-groupoids, so that $\int^{-1}\pi$ is even a weak cartesian structure as required. □

Proof (of theorem 2.2.2.1.4). In order for theorem 2.2.2.1.3 to define a morphism of ∞-operads, by proposition 2.2.1.2.3, the monoidal ∞-functor $\mathfrak{Tw}(\mathfrak{Env}(\mathfrak{O}))^{\otimes} \to \mathfrak{G}^{\mathrm{op}\amalg}$ must send any composite gf to the fibred product of the images of $f\colon X \to Y$ and $g\colon Y \to Z$ over $\mathbb{1}_Y$. But the objects of $\mathfrak{Tw}(\mathfrak{Env}(\mathfrak{O}))^{\otimes}$ are families of active morphisms of $\mathfrak{O}^{\otimes}$, so this condition is exactly the coherence condition for $\mathfrak{O}^{\otimes}$. □

Remark 2.2.2.1.7. Let $\sigma \in \mathfrak{O}(n)$ be given, with $\mathfrak{O}^{\otimes}$ coherent. Then the morphism of ∞-operads provides a cocorrespondence between $\coprod_n \mathfrak{O}(2)$ and $\mathfrak{O}(2)$, where we recall that $\mathfrak{O}(2) \simeq \mathrm{Ext}(\mathbb{1}_C)$. The cocorrespondence is in fact given by

$$\coprod_n \mathrm{Ext}(\mathbb{1}_C) \to \mathrm{Ext}(\sigma) \leftarrow \mathrm{Ext}(\mathbb{1}_C), \tag{2.26}$$

where C is the colour of $\mathfrak{O}^{\otimes}$. Indeed, we see from the proof of property 2.2.1.2.2 that the associated ∞-functor $\mathfrak{Env}(\mathfrak{O})^{\otimes} \to \left(\mathfrak{S}^{\amalg}\right)^{\mathrm{cocorr}}$ sends an object $C^{\oplus n}$ to the fibre of π over $\mathbb{1}_{C^{\oplus n}}$, which is $\coprod_n \mathrm{Ext}(\mathbb{1}_C)$, and a morphism $\sigma \in \mathrm{Map}(C^{\oplus n}, C)$ to the cospan mentioned above.

Notice that this diagram can easily be identified with the pullback of the relative cocorrespondence

$$\begin{tikzcd} \coprod_n \mathfrak{O}(2) \times \mathfrak{O}(n) \arrow[r] \arrow[dr] & \mathfrak{O}(n+1) \arrow[d] & \mathfrak{O}(2) \times \mathfrak{O}(n) \arrow[l] \arrow[dl] \\ & \mathfrak{O}(n) & \end{tikzcd} \tag{2.27}$$

along $* \xrightarrow{\sigma} \mathfrak{O}(n)$.

This formulation allows us to see that the coherence condition for theorem 2.2.2.1.4 is indeed identical to the condition of being of configuration type of [Toë13, Proposition 3.5], which was expressed as the following diagram being homotopy cartesian:

$$\begin{tikzcd} \mathfrak{O}(n) \times \mathfrak{O}(m+1) \coprod_{\mathfrak{O}(n)\times\mathfrak{O}(2)\times\mathfrak{O}(m)} \mathfrak{O}(n+1) \times \mathfrak{O}(m) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_h ?"] & \mathfrak{O}(n+m) \arrow[d] \\ \mathfrak{O}(n) \times \mathfrak{O}(m) \arrow[r] & \mathfrak{O}(n+m-1) \end{tikzcd} \tag{2.28}$$

where the horizontal morphisms are induced by partial compositions and the vertical morphisms are induced by forgetting the last input (and the arrows defining the coproduct are also partial compositions). Taking the fibre product with a pair of active morphisms $* \xrightarrow{(\sigma,\tau)} \mathfrak{O}(n) \times \mathfrak{O}(n)$ we obtain the square

$$\begin{tikzcd} \mathrm{Ext}(\sigma) \coprod_{\mathfrak{O}(2)} \mathrm{Ext}(\tau) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_h ?"] & \mathrm{Ext}(\tau \circ_i \sigma) \arrow[d] \\ \{(\sigma, \tau)\} \arrow[r] & \{\tau \circ_i \sigma\} \end{tikzcd} \tag{2.29}$$

and we recover the coherence condition $\mathrm{Ext}(\tau \circ_i \sigma) = \mathrm{Ext}(\sigma) \coprod_{\mathrm{Ext}(\mathbb{1}_C)} \mathrm{Ext}(\tau)$.

Corollary 2.2.2.1.8. *Let* $X \in \mathfrak{S}$ *be a space and* $\mathfrak{O}^{\otimes}$ *be a coherent reduced* ∞*-operad. Define the* ***moduli space of*** $\mathfrak{O}^{\otimes}$***-branes in*** X *as* $\mathcal{B}_{\mathfrak{O}^{\otimes}}(X) \coloneqq \mathrm{Map}(\mathfrak{O}(2), X)$. *Then* $\mathcal{B}_{\mathfrak{O}^{\otimes}}(X)$ *has a structure of* $\mathfrak{O}^{\otimes}$*-algebra in correspondences.*

Proof. The presheaf represented by X gives an ∞-functor $\mathrm{Map}(-, X)\colon \mathfrak{S}^{\mathrm{op}} \to \mathfrak{S}$, which turns coproducts into products, and thus a monoidal ∞-functor $\mathfrak{S}^{\mathrm{op}\amalg} \to \mathfrak{S}^{\times}$, which also passes to correspondences (as it sends pushforwards to pullbacks). By composing with the brane action for $\mathfrak{O}^{\otimes}$, we obtain a morphism of operads $\mathfrak{O}^{\otimes} \to (\mathfrak{S}^{\times})^{\mathrm{corr}}$ sending the colour of $\mathfrak{O}^{\otimes}$ to the space $\mathcal{B}_{\mathfrak{O}^{\otimes}}(X)$. $\square$

2.2.2.2 Brane action in an ∞-topos

Combining remark 2.2.2.1.7 with the discussion at the end of section 2.1.2.1, a monochromatic ∞-operad $\mathcal{O}^{\otimes}$ in an ∞-topos $\mathfrak{T} = \mathfrak{Sh}_{\tau}(\mathfrak{C})$ is coherent (that is, sends any object of $\mathfrak{C}$ to a coherent ∞-operad) if and only if the square

$$\begin{tikzcd} \mathcal{O}_n \times \mathcal{O}_{m+1} \coprod_{\mathcal{O}_n \times \mathcal{O}_2 \times \mathcal{O}_m} \mathcal{O}_{n+1} \times \mathcal{O}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_h ?"] & \mathcal{O}_{n+m} \arrow[d] \\ \mathcal{O}_n \times \mathcal{O}_m \arrow[r] & \mathcal{O}_{n+m-1} \end{tikzcd} \tag{2.30}$$

is a homotopy cartesian square in $\mathfrak{T}$.

For the rest of the section, we operate under the assumption that for every $Z \in \mathfrak{C}$, the ∞-operad $\mathcal{O}^{\otimes}(Z)$ is reduced with unique colour C_Z.

For every object $Z \in \mathfrak{C}$, paragraph 2.2.2.1.3 provides a lax brane action $\mathcal{O}^{\otimes}(Z) \rightsquigarrow \left(\mathfrak{S}^{\mathrm{op}\amalg}\right)^{\mathrm{cocorr}}$, which by theorem 2.2.2.1.4 is a morphism of ∞-operads if and only if $\mathcal{O}^{\otimes}(Z)$ is coherent. To obtain a brane action on $\mathcal{O}^{\otimes}$, we thus need to study the compatibilities between these brane actions in spaces.

Lemma 2.2.2.2.1 (Functoriality of brane actions). *[MR18a, § 2.1.3] Let $\mathcal{F}\colon \mathfrak{O}^{\otimes} \to \mathfrak{P}^{\otimes}$ be a map of reduced ∞-operads. Then $\mathcal{F}$ induces an ∞-functor of fibrations in spaces from $\pi_{\mathfrak{O}}\colon \mathrm{B}\mathfrak{O} \to \mathfrak{Tw}(\mathfrak{Env}(\mathfrak{O}))^{\otimes}$ to $\pi_{\mathfrak{P}}\colon \mathrm{B}\mathfrak{P} \to \mathfrak{Tw}(\mathfrak{Env}(\mathfrak{P}))^{\otimes}$.*

Proof. Since $\mathfrak{Env}^{\otimes}(-)$, $\mathfrak{Tw}(-)^{\otimes}$ and $\mathfrak{Fun}([1], -)$ are (covariant) ∞-functors, we immediately obtain a map of fibrations from $\mathrm{ev}_{0,\mathfrak{O}}$ to $\mathrm{ev}_{0,\mathfrak{P}}$, and simply need to check that its underlying ∞-functor sends the sub-∞-category $\mathrm{B}\mathfrak{O}$ to $\mathrm{B}\mathfrak{P}$.

But since $\mathcal{F}$ is a morphism of ∞-operad, and in particular a morphism of ∞-categories over Γ, it sends a semi-inert morphism in $\mathfrak{O}^{\otimes}$ to a semi-inert morphism in $\mathfrak{P}^{\otimes}$ (as $\mathcal{F}$ preserves inert morphisms and projections to Γ). This ensures that $\mathcal{F}$ sends objects of $\mathrm{B}\mathfrak{O}$ to objects of $\mathrm{B}\mathfrak{P}$, and it will also preserve morphisms as the condition is on the map in Γ that they lift. $\square$

In our sheafified setting, if $U \to V$ is an arrow in $\mathfrak{C}$, it will by definition induce a map of ∞-operads $\mathcal{O}^{\otimes}(V) \to \mathcal{O}^{\otimes}(U)$, and thus a map between the associated brane actions on $\mathcal{O}^{\otimes}(V)(2)$ and $\mathcal{O}^{\otimes}(U)(2)$. It follows that, by the Yoneda lemma, the brane action should be visible at the level of the components $\mathcal{O}_2$.

Construction 2.2.2.2.2. Let $Z \in \mathfrak{C}$. We know from remark 2.2.2.1.7 that the brane action on $\mathcal{O}^{\otimes}(Z)(2) \simeq \mathrm{Map}_{\mathfrak{T}}(Z, \mathcal{O}_2) \simeq \mathrm{Map}_{\mathfrak{T}_{/Z}}(Z, \mathcal{O}_2 \times Z)$ is given, for $\sigma\colon Z \to \mathcal{O}_n$ (*i.e.* $\sigma \in \mathcal{O}^{\otimes}(Z)(n)$), by the cocorrespondences

$$\coprod_n \mathcal{O}^{\otimes}(Z)(2) \to \mathrm{Ext}(\sigma) \leftarrow \mathcal{O}^{\otimes}(Z)(2). \tag{2.31}$$

As we have seen, the space $\mathcal{O}^{\otimes}(Z)(2)$ can be written as $\mathrm{Map}_Z(Z, \mathcal{O}_2 \times Z)$. Concomitantly, by[Lur09, Proposition 5.1.2.3] we have

$$\begin{aligned}\mathrm{Ext}(\sigma) &:= * \times_{\mathcal{O}^{\otimes}(Z)(n)} \mathcal{O}^{\otimes}(Z)(n+1)\\ &\simeq \mathrm{Map}_Z(Z, Z) \times_{\mathrm{Map}_Z(Z,\mathcal{O}_n)} \mathrm{Map}_Z(Z, \mathcal{O}_{n+1})\\ &\simeq \mathrm{Map}_Z(Z, Z \times_{\mathcal{O}_n} \mathcal{O}_{n+1}).\end{aligned} \tag{2.32}$$

We finally deduce that the universal (relative) cocorrespondences given by the Yoneda lemma are

$$\begin{tikzcd} \coprod_n \mathcal{O}_2 \times \mathcal{O}_n \arrow[r] \arrow[dr] & \mathcal{O}_{n+1} \arrow[d] & \mathcal{O}_2 \times \mathcal{O}_n \arrow[l] \arrow[dl] \\ & \mathcal{O}_n & \end{tikzcd}, \tag{2.33}$$

from which (2.31) is obtained by first taking the pullback along $\sigma\colon Z \to \mathcal{O}_n$ and then taking sections by $\mathrm{Map}_Z(Z, -)$.

We wish to encode these diagrams as a morphism of ∞-operads in $\mathfrak{T}$.

Lemma 2.2.2.2.3. *[MR18a, Proposition 2.2.3] Let $\mathfrak{T} = \mathfrak{Sh}_\tau(\mathfrak{C})$ be a stack ∞-topos. The assignment*

$$\begin{aligned}\left((\mathfrak{T}_{/-})^{\amalg}\right)^{\mathrm{cocorr}} : \mathfrak{C}^{\mathrm{op}} &\to \mathfrak{Op}_\infty\\ Z &\rightsquigarrow \left((\mathfrak{T}_{/Z})^{\amalg}\right)^{\mathrm{cocorr}}\end{aligned} \tag{2.34}$$

defines an operad in $\mathfrak{T}$.

Proof. First, [Lur09, p. 6.3.5.1] ensures that for any $Z \in \mathfrak{T}$, the slice ∞-category $\mathfrak{T}_{/Z}$ is an ∞-topos. Since $\mathfrak{T}$ is in particular presentable with generating small category $\mathfrak{C}$, the proof of [Lur09, Theorem 6.1.6.8] implies that the class of morphisms in $\mathfrak{T}$ to (images under the Yoneda embedding of) objects of $\mathfrak{C}$ is local, and admits a classifying object written $\mathfrak{T}_{/-}$: for any $Z \in \mathfrak{C}$, the space $\mathrm{Map}(Z, \mathfrak{T}_{/-})$ is (categorically) equivalent to the maximal ∞-groupoid of $\mathfrak{T}_{/Z}$. Furthermore, this locality property also ensures that $\mathfrak{T}_{/-}$ is a sheaf on $\mathfrak{C}$.

Since both functors $(-)^{\mathrm{op}}$ and $(-)^{\mathrm{corr}}$ admit left adjoints, they preserve limits so we obtain a sheaf of ∞-operads $\left((\mathfrak{T}_{/-})^{\amalg}\right)^{\mathrm{cocorr}} : \mathfrak{C}^{\mathrm{op}} \to \mathfrak{Op}_\infty$. ☐

Corollary 2.2.2.2.4. *We have similarly a categorical ∞-operad $\mathfrak{Span}\left((\mathfrak{T}_{/-})^{\amalg}\right)$ in $\mathfrak{T}$.* ☐

Theorem 2.2.2.2.5 (Lax brane action). *[MR18a, Proposition 2.2.4] Let $\mathfrak{T} = \mathfrak{Sh}_\tau(\mathfrak{C})$ be a stack ∞-topos, and $\mathcal{O}^\otimes \colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{Op}_\infty$ an operad in $\mathfrak{T}$ taking values in reduced ∞-operads. There is a lax morphism of operads in $\mathfrak{T}$ from $\mathcal{O}^\otimes$ to $\left((\mathfrak{T}_{/-})^{\amalg}\right)^{\mathrm{cocorr}}$, sending for each $Z \in \mathfrak{C}$ the colour C_Z of $\mathcal{O}^\otimes(Z)$ to the space $\mathrm{Ext}(\mathbb{1}_{C_Z})$.*

The morphism is given in the following way. As a morphism of operads in $\mathfrak{T}$ is nothing but a morphism in $\mathfrak{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Op}_\infty)$ (a natural transformation of sheaves), we may use the adjunctions $\mathfrak{Env}^\otimes$ and $\mathfrak{Tw}^\otimes \dashv (-^\times)^{\mathrm{corr}}$ to express the required morphism as a natural transformation $\mathfrak{Tw}(\mathfrak{Env}(\mathcal{O}))^\otimes \to (\mathfrak{T}_{/-}{}^{\mathrm{op}})^\times$, that is a natural transformation $\mathfrak{Tw}(\mathfrak{Env}(\mathcal{O}))^\otimes \to \mathfrak{T}_{/-}{}^{\mathrm{op}}$ verifying the definition of a weak cartesian structure. Passing to the Grothendieck constructions, this becomes a morphism of cartesian fibrations over $\mathfrak{T}$

$$\begin{tikzcd}
\int \mathfrak{Tw}(\mathfrak{Env}(\mathcal{O}))^\otimes \arrow[rr] \arrow[dr] & & \int \mathfrak{T}_{/-}{}^{\mathrm{op}} \arrow[dl] \\
& \mathfrak{T} &
\end{tikzcd} . \tag{2.35}$$

It can then be seen (*cf.* [MR18a, Remark 2.2.5]) that this corresponds a a fibration in spaces

$$\mathrm{B}^{\mathfrak{T}}(\mathcal{O}) \to \int \mathfrak{Tw}(\mathfrak{Env}(\mathcal{O}))^\otimes \times_{\mathfrak{T}} \mathfrak{Fun}([1], \mathfrak{T}). \tag{2.36}$$

Theorem 2.2.2.2.6 (Brane action). *The lax morphism defines a morphism of operads in $\mathfrak{T}$ if and only if $\mathcal{O}^\otimes$ is coherent.* ☐

For any object $F \in \mathfrak{T}$, there is an ∞-functor $\mathfrak{T} \to \mathfrak{Fun}^{\lim}(\mathfrak{T}^{\mathrm{op}}_{/-}, \mathfrak{T}_{/-})$, $Z \rightsquigarrow \mathbb{R}\mathcal{M}ap_{/Z}(-, F \times Z)$, which passes to a map of ∞-operads in $\mathfrak{T}$

$$\mathbb{R}\mathcal{M}ap_{/-}(\bullet, F \times -) \colon \left((\mathfrak{T}_{/-})^{\amalg}\right)^{\mathrm{cocorr}} \to \left((\mathfrak{T}_{/-})^{\times}\right)^{\mathrm{corr}}. \tag{2.37}$$

Corollary 2.2.2.2.7. *[MR18a, § 2.2.2] For $\mathcal{X} \in \mathfrak{T}$, write $\mathcal{B}_{\mathcal{O}^\otimes}(\mathcal{X}) = \mathbb{R}\mathcal{M}ap(\mathcal{O}_2, \mathcal{X})$ for the moduli stack of $\mathcal{O}^\otimes$-branes in $\mathcal{X}$. Then $\mathcal{B}_{\mathcal{O}^\otimes}(\mathcal{X})$ has an induced structure of $\mathcal{O}^\otimes$-algebra in correspondences in $\mathfrak{T}$.*

As in (2.36), the brane action is given by a cocartesian fibration in spaces

$$\mathrm{B}^{\mathfrak{T}}(\mathcal{O}, \mathcal{X}) \to \int^{\mathrm{co}} \mathfrak{Tw}(\mathfrak{Env}(\mathcal{O}))^\otimes \times_{\mathfrak{T}^{\mathrm{op}}} \mathfrak{Fun}([1], \mathfrak{T})^{\mathrm{op}}, \tag{2.38}$$

such that the fibre over $(\sigma \colon \mathcal{Z} \to \mathcal{O}_n, u \colon \mathcal{Y} \to \mathcal{Z})$ is informally given by the space $\mathrm{Map}_{/\mathcal{Z}}(\mathcal{Y}, \mathbb{R}\mathcal{M}ap_{/\mathcal{Z}}(\mathcal{Z} \times_{\mathcal{O}_n} \mathcal{O}_{n+1}, \mathcal{X} \times \mathcal{Z}))$.

Part II

Gromov–Witten theory

Chapter 3

The operad of stable maps

3.1 The Deligne–Mumford modular operad

3.1.1 Stable curves

3.1.1.1 Definition and moduli space

Definition 3.1.1.1.1 (Family of curves)**.** A **curve of genus** g **with** n **marked points** is the data of a projective curve C, that is a smooth variety projective over $\operatorname{Spec} k$ of pure dimension 1, of arithmetic genus $\dim H^0(C, \Omega_C) = g$, with a choice of n marked points $x_1, \dots, x_n \in C$. A **morphism** of curves with n marked points is a morphism of the underlying schemes over $\operatorname{Spec} k$ sending each marked point of the source curve to the corresponding marked point (with respect to the labelling) in the target curve.

Let S be a scheme. A **family** of curves of genus g with n marked points over S is a flat morphism $C \to S$ with n sections $\sigma_1, \dots, \sigma_n \colon S \to C$, such that each fibre C_s, endowed with the images $(\sigma_i(s))_{1 \leq i \leq n}$ of the sections, is a smooth curve of genus g with n marked points. A **morphism** $(C \to S; \sigma_1, \dots \sigma_n) \to (C' \to S; \sigma'_1, \dots \sigma'_n)$ of curves with n marked points over S is a morphism $f \colon C \to C'$ of the underlying S-schemes such that $f \circ \sigma_i = \sigma'_i$ for $i = 1, \dots, n$.

We let $\mathcal{M}_{g,n}$ denote the moduli stack parameterising curves of genus g with n marked points, that is the stack whose S-points under an affine scheme S are given by the groupoid whose objects are families of curves over S and morphisms their automorphisms. Its structure morphism $\mathcal{M}_{g,n} \to \operatorname{Spec} k$ is not proper. Intuitively, this is because of the degeneracies which appear when moving marked points closer to each other or when pinching cycles around the curve. We then introduce a more general type of curves in which we allow certain singularities so as to compactify the moduli stack.

Definition 3.1.1.1.2 (Prestable curve)**.** Let C be an at worst nodal, reduced, connected, projective scheme of dimension 1 and genus g with n marked points $x_1,\dots,x_n$. Then $(C; x_1, \dots, x_n)$ is said to be **prestable** if all the marked points are disjoint from the nodes.

We call **special points** the points of C which are either nodal or marked points.

Definition 3.1.1.1.2.1 (Stable curve)**.** A prestable curve $(C; x_1, \dots, x_n)$ is called **stable** if each component of genus 0 (resp. 1) has at least 3 (reps. 1) special points.

We have a similar definition in the relative setting.

Lemma 3.1.1.1.2.2. *A prestable curve is stable if and only if its automorphism group is finite.*

Theorem 3.1.1.1.3. *[DM69, Proposition 5.1, Theorem 5.2] The stack of stable curves $\overline{\mathcal{M}_{g,n}}$ is a smooth proper Deligne–Mumford stack of finite type over* k.

By definition, $\overline{\mathcal{M}_{g,n}}$ carries a **universal curve** $\overline{\mathcal{C}_{g,n}} \to \overline{\mathcal{M}_{g,n}}$ with universal sections, inducing any other stable curve $C \to S$ by pullback along a unique arrow $S \to \overline{\mathcal{M}_{g,n}}$.

Remark 3.1.1.1.4 (Dual graph). Let $(C; x_1, \dots, x_n)$ be a prestable curve. We can associate to it a labelled graph $\Gamma_{(C;x_1,\dots,x_n)}$. A vertex of $\Gamma_{(C;x_1,\dots,x_n)}$ is a connected component of C, to which we associate as label the genus of the component. Two vertices are connected by an edge whenever there is a node of C connecting the corresponding components, and the marked points of C are translated to legs at the corresponding vertices. The genus of the graph is the sum of the genera given as labels and the number of loops appearing; it coincides with the genus of C.

3.1.1.2 Gluing and stabilisation of curves

In this section we describe the morphisms between moduli spaces of stable curves with different genera and numbers of marked points.

Let $n \geq 3$. There is a **contraction** morphism $\sigma \colon \overline{\mathcal{M}_{g,n+1}} \to \overline{\mathcal{M}_{g,n}}$. On the level of points, acting on a point representing a stable curve with $n+1$ marked points, this morphism forgets the marking x_{n+1} of the last point, and stabilises the curve if forgetting the point has made it unstable. More precisely, if x_{n+1} was on a rational component C_i with only 2 other special points, the stabilisation morphism will contract this component and redistribute the special as follows:

- if the special points are two nodes joining C_i to other components C_j and C_k respectively, so that the original component was a rational bridge with one marked point, then the C_i will be replaced by a node joining C_j and C_k;
- if the special points are one node and one marked point, so that the original component was a rational tail with two marked point, then C_i is deleted and the other marked point is inserted at the position of the original node.

Remark 3.1.1.2.1. We could as well define moduli spaces $\overline{\mathcal{M}_{g,I}}$ of curves with marked points indexed by a finite set I, and describe similar stabilisation morphisms $\overline{\mathcal{M}_{g,I}} \to \overline{\mathcal{M}_{g,I\setminus i}}$ for any $i \in I$. We have simply chosen to simplify the combinatorics of the morphisms by using the natural order to select the last marked point (equivalently, we have restricted to a skeleton of the category of finite set and reduced to invariants with regard to the automorphisms).

Note that the contraction morphism is of relative dimension 1, as the fibers are the loci of possible positions for the additional marked point, that is the logarithmically smooth loci of the divisor of marked points. The following is well-known.

Property 3.1.1.2.2. *The contraction morphism is the universal curve over $\overline{\mathcal{M}_{g,n}}$.*

Fix now two genera g_1, g_2 and two numbers of marked points n_1, n_2. There is a **gluing** morphism $\gamma\colon \overline{\mathcal{M}_{g_1,n_1}} \times \overline{\mathcal{M}_{g_2,n_2}} \to \overline{\mathcal{M}_{g_1+g_2,n_1+n_2-2}}$. At the level of points, it simply glues the first marked point of the curve with n_2 markings to the last marked point of the curve with n_1 markings, and forgets the marking the resulting point (since it is nodal). The fact that the points are glued to a node ensures that the number of special points on each component, and thus the stability, is unaffected.

Finally, given a genus g and a number of marked points n, there is a "**gluing to a loop**" morphism $\lambda\colon \overline{\mathcal{M}_{g,n}} \to \overline{\mathcal{M}_{g+1,n-2}}$, which at the level of points glues the last two marked points of a curve into a node, thereby adding a loop to the curve.

3.1.2 Operadic structure

3.1.2.1 Modular operads as a kind of generalised operads

A cyclic operad (whose definition we recall in section 3.1.2.2) is a symmetric (monochromatic) operad with an additional operation permuting the inputs and the outputs. Whereas operads are usually represented using rooted trees, cyclic operads will then correspond to connected trees, or connected graphs of genus 0. We wish for modular operads to provide a higher genus generalisation of cyclic operads: while the latter generalise operads by allowing to exchange inputs and output, they still only allow a linear form of composition, as contractions (composition by using the output as an input) increase the genus.

It seems clear (and will be shown in section 3.1.2.3) that the contraction morphisms of the moduli stacks of stable curves ought to provide a prototypical example for modular operads.

Remark 3.1.2.1.1 (Monads and generalised operads). Recall (from *e.g.* [LV12]) that a monochromatic operad in a monoidal category $(\mathfrak{V}, \otimes)$ is nothing but a monad on $\mathfrak{V}$ with the additional structure of an $\mathbb{S}$-module. In fact more general forms of generalised operads can usually be described as monoids over monads (or 2-monads) on certain categorical constructions (so-called "virtual double categories", multi-objects versions of pseudo-double categories).

Hence we will pursue this point of view to define modular operads.

Let $(\mathfrak{V}, \otimes, I)$ be a monoidal category with final object $*$.

Definition 3.1.2.1.2 (Stable $\mathbb{S}$-module)**.** A stable $\mathbb{S}$-module $\mathcal{M}$ in $\mathfrak{V}$ is a family of objects $(\mathcal{M}((g,n)))_{n,g\geq 0}$ with an action of $\mathbb{S}_n$ on $\mathcal{M}((g,n))$ for every g, and such that $\mathcal{M}((g,n)) = *$ whenever $2g-2+n \leq 0$.

A morphism of stable $\mathbb{S}$-modules is given by a collection of $\mathbb{S}_\bullet$-equivariant morphisms between the components.

Indeed, this is the case for the collection of moduli spaces of stable maps, which are empty whenever $2g-2+n \leq 0$.

Given that stable curves can be represented by their dual graphs, and that free operads can often be constructed by appropriately summing over graphs, we will define a monad on the category of stable $\mathbb{S}$-modules from stable graphs.

Definition 3.1.2.1.3 (Stable graph)**.** Let (G, g) be a **labelled graph**, that is a connected graph G together with a map g from the set of vertices of G to $\mathbb{N}$. For any vertex v we denote $n(v)$, called its **valence**, the number of legs of G attached to v.

The labelled graph is said to be **stable** if $2(g(v)-1)+n(v) > 0$ for every vertex v of G.

The genus of a labelled graph (G, g) is defined[GK98, eq. 2.9, 2.10] as

$$g(G) = 1+\frac{1}{2}\left(-n+\sum_{v}(2(g(v)-1)+n(v))\right) = \sum_{v}(g(v)-1)+\#\{\text{edges}(G)\}+1. \quad (3.1)$$

Construction 3.1.2.1.4 (The graphs monad)**.** Let $\Gamma((g,n))$ be the category whose objects are stable graphs of genus g with n legs equipped with a linear ordering of the legs, and whose morphisms are the morphisms of labelled (stable) graphs respecting the ordering of the legs. By [GK98, Lemma 2.16], the set $[\Gamma((g,n))]$ of isomorphism classes of objects of this category is finite.

Let $\mathcal{M}$ be a stable $\mathbb{S}$-module in $\mathfrak{V}$. For any stable graph $G \in \Gamma((g,n))$, we set $\mathcal{M}((G)) = \bigotimes_v \mathcal{M}((g(v), n(v)))$. Then, for any g, n, we let

$$\mathbb{G}\mathcal{M}((g,n)) = \varinjlim_{G \in \mathfrak{Iso}(\Gamma((g,n)))} \mathcal{M}((G)) \simeq \bigoplus_{G\in[\Gamma((g,n))]} \mathcal{M}((G))_{\mathrm{Aut}(G)}. \quad (3.2)$$

This construction is functorial, and the induced endofunctor $\mathbb{G}$ of the category of stable $\mathbb{S}$-modules has in fact the structure of a monad by [GK98, § 2.17].

Definition 3.1.2.1.5 (Modular operad)**.** A **modular operad** in $\mathfrak{V}$ is an algebra over the monad $\mathbb{G}$ (in the category of $\mathbb{S}$-modules of $\mathfrak{V}$).

3.1.2.2 Modular operads as graded cyclic operads with contractions

Definition 3.1.2.2.1 (Cyclic operad)**.**

- A **cyclic $\mathbb{S}$-module** $\mathcal{C}$ is a collection of objects $(\mathcal{C}(n))_{n\geq 0}$ with an action of $\mathbb{S}_{n+}$ on $\mathcal{C}(n)$ (where $\mathbb{S}_{n+} \cong \mathbb{S}_{n+1}$ is the group of permutations of $\{0, \dots, n\}$). In particular each $\mathcal{C}(n)$ has an action of $\mathbb{S}_n$ (so $\mathcal{C}$ is an $\mathbb{S}$-module), and an action of the cyclic subgroup generated by the permutation $(01\cdots n)$.
- A **cyclic operad** is a cyclic $\mathbb{S}$-module $\mathcal{C}$ whose underlying $\mathbb{S}$-module has a structure of operad and such that, using the partial composition notation, for any $\sigma \in \mathcal{C}(m), \tau \in \mathcal{C}(n)$, we have $(\sigma \circ_m \tau)^{(01\cdots(n+m-1))} = \tau^{(01\cdots n)} \circ_1 \sigma^{(01\cdots m)}$.

We usually write $\mathcal{C}((n)) \coloneqq \mathcal{C}(n-1)$ for $n > 0$. Note that partial composition will then map $\mathcal{C}((n)) \otimes \mathcal{C}((m)) \xrightarrow{\circ_i} \mathcal{C}(n+m-2)$.

Remark 3.1.2.2.2. The category of stable $\mathbb{S}$-modules has a natural forgetful functor $\mathcal{C}$ to the category of cyclic $\mathbb{S}$-modules whose objects of n-ary operations is $*$ for $n < 2$, which forgets the higher genus components, that is $\mathcal{C}(\mathcal{M})((n)) \coloneqq \mathcal{M}((0, n))$. It admits as left adjoint the free stable $\mathbb{S}$-module functor $\mathcal{F}$, which to a cyclic $\mathbb{S}$-module $\mathcal{C}$ with $\mathcal{C}((0)) = \mathcal{C}((1)) = *$ associates $\mathcal{F}(\mathcal{C})$ such that $\mathcal{F}(\mathcal{C})((g, n))$ is $\mathcal{C}((n))$ if $g = 0$ and $*$ if $g > 0$.

Definition 3.1.2.2.3 (Graded cyclic operad)**.** A **grading** on a cyclic operad $\mathcal{C}$ is an $\mathbb{S}_n$-invariant decomposition $\mathcal{C}((n)) = \coprod_{g \geq 0} \mathcal{C}((g, n))$ such that partial composition maps $\mathcal{C}((g, n)) \otimes \mathcal{C}((h, m)) \xrightarrow{\circ_i} \mathcal{C}((g+h, n+m-2))$.

We say that a graded cyclic operad $\mathcal{C}$ is **stable** if $\mathcal{C}((g, n)) = *$ whenever $2g-2+n \leq 0$.

Construction 3.1.2.2.4 (Contraction maps for modular operads)**.** Let $G \in \Gamma((g, n))$ be a stable graph. Let $i, j \leq n$. Gluing together the legs i and j of G adds a loop while forgetting two legs, and hence gives a graph $\chi_{i,j} G \in \Gamma((g+1, n-2))$. Since forgetting legs preserves the relative ordering of the remaining legs, this operation commutes with the isomorphisms in $\Gamma((g, n))$ and defines a functor $\mathfrak{Iso}\left(\Gamma((g, n))\right) \to \Gamma((g+1, n-2))$. In particular, for any stable $\mathbb{S}$-module $\mathcal{M}$, we will obtain morphisms $\chi_{i,j} \colon \mathbb{G}\mathcal{M}((g, n)) \to \mathbb{G}\mathcal{M}((g+1, n-2))$, giving if $\mathcal{M}$ is a modular operad morphisms $\chi_{i,j} \colon \mathcal{M}((g, n)) \to \mathcal{M}((g+1, n-2))$.

Theorem 3.1.2.2.5 (Other definition of modular operads)**.** *[GK98, Theorem 3.7] The data of a modular operad is equivalent to that of a stable graded cyclic operad with contraction maps $\chi_{i,j} \colon \mathcal{C}((g, n)) \to \mathcal{C}((g+1, n-2))$ satisfying the following coherence conditions for any $n, m \geq 0$, $i, j, k, \ell \leq n$ distinct and $\sigma \in \mathcal{C}(m), \tau \in \mathcal{C}(n)$:*

1. $\chi_{i,j}\chi_{k,\ell} = \chi_{k,\ell}\chi_{i,j}$;
2. $\chi_{1,2}(\sigma \circ_m \tau) = (\chi_{1,2}\sigma) \circ_{m-2} \tau$;
3. $\chi_{m,m+1}(\sigma \circ_m \tau) = \sigma \circ_m (\chi_{1,2}\tau)$;
4. $\chi_{m-1,m}(\sigma \circ_m \tau) = \chi_{m+n-2,m+n-1}(\sigma \circ_{m-1} \tau^{(01\cdots n)})$.

Example 3.1.2.2.6 (Endomorphisms modular operad). Suppose $V \in \mathfrak{V}$ is equipped with a trace map $T \colon V \otimes V \to I$ (for example $(\mathfrak{V}, \otimes, I) = (\mathfrak{Mod}_k, \otimes, k)$ is the category of k-vector spaces and T is a bilinear form on V). We then define a modular operad $\mathcal{E}[V]$ by $\mathcal{E}[V]((g, n)) = V^{\otimes n+1}$ for any stable g, n, with $\mathbb{S}_n$-action given by permutation of the factors, and the compositions and contractions induced by T.

An **algebra** over a modular operad $\mathcal{M}$ is an object V with a trace map and a morphism of modular operads $\mathcal{M} \to \mathcal{E}[V]$.

3.1.2.3 Cohomological field theories and the operad of moduli of stable curves

Proposition 3.1.2.3.1. *The collection $\left(\overline{\mathcal{M}_{g,n}}\right)_{g,n}$ defines a modular operad in the category of algebraic stacks (with the cartesian monoidal structure), with $\mathbb{S}_\bullet$-action given by permutation of the marked points and composition given by gluing of curves at the marked points.*

Proof. Put $\mathcal{M}((g,n)) = \overline{\mathcal{M}_{g,n+1}}$, with the $\mathbb{S}_{n+}$-action (at the level of points) permuting the marked points of a stable curve. Clearly the stability condition on curves is equivalent to $\mathcal{M}$ being stable as a graded cyclic $\mathbb{S}$-module. The operad structure is given, in terms of partial composition, by the gluing maps $\gamma\colon \mathcal{M}((g_1,n_1)) \times \mathcal{M}((g_2,n_2)) \to \mathcal{M}((g_1+g_2, n_1+n_2-1))$. The contraction maps are given by the gluing to a loop $\lambda\colon \mathcal{M}((g,n)) \to \mathcal{M}((g,n-2))$. It is straightforward to check that these maps satisfy the compatibilities for a modular operad. □

The structure of modular operad of the moduli stacks of stable curves induces a modular operad in graded abelian groups on the collection $\left(A_\bullet\overline{\mathcal{M}_{g,n}}\right)_{g,n}$.

Definition 3.1.2.3.2 (Cohomological field theory)**.** A **cohomological field theory** (or **CohFT**) is an algebra over the modular operad $\left(A_\bullet\overline{\mathcal{M}_{g,n}}\right)_{g,n}$ in the category of graded abelian groups.

Remark 3.1.2.3.3. By remark 3.1.2.2.2, the collection of genus zero moduli stacks defines a cyclic operad in DM-stacks $(\overline{\mathcal{M}_{0,n}})_n$. To force unitality, we replace $\overline{\mathcal{M}_{0,2}} = \emptyset$ by a single point $*$, to be thought of as parameterising a projective line $\mathbb{P}^1_k$ with 2 marked points "stabilised" by removing automorphisms. We then obtain a unital operad in DM stacks $\overline{\mathcal{M}}$ with $\overline{\mathcal{M}}(0) = \emptyset$, $\overline{\mathcal{M}}(1) = *$ (acting as the unit with regard to the composition) and $\overline{\mathcal{M}}(n) = \overline{\mathcal{M}_{0,n+1}}$ for $n \geq 2$.

A **tree-level CohFT** is defined as an algebra over the operad in graded abelian groups $A_\bullet\overline{\mathcal{M}}$.

Since $\overline{\mathcal{M}}((g,n)) = \overline{\mathcal{M}_{g,n+1}}$, a CohFT on an abelian group G will be given by morphisms $A_\bullet\overline{\mathcal{M}_{g,n+1}} \to G^{\otimes n+1}$. Using the Poincaré pairing on to dualise and the trace of G to partially dualise, this is equivalent to maps $(G^\vee)^{\otimes n+1} \to A^\bullet\overline{\mathcal{M}_{g,n+1}}$.

The structure of a CohFT is thus given by the following data: an abelian group H with a non-degenerate pairing $\langle\cdot,\cdot\rangle$, an element $1\colon \mathbb{Z} \to H$, and for each (g,n) an $\mathbb{S}_n$-equivariant homomorphism $F_{g,n}\colon H^{\otimes n} \to A^\bullet\overline{\mathcal{M}_{g,n}}$ such that:

- for any $\alpha_1, \dots, \alpha_{n_1+n_2-2} \in H$,

$$\begin{aligned}\gamma^* F_{g_1+g_2,n_1+n_2-2}&(\alpha_1 \otimes \cdots \otimes \alpha_{n_1+n_2-2}) \\ &= \sum_{i,j} F_{g_1,n_1}(\alpha_1 \otimes \cdots \otimes \alpha_{n_1-1} \otimes h_i)\eta^{i,j} F_{g_2,n_2}(h_j \otimes \alpha_{n_1} \otimes \cdots \otimes \alpha_{n_1+n_2})\end{aligned} \tag{3.3}$$

  where $(h_i)_i$ is a basis of H and $(\eta^{i,j})_{i,j}$ the inverse of the pairing matrix $\eta_{i,j} = \langle h_i, h_j\rangle$;

- for any $\alpha_1, \dots, \alpha_n \in H$,

$$\lambda^* F_{g+1,n}(\alpha_1 \otimes \cdots \otimes \alpha_n) = \sum_{i,j} F_{g,n+2}(\alpha_1 \otimes \cdots \otimes \alpha_n \otimes h_i \otimes h_j)\eta^{i,j} \tag{3.4}$$

- for any $\alpha_1, \dots, \alpha_n \in H$,

$$\sigma^* F_{g,n}(\alpha_1 \otimes \cdots \otimes \alpha_n) = F_{g,n+1}(\alpha_1 \otimes \cdots \otimes \alpha_n \otimes 1) \tag{3.5}$$

- for any $\alpha_1, \alpha_2 \in H$,

$$\int_{[\overline{\mathcal{M}_{0,3}}]} F_{0,3}(\alpha_1, \alpha_2, 1) = \langle \alpha_1, \alpha_2 \rangle. \tag{3.6}$$

3.2 Stable maps and Gromov–Witten theory

3.2.1 The moduli stack of stable maps

Let X be a smooth projective k-variety, and $\beta \in A_1 X$ be a cycle class in X.

3.2.1.1 Definition of the moduli stack

Definition 3.2.1.1.1 (Stable map)**.** A **genus** g **stable map with** n **marked points to** X **with class** β is the data of a prestable curve $(C; x_1, \dots, x_n)$ of genus g with n marked points and a morphism $f\colon C \to X$ such that $f_*[C] = \beta$, respecting the Kontsevich stability condition: any irreducible component sent to a point must be stable as a marked curve.

Our aim is to study the moduli stack $\overline{\mathcal{M}_{g,n}}(X, \beta)$ of stable maps to X with class β. This moduli space of stable maps fits into the natural mapping space diagram

$$\begin{tikzcd} & \overline{\mathcal{M}_{g,n}}(X,\beta) \arrow[dl, "\mathrm{Stab}"'] \arrow[dr, "\mathrm{ev}"] & \\ \overline{\mathcal{M}_{g,n}} & & X^n \end{tikzcd}, \tag{3.7}$$

where $\mathrm{Stab}\colon \overline{\mathcal{M}_{g,n}}(X, \beta) \to \overline{\mathcal{M}_{g,n}}$ is the morphism which forgets the map and stabilises the source curve, and $\mathrm{ev}\colon \overline{\mathcal{M}_{g,n}}(X, \beta) \to X^n$ is the evaluation morpism $(\mathrm{ev}_1, \dots, \mathrm{ev}_n)$ induced by the morphisms of evaluation of the map at the ith marked points $\mathrm{ev}_i\colon \overline{\mathcal{M}_{g,n}}(X, \beta) \to X$.

A class $\beta \in A_1 X$ is said to be **effective** if there is a stable map $f\colon C \to X$ such that $f_*[C] = \beta$. We denote $NE(X)$ the Mori cone of (numerically) effective classes ([Deb16]). It has the structure of a semigroup; furthermore it has the properties required by [Cos06], that is:

indecomposable zero: $\beta + \gamma = 0$ implies $\beta = \gamma = 0$;

finite decomposition: for every $\alpha \in NE(X)$, the set $\{(\beta, \gamma) \in NE(X)^2 \mid \beta + \gamma = \alpha\}$ is finite.

Let $\beta \in NE(X)$. We let $\mathfrak{M}_{g,n,\beta}$ be the moduli stack, constructed formally in [Cos06, § 2.0], parameterising prestable curves $(C; x_1, \dots, x_n)$ of genus g with n marked points and with each irreducible component C_i labelled by a class $\beta_i \in A_1X$ such that $\sum_i \beta_i = \beta$, with the stability condition that C_i must be stable as soon as $\beta_i = 0$.

Property 3.2.1.1.2. *[Cos06, Proposition 2.0.1] The stack $\mathfrak{M}_{g,n,\beta}$ is a smooth algebraic stack.*

Let $\mathfrak{C}_{g,n,\beta} \to \mathfrak{M}_{g,n,\beta}$ denote the universal curve over the moduli stack. There is also the forgetful morphism $\mathfrak{M}_{g,n+1,\beta} \to \mathfrak{M}_{g,n,\beta}$, which forgets the last marked point and stabilises if needed the resulting unstable components.

Proposition 3.2.1.1.3. *[Cos06, Proposition 2.2.2] There is an isomorphism $\mathfrak{C}_{g,n,\beta} \cong \mathfrak{M}_{g,n+1,\beta}$ of stacks over $\mathfrak{M}_{g,n,\beta}$.*

Corollary 3.2.1.1.4. *[Beh97, Proposition 4] The stack $\overline{\mathcal{M}_{g,n}}(X, \beta)$ is an open substack of $\mathfrak{Hom}_{/\mathfrak{M}_{g,n,\beta}}(\mathfrak{M}_{g,n+1,\beta}, X \times \mathfrak{M}_{g,n,\beta})$ (where $\mathfrak{Hom}_{/\bullet}$ denotes the relative internal mapping stack).*

Proof. This follows from the fact that $\mathfrak{M}_{g,n+1,\beta} \to \mathfrak{M}_{g,n,\beta}$ is the universal curve. Let S be a $\mathfrak{M}_{g,n,\beta}$-scheme. We have

$$\begin{aligned}\mathfrak{Hom}_{/\mathfrak{M}_{g,n,\beta}}(\mathfrak{M}_{g,n+1,\beta}, X \times \mathfrak{M}_{g,n,\beta})(S) &\coloneqq \hom_{/S}\left(\mathfrak{M}_{g,n+1,\beta} \underset{\mathfrak{M}_{g,n,\beta}}{\times} S, X \times \mathfrak{M}_{g,n,\beta} \underset{\mathfrak{M}_{g,n,\beta}}{\times} S\right) \\ &= \hom_{/S}\left(\mathfrak{C}_{g,n,\beta} \underset{\mathfrak{M}_{g,n,\beta}}{\times} S, X \times S\right) \\ &= \hom\left(\mathfrak{C}_{g,n,\beta} \underset{\mathfrak{M}_{g,n,\beta}}{\times} S, X\right).\end{aligned} \tag{3.8}$$

But by property of the universal curve, the pullback $\mathfrak{C}_{g,n,\beta} \times_{\mathfrak{M}_{g,n,\beta}} S$ is a family of curves $C \to S$ selected by the structure map $S \to \mathfrak{M}_{g,n,\beta}$.

Therefore this category is exactly the category of all S-parameterised families of prestable maps to X whose source is compatible with β. The degree condition $f_*[C] = \beta$ (and $f_*[C_i] = \beta_i$ for irreducible components C_i with marking β_i from a decomposition of β) is semi-continuous so $\overline{\mathcal{M}_{g,n}}(X, \beta)$ is open. □

Corollary 3.2.1.1.5. *[Beh97, discussion between Propositions 4 and 5] The stack $\overline{\mathcal{M}_{g,n}}(X, \beta)$ has a canonical relative perfect obstruction theory of virtual dimension*

$$\operatorname{vdim} \overline{\mathcal{M}_{g,n}}(X, \beta) = (\dim X - 3)(1 - g) + n + \int_\beta c_1(\mathcal{T}_X). \tag{3.9}$$

Proof. A perfect obstruction theory on the mapping stack from the universal curve is given by the relative version of example 1.1.2.2.3. Since $\overline{\mathcal{M}_{g,n}}(X, \beta)$ is open, it inherits an induced relative perfect obstruction theory.

The computation of the virtual dimension then follows by the Riemann–Roch theorem and arguments from deformation theory (as the virtual dimension is locally constant). See for example the explanation in [Nab15, §3.4]. □

3.2.1.2 Operadic structure

Construction 3.2.1.2.1 (Operad of decorated prestable maps). For any $n \geq 2$, any genus $g \geq 0$ and any class $\beta \in NE(X)$, set $\mathfrak{M}((g, n, \beta)) := \mathfrak{M}_{g,n+1,\beta}$. Similarly, for any $g \geq 1$ and $n = 1$, or $n = 0$ and $g \geq 2$, set also $\mathfrak{M}((g, n, \beta)) := \mathfrak{M}_{g,n+1,\beta}$. For $g = 1$ and $n = 0$, or $g = 0$ and $n \leq 1$, set $\mathfrak{M}((g, n, 0)) = \operatorname{Spec} k$ and $\mathfrak{M}((g, n, \beta)) = \emptyset$ for any $\beta \neq 0$.

This produces a $NE(X)$-graded stable $\mathbb{S}$-module $\mathfrak{M} = (\mathfrak{M}((g, n, \beta)))_{(g,n)\in\mathbb{N}^2,\beta\in NE(X)}$ in the cartesian monoidal category of stacks in groupoids.

Proposition 3.2.1.2.2. *[MR18a, Proposition 3.1.4] The graded stable $\mathbb{S}$-module $\mathfrak{M}$ is a $NE(X)$-graded modular operad in algebraic stacks.*

Proof. Similarly to the case of proposition 3.1.2.3.1, we obtain the operations defining a modular operad from the operations on curves described in section 3.1.1.2. There only remains to check that these operations are compatible with the grading. If $\sigma_1 \in \mathfrak{M}_{g_1,n_1,\beta_1}$ and $\sigma_2 \in \mathfrak{M}_{g_2,n_2,\beta_2}$, then the curve obtained by gluing will have the marking β_1 on the components coming from σ_1 and marking β_2 on the those coming from σ_2, which from the definition of prestable curves with markings means that it will be a curve parameterised by $\mathfrak{M}_{g_1+g_2,n_1+n_2-1,\beta_1+\beta_2}$. It is even easier to see that the other operations respect the grading as well. $\square$

Corollary 3.2.1.2.3. *The induced $NE(X)$-graded cyclic $\mathbb{S}$-module $\mathfrak{M}_0 = (\mathfrak{M}((0, n, \beta)))_{n\in\mathbb{N},\beta\in NE(X)}$ is a $NE(X)$-graded cyclic operad in algebraic stacks.* $\square$

Proposition 3.2.1.2.4. *There is a morphism of $NE(X)$-graded modular operads $\mathfrak{M} \to \overline{\mathcal{M}} \times NE(X)$.*

Similarly, there is an induced morphism of operads $\mathfrak{M}_0 \to \overline{\mathcal{M}_0} \times NE(X)$.

Proof. First we note the following. Let B be a monoid with indecomposable zero and finite decompositions. There is a canonical functor from the category of B-graded modular operads to that of modular operads, forgetting the B-graded structure by taking B-indexed coproducts: $\mathcal{G}((g, n)) := \coprod_{b\in B} \mathcal{G}((g, n, b))$. This functor admits a "trivial B-grading" right adjoint, given by product with B, with the obvious induced grading: $\mathcal{M}((g, n, b)) := \mathcal{M}((g, n)) \times \{b\}$ (and for more general enrichments of the monoidal category, replacing the product with an appropriate cotensor). Indeed, the adjunction property is exactly the universal property for the coproduct in the forgetful functor (explicitly, giving a collection of maps $\mathcal{G}((g, n, b)) \to \mathcal{M}((g, n)), b \in B$ is the same as giving maps $\mathcal{G}((g, n, b)) \to \mathcal{M}((g, n)) \times \{b\}, b \in B$). So it is enough here to exhibit the morphism of (non-graded) operads.

The morphism considered here is as usual given at the level of points by stabilisation. For a given prestable curve with marking β, we obtain a stable curve by forgetting the class β and stabilising the underlying curve. This construction clearly preserves the operadic structure (or in other words, commutes with all operations on marked points), so it defines a morphism of modular operads. $\square$

3.2.2 Gromov–Witten theory

3.2.2.1 Gromov–Witten classes

Passing to the Chow groups, the diagram (3.7) induces morphisms $\mathrm{ev}^*\colon A^\bullet X^n = (A^\bullet X)^{\otimes n} \to A^\bullet\overline{\mathcal{M}_{g,n}}(X,\beta)$ and $\mathrm{Stab}_*\colon A_\bullet\overline{\mathcal{M}_{g,n}}(X,\beta) \to A_\bullet\overline{\mathcal{M}_{g,n}}$.

By corollary 3.2.1.1.5, the stack $\overline{\mathcal{M}_{g,n}}(X,\beta)$ is canonically endowed with a virtual fundamental class $\left[\overline{\mathcal{M}_{g,n}}(X,\beta)\right]^{\mathrm{vir}} \in A_{\mathrm{vdim}}\overline{\mathcal{M}_{g,n}}(X,\beta)$. We can then define the **Gromov–Witten class**

$$I_{g,n,\beta} \coloneqq \mathrm{Stab}_* \circ (\left[\overline{\mathcal{M}_{g,n}}(X,\beta)\right]^{\mathrm{vir}} \frown \mathrm{ev}^*)\colon (A^\bullet X)^{\otimes n} \cong A^\bullet X^n \to A_\bullet\overline{\mathcal{M}_{g,n}}, \tag{3.10}$$

that is

$$I_{g,n,\beta}(\gamma_1\otimes\cdots\otimes\gamma_n) = \mathrm{Stab}_*\left(\left[\overline{\mathcal{M}_{g,n}}(X,\beta)\right]^{\mathrm{vir}} \frown \mathrm{ev}^*(\gamma_1,\dots,\gamma_n)\right) \tag{3.11}$$

for $\gamma_1,\dots,\gamma_n \in A^\bullet X$.

Remark 3.2.2.1.1. Our capping with the virtual fundamental class explains why we use the bivariant rings $A^\bullet$ on X and the Chow groups $A_\bullet$ for $\overline{\mathcal{M}_{g,n}}$, as the cap-product induces $\left[\overline{\mathcal{M}_{g,n}}(X,\beta)\right]^{\mathrm{vir}} \frown\colon A^\bullet\overline{\mathcal{M}_{g,n}}(X,\beta) \to A_{\mathrm{vdim}-\bullet}\overline{\mathcal{M}_{g,n}}(X,\beta)$.

Remember that the modular operad of stable curves has $\overline{\mathcal{M}}((g,n)) = \overline{\mathcal{M}_{g,n+1}}$. The correspondence of (3.7) for $n+1$ marked points can also be written as

$$\begin{tikzcd} & \overline{\mathcal{M}_{g,n+1}}(X,\beta) \arrow[dl, "(\mathrm{Stab},\mathrm{ev})"'] \arrow[dr, "\mathrm{ev}_{n+1}"] & \\ \overline{\mathcal{M}}((g,n)) \times X^n & & X \end{tikzcd}, \tag{3.12}$$

where we also recall that $X^{n+1} = \mathcal{E}[X]((g,n))$ is the component of the modular endomorphisms operad of example 3.1.2.2.6 for any g.

Theorem 3.2.2.1.2. *The Gromov–Witten classes provide, through the correspondence of* (3.12), *a structure of CohFT on* $A^\bullet X$ *(with the intersection pairing as contracting bilinear form).*

Proof. This amounts to saying that the Gromov-Witten classes verify the axioms □

Corollary 3.2.2.1.3. *The genus zero Gromov–Witten classes provide a structure of tree-level CohFT on* $A^\bullet X$. □

We also define the **Gromov–Witten invariants** as the degrees of (the Poincaré duals of) the Gromov–Witten classes (taken by cap-product with the fundamental class of $\overline{\mathcal{M}_{g,n}}$):

$$\langle I_{g,n\beta}\rangle = \int_{\left[\overline{\mathcal{M}_{g,n}}\right]} I_{g,n\beta}, \tag{3.13}$$

which by Fulton's functorial change of variables is

$$\langle I_{g,n\beta}\rangle(\gamma_1 \otimes \cdots \otimes \gamma_n) = \int_{\left[\overline{\mathcal{M}_{g,n}}(X,\beta)\right]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1) \smile \cdots \smile \mathrm{ev}_n^*(\gamma_n). \tag{3.14}$$

Remember also that cap-product with the (non-virtual) fundamental class is given by pushforward along the structure morphism to the point $\operatorname{Spec} k$.

So as to solve issues of convergence, we often work formally, over the Novikov ring which separates the effective classes. The semigroup ring $k[NE(X)]$ of the monoid $NE(X)$ has a maximal ideal $\mathfrak{m}$ generated by the monomials in a choice of basis elements, and we define the **Novikov ring** Λ as its formal completion $\widehat{NE(X)}$. It consists of formal series in the elements of $NE(X)$. We usually write Q^β for the element of Λ corresponding to $\beta \in NE(X)$, with the multiplication law $Q^\beta Q^{\beta'} = Q^{\beta+\beta'}$, so that $\Lambda = \{\sum_{\beta\in NE(X)} \alpha_\beta Q^\beta \mid \alpha_\beta \in k\}$.

It is often convenient to fix a (homogeneous) basis of $A^\bullet X$. We write $\{T_i\}_{0\leq i\leq r}$ (where $r = \dim A^\bullet X - 1$) for the basis elements, such that $T_0 = 1 \in A^0X$ is the unit for the cup-product (the Poincaré dual to the fundamental class $[X] \in A_{\dim X}X$), and x_i for the generic coordinates, so that a generic class is $\gamma = \sum_i x_i T_i$. We also denote $(g_{i,j})_{i,j}$ the metric given by the intersection pairing: $g_{i,j} = \int_{[X]} T_i \smile T_j$, and $(g^{i,j})_{i,j}$ for its inverse matrix. We define the Gromov–Witten potential as

$$\begin{aligned}\Phi(x_1,\ldots,x_r) &= \sum_{\substack{n\geq 0\\ \beta\in NE(X)}} Q^\beta \frac{1}{n!}\langle I_{0,n,\beta}\rangle(\gamma^{\otimes n})\\ &= \sum_{\substack{d_0,\ldots,d_r,\sum_i d_i\geq 0\\ \beta\in NE(X)}} Q^\beta \frac{\prod_i x_i^{d_i}}{\prod_i d_i!}\langle I_{0,\sum_i d_i,\beta}\rangle(T_0^{\otimes d_0}\otimes\cdots\otimes T_r^{\otimes d_r}).\end{aligned} \tag{3.15}$$

We see easily that its third derivatives are

$$\Phi_{i,j,k} = \sum_{d_0,\ldots,d_r,\beta} Q^\beta \frac{\prod_i x_i^{d_i}}{\prod_i d_i!}\langle I_{0,\sum_i d_i+3,\beta}\rangle(T_0^{d_0},\ldots,T_r^{d_r},T_i,T_j,T_k). \tag{3.16}$$

We now define the **quantum cup-product** on $A^\bullet X \otimes \Lambda[[x_0,\ldots,x_r]]$ to have the $\Phi_{i,j,k}$ as structure constants, that is $T_i \bullet T_j := \sum_{0\leq k,\ell\leq r} \Phi_{i,j,k} g^{k,\ell} T_\ell$.

The **small quantum product** on $A^\bullet X \otimes \Lambda$ is similarly defined by only considering in the sum the terms with $\sum_i d_i = 0$. Then the small quantum product of $\gamma_1 = \sum x_i T_i$ and $\gamma_2 = \sum y_i T_i$ is

$$\begin{aligned}\gamma_1 \bullet \gamma_2 &= \sum_{i,j,k,\ell} x_i y_j \Phi^{\mathrm{small}}_{i,j,k} g^{k,\ell} T_\ell\\ &= \sum_{i,j,k,\ell} x_i y_j \sum_\beta Q^\beta \langle I_{0,3,\beta}\rangle(T_i\otimes T_j\otimes T_k) g^{k,\ell} T_\ell\\ &= \sum_{k,\ell,\beta} Q^\beta \left(\int_{\left[\overline{\mathcal{M}_{0,3}}(X,\beta)\right]^{\mathrm{vir}}} \mathrm{ev}_1^*\gamma_1 \smile \mathrm{ev}_2^*\gamma_2 \smile \mathrm{ev}_3^* T_k\right) g^{k,\ell} T_\ell.\end{aligned} \tag{3.17}$$

Proposition 3.2.2.1.4. *The small quantum product is given in a coordinate-free way by*

$$\gamma_1 \bullet \gamma_2 = \sum_{\beta} Q^{\beta} \operatorname{ev}_{3,*}\left(\operatorname{ev}_1^* \gamma_1 \smile \operatorname{ev}_2^* \gamma_2 \frown \left[\overline{\mathcal{M}_{0,3}}(X,\beta)\right]^{\mathrm{vir}}\right). \tag{3.18}$$

Proof. Taking the projection of eq. (3.17) onto T_p gives

$$\begin{aligned}\int_X \gamma_1 \bullet \gamma_2 \smile T_p &= \int_{[X]} \sum_{k,\ell,\beta} Q^{\beta} \left(\int_{\left[\overline{\mathcal{M}_{0,3}}(X,\beta)\right]^{\mathrm{vir}}} \operatorname{ev}_1^* \gamma_1 \smile \operatorname{ev}_2^* \gamma_2 \smile \operatorname{ev}_3^* T_k\right) g^{k,\ell} T_\ell \smile T_p \\ &= \sum_{\beta,\ell} Q^{\beta} \sum_k \left(\int_{\left[\overline{\mathcal{M}_{0,3}}(X,\beta)\right]^{\mathrm{vir}}} \operatorname{ev}_1^* \gamma_1 \smile \operatorname{ev}_2^* \gamma_2 \smile \operatorname{ev}_3^* T_k\right) g^{k,\ell} g_{\ell,p} \\ &= \sum_{\beta} Q^{\beta} \left(\int_{\left[\overline{\mathcal{M}_{0,3}}(X,\beta)\right]^{\mathrm{vir}}} \operatorname{ev}_1^* \gamma_1 \smile \operatorname{ev}_2^* \gamma_2 \smile \operatorname{ev}_3^* T_p\right)\end{aligned} \tag{3.19}$$

But, by functorial change of variables, for all $\beta \in NE(X)$:

$$\begin{aligned}&\int_{\left[\overline{\mathcal{M}_{0,3}}(X,\beta)\right]^{\mathrm{vir}}} \operatorname{ev}_1^* \gamma_1 \smile \operatorname{ev}_2^* \gamma_2 \smile \operatorname{ev}_3^* T_p \\ &\coloneqq \int_{\left[\overline{\mathcal{M}_{0,3}}(X,\beta)\right]} \operatorname{ev}_1^* \gamma_1 \smile \operatorname{ev}_2^* \gamma_2 \smile \operatorname{ev}_3^* T_p \frown \left[\overline{\mathcal{M}_{0,3}}(X,\beta)\right]^{\mathrm{vir}} \\ &= \int_{[X]} \operatorname{ev}_{3,*}\left(\operatorname{ev}_1^* \gamma_1 \smile \operatorname{ev}_2^* \gamma_2 \smile \operatorname{ev}_3^* T_p \frown \left[\overline{\mathcal{M}_{0,3}}(X,\beta)\right]^{\mathrm{vir}}\right) \\ &= \int_{[X]} \operatorname{ev}_{3,*}\left(\operatorname{ev}_1^* \gamma_1 \smile \operatorname{ev}_2^* \gamma_2 \frown \left[\overline{\mathcal{M}_{0,3}}(X,\beta)\right]^{\mathrm{vir}}\right) \smile T_p\end{aligned} \tag{3.20}$$

by the projection formula of [Ful98, A_{123}, p.323] (and where in the last line we implicitly use Poincaré duality twice, along with the functoriality of the virtual fundamental class[BF97]). □

3.2.2.2 Quantum K-theory

Mimicking the expression of eq. (3.18), we would define a quantum product on $G_0(X) \otimes \Lambda$ by

$$[\mathcal{E}] \bullet [\mathcal{F}] \coloneqq \sum_{\beta \in NE(X)} Q^{\beta} \operatorname{ev}_{3,*}\left(\operatorname{ev}_1^*[\mathcal{E}] \otimes \operatorname{ev}_2^*[\mathcal{F}] \otimes \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}_{0,3}}(X,\beta)}\right]\right). \tag{3.21}$$

This is however not associative.

We define the K-theoretic quantum product by

$$[\mathcal{E}] \bullet [\mathcal{F}] \coloneqq \sum_{\beta \in NE(X)} Q^{\beta} \operatorname{ev}_{3,*}\left(\operatorname{ev}_1^*[\mathcal{E}] \otimes \operatorname{ev}_2^*[\mathcal{F}] \otimes \sum_{\substack{r \geq 0 \\ \beta_0,\ldots,\beta_r \\ \sum_i \beta_i = \beta}} (-1)^r \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}_{0,3}}(X,\beta_0)}\right] \otimes \bigotimes_{i=1}^{r} \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}_{0,2}}(X,\beta_1)}\right]\right), \tag{3.22}$$

and we will dedicate the rest of the section to understanding the additional terms.

Lemma 3.2.2.2.1 (Refined Gysin morphism in bivariant K-theory). *[Lee04, § 2.1, (1)][MR18b, eq. 5.4.3] Consider a regular embedding* $f\colon B' \to B$ *and a cartesian square*

$$\begin{tikzcd} V' \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner"] & V \arrow[d, "p"] \\ B' \arrow[r, "f"'] & B \end{tikzcd} \quad . \tag{3.23}$$

Then there is an induced map $f^!\colon G_0(V) \to G_0(V')$.

This map can in fact be constructed from the homotopy fibre in the following way. There is a canonical map $j\colon V' \to V \times^h_B B'$ *from the universal property of the homotopy fibre product, which is the closed embedding of* V' *into its derived enhancement. Write* $\bar{f}$ *the canonical map* $V \times^h_B B' \to V$. *Then* $f^! = (j_*)^{-1} \circ \bar{f}^*$.

3.2.2.2.2 Consider the gluing morphisms $\gamma\colon \overline{\mathcal{M}_{g_1,n_1}} \times \overline{\mathcal{M}_{g_2,n_2}} \to \overline{\mathcal{M}_{g_1+g_2,n_1+n_2-2}}$ as well as the forgetful stabilisation morphism $\mathrm{Stab}_\beta\colon \overline{\mathcal{M}_{g,n}}(X,\beta) \to \overline{\mathcal{M}_{g,n}}$, where we write $g = g_1 + g_2$ and $n = n_1 + n_2 - 2$. Let us describe the points of the fibred product

$$Z_\beta \coloneqq (\overline{\mathcal{M}_{g_1,n_1}} \times \overline{\mathcal{M}_{g_2,n_2}}) \times_{\overline{\mathcal{M}_{g,n}}} \overline{\mathcal{M}_{g,n}}(X,\beta) \tag{3.24}$$

by comparing the fibres of these morphisms.

Let $\sigma\colon \operatorname{Spec} k \to \overline{\mathcal{M}_{g,n}}$ classify a prestable curve of genus g with n marked points. A point in the fibre of γ over σ will simply correspond to a decomposition of the curve σ in two prestable curves (σ_1, σ_2) in a way compatible with the genera and the markings. However, a point in the fibre of Stab_β will classify a stable decorated curve of total class β, whose *stabilisation* coincides with σ: components with a non-zero class β_i may be highly unstable and differ from σ as they will be contracted. In particular, a point of $\mathrm{Stab}_\beta^{-1}(\sigma)$ may correspond to a pair of curves in $\gamma^{-1}(\sigma)$, not directly glued together at marked points but connected by a tree of rational bridges, each a $\mathbb{P}^1$ with two nodes and no markings, and thus a non-zero class β_i. The only requirement is that marked points glued together be sent to the same point in X by their evaluation maps. The fibre of Stab_β, this time over a point (σ_1, σ_2) of the fibre $\gamma^{-1}(\sigma)$, can then be described as

$$\coprod_{r\geq 0} \coprod_{\beta_0+\cdots+\beta_{r+1}=\beta} X^{\sigma_1}_{g_1,n_1,\beta_0} \times_X X_{0,2,\beta_1} \times_X \cdots \times_X X_{0,2,\beta_r} \times_X X^{\sigma_2}_{g_2,n_2,\beta_{r+1}} \tag{3.25}$$

where we write $X_{g_i,n_i,\beta_i} \coloneqq \overline{\mathcal{M}_{g_i,n_i}}(X,\beta_i)$ and $X^{\sigma_i}_{g_i,n_i,\beta_i}$ for the fibre of its stabilisation to $\overline{\mathcal{M}_{g_i,n_i}}$ over σ_i. Note also that as $NE(X)$ has finite decompositions, the number of decompositions $\beta = \beta_0 + \cdots + \beta_{r+1}$, that is the number of terms in the coproduct, is finite. Hence a point of the fibre product Z_β consists of a pair (σ_1, σ_2) and an element of a decomposition as in (3.25).

3.2.2.2.3 From the universal properties of the coproducts and of the fibred product, there are then maps $\Psi_{r,\beta}$ fitting in the commutative diagram

$$\begin{tikzcd}
\coprod\limits_{\sum_{i=0}^{r+1}\beta_i=\beta} X_{g_1,n_1,\beta_0} \underset{X}{\times} \prod\limits_{X,i=1}^{r} X_{0,2,\beta_i} \underset{X}{\times} X_{g_2,n_2,\beta_{r+1}} \arrow[dr, "\Psi_{r,\beta}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Z_\beta \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_{g,n,\beta} = \overline{\mathcal{M}_{g,n}}(X,\beta) \arrow[d] \\
& \overline{\mathcal{M}_{g_1,n_1}} \times \overline{\mathcal{M}_{g_2,n_2}} \arrow[r, "\Phi"'] & \overline{\mathcal{M}_{g,n}}
\end{tikzcd} \tag{3.26}$$

By [Lee04, Proposition 11], there is an equality of G-theory classes

$$\sum_{r\geq 0}(-1)^r\Psi_{r,\beta,*}\left(\sum_{\sum_i \beta_i=\beta}\left[\mathcal{O}^{\mathrm{vir}}_{X_{g_1,n_1,\beta_0}\times_X \prod_{X\, i=1}^{r} X_{0,2,\beta_i}\times_X X_{g_2,n_2,\beta_{r+1}}}\right]\right) = \Phi^!\left[\mathcal{O}^{\mathrm{vir}}_{X_{g,n,\beta}}\right]. \tag{3.27}$$

3.2.2.2.4 Consider now the morphism $X_{g_1,n_1,\beta_0}\times X_{0,2,\beta_1}\times X_{g_2,n_2,\beta_2}\to(X\times X)^2$ given by the evaluations $\mathrm{ev}_{n_1}\times\mathrm{ev}_1\colon X_{g_1,n_1,\beta_0}\times X_{0,2,\beta_1}\to X\times X$ and $\mathrm{ev}_2\times\mathrm{ev}_1\colon X_{0,2,\beta_1}\times X_{g_2,n_2,\beta_2}\to X\times X$. Writing $\Delta^2\colon X^2\to(X\times X)^2$, the fibred product over $(X\times X)^2$ is clearly $X_{g_1,n_1,\beta_0}\times_X X_{0,2,\beta_1}\times_X X_{g_2,n_2,\beta_2}$. Then the product property [Lee04, Proposition 6] gives

$$\Delta^{2,!}\left(\left[\mathcal{O}^{\mathrm{vir}}_{X_{g_1,n_1,\beta_0}}\right]\boxtimes\left[\mathcal{O}^{\mathrm{vir}}_{X_{0,2,\beta_1}}\right]\boxtimes\left[\mathcal{O}^{\mathrm{vir}}_{X_{g_2,n_2,\beta_2}}\right]\right)=\left[\mathcal{O}^{\mathrm{vir}}_{X_{g_1,n_1,\beta_0}\times_X X_{0,2,\beta_1}\times_X X_{g_2,n_2,\beta_2}}\right]. \tag{3.28}$$

It is straightforward to observe that similar formulæ hold over the r-fold product of diagonals $\Delta^r\colon X^r\to(X\times X)^r$ for the higher decompositions of β. We can finally rewrite (3.27) as:

$$\sum_{r\geq 0}(-1)^r\Psi_{r,\beta,*}\left(\sum_{\sum_i\beta_i=\beta}\Delta^{r,!}\left(\left[\mathcal{O}^{\mathrm{vir}}_{X_{g_1,n_1,\beta_0}}\right]\mathop{\boxtimes}_{i=1}^{r}\left[\mathcal{O}^{\mathrm{vir}}_{X_{0,2,\beta_i}}\right]\boxtimes\left[\mathcal{O}^{\mathrm{vir}}_{X_{g_2,n_2,\beta_{r+1}}}\right]\right)\right)=\Phi^!\left[\mathcal{O}^{\mathrm{vir}}_{X_{g,n,\beta}}\right]. \tag{3.29}$$

3.2.2.2.5 As in the case of quantum cohomology, it is convenient to introduce a basis $\kappa_0,\dots,\kappa_s$ of $G_0(X)$ such that $\kappa_0=[\mathcal{O}_X]$, as well as the pairing metric $h_{i,j}\coloneqq\chi(\kappa_i\otimes\kappa_j)$ with inverse matrix $(h^{i,j})$.

We also define the K-theoretic Gromov–Witten invariants

$$\langle I_{g,n,\beta}\rangle([\mathcal{E}_1]\otimes\cdots\otimes[\mathcal{E}_n])\coloneqq\chi\left(\mathrm{Stab}_*\left(\mathrm{ev}_1^*[\mathcal{E}_1]\otimes\cdots\otimes\mathrm{ev}_n^*[\mathcal{E}_n]\otimes\left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}_{g,n}}(X,\beta)}\right]\right)\right), \tag{3.30}$$

where χ is the Euler characteristic, given by pushforward along the map to $\operatorname{Spec} k$.

Now for a general element $E = \sum_i t_i \kappa_i \in G_0(X)$, the K-theoretic Gromov–Witten potential $\Phi(E) \in G_0(X) \otimes \Lambda[[t_0, \ldots, t_s]]$ is defined as

$$\begin{aligned}\Phi(t_0, \ldots, t_s) :=& \frac{1}{2} h(E, E) + \sum_{\substack{n \geq 0 \\ \beta \in NE(X)}} Q^\beta \frac{1}{n!} \langle I_{g,n,\beta} \rangle (E^{\otimes n}) \\ =& \frac{1}{2} \sum_{i,j} t_i t_j h_{i,j} + \sum_{\substack{d_0, \ldots, d_s, \sum_i d_i \geq 0 \\ \beta \in NE(X)}} Q^\beta \frac{\prod_i t_i^{d_i}}{\prod_i d_i!} \left\langle I_{0, \sum_i d_i, \beta} \right\rangle (\kappa_0^{\otimes d_0} \otimes \cdots \otimes \kappa_s^{\otimes d_s}).\end{aligned} \tag{3.31}$$

We finally introduce the "quantised metric"

$$\widetilde{h}_{i,j}(E) = \partial_{\kappa_i} \partial_{\kappa_j} \Phi(E) \in \mathbb{Z} \otimes \Lambda[[E]], \tag{3.32}$$

and, using its inverse $(\widetilde{h}^{i,j})$, the quantum product

$$\kappa_i \bullet_E \kappa_j = \sum_{k,\ell} \left(\partial_{\kappa_i} \partial_{\kappa_j} \partial_{\kappa_k} \Phi(E) \right) \widetilde{h}^{k,\ell}(E) \kappa_\ell. \tag{3.33}$$

Once again we specialise to a **small quantum product** by setting E to zero: for $\mathcal{E}_1 = \sum_i x_i \kappa_i$ and $\mathcal{E}_2 = \sum_i y_i \kappa_i$,

$$\mathcal{E}_1 \bullet \mathcal{E}_2 = \sum_{k,\ell} \sum_\beta Q^\beta \cdot \chi \left(\mathrm{Stab}_* \left(\mathrm{ev}_1^* \mathcal{E}_1 \otimes \mathrm{ev}_2^* \mathcal{E}_2 \otimes \mathrm{ev}_3^* \kappa_k \otimes \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}_{0,3}(X,\beta)}} \right] \right) \right) \widetilde{h}^{k,\ell} \kappa_\ell. \tag{3.34}$$

Following [Lee04, Remark 10], developing the inverse of the metric and applying base-change formulæ, this is the same as

$$\mathcal{E}_1 \bullet \mathcal{E}_2 = \sum_\beta Q^\beta \cdot \mathrm{ev}_{3,*} \left(\mathrm{ev}_1^* \mathcal{E}_1 \otimes \mathrm{ev}_2^* \mathcal{E}_2 \otimes \Phi^! \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}_{0,3}(X,\beta)}} \right] \right), \tag{3.35}$$

where $\Phi^!$ is as in (3.27). Finally, applying (3.29) with the decomposition $\overline{\mathcal{M}_{0,3}} \times \overline{\mathcal{M}_{0,2}} \to \overline{\mathcal{M}_{0,3}}$, we obtain the product defined in (3.22).

Remark 3.2.2.2.6 (Comparison with quantum cohomology). Notice that the difference between eq. (3.22) and eq. (3.18) consists of the terms with $r > 0$; indeed a formula similar to eq. (3.29) with the sum stopped at $r = 0$ holds for the virtual fundamental class. This comes from the fact that the higher decompositions of the fibres of Z_β are intersections of divisors in $X_{g,n,\beta}$, which are not seen in the part of the Chow ring of degree the virtual dimension. See also the derived geometric interpretation in remark 4.1.1.2.5.

3.3 Digression: Quasimap moduli space

In this section we discuss a generalisation of Gromov–Witten theory in which the stability condition is allowed to vary, giving a family of theories parameterised by $\mathbb{Q}_{\geq 0}$.

3.3.1 Preliminary considerations

3.3.1.1 Stability conditions in Gromov–Witten theory

To begin with, let us recast the combinatorial stability condition for stable curves in terms of the algebraic geometry of the curve.

Let $(C; x_1, \ldots, x_n)$ be a prestable curve. Let ω_C be the dualising sheaf of C. The **logarithmic dualising sheaf** of the prestable curve is

$$\omega_{(C;x_1,\ldots,x_n),\log} \coloneqq \omega_C \left(\sum_{i=1}^{n} x_i \right), \tag{3.36}$$

which we will write simply as $\omega_{\log}$ whenever there is no ambiguity. Recall that if C is smooth its dualising sheaf is the canonical sheaf $\bigwedge^1 \Omega^1_C = \Omega^1_C$, and if C is nodal with nodes $y_1, \ldots, y_r$ forming a divisor $D = \sum_i [y_i]$ it is the twisted $\Omega^1_C(D)$. It is enough to study the irreducible components individually.

Suppose $C \simeq \mathbb{P}^1$ is a rational component (of genus 0). Then its dualising sheaf is the cotangent sheaf $\omega_{\mathbb{P}^1} = \bigwedge^1 \Omega^1_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$. Adding a divisor of $m = n + \#\{\text{nodes}\}$ marked or special (*i.e.*, or nodal) points gives

$$\omega_{\log} = \mathcal{O}_{\mathbb{P}^1}(-2 + m). \tag{3.37}$$

If $m = 0$ or $m = 1$, the logarithmic sheaf has no global sections; if $m = 2$, the logarithmic sheaf is the structure sheaf whose global sections are constant. If $m \geq 3$, that is $(C; x_1, \ldots, x_n)$ is stable, then $\omega_{\log}$ is a tensor power of the twisting sheaf, which is very ample. Hence a prestable curve of genus 0 is stable if and only if its logarithmic sheaf is very ample, if and only if it is ample, if and only if it is of (strictly) positive degree $\deg \mathcal{O}_{\mathbb{P}^1}(m-2) = m - 2$.

Note that, more generally, a line bundle $\mathcal{L}$ on a projective smooth integral curve over k is ample if and only if $\deg \mathcal{L} \geq 0$ (see [Vak17, 19.2.E]). We can then use the Riemann-Roch theorem to describe the invertible sheaves on curves.

Suppose now C is a smooth curve of genus 1; by smoothness the dualising sheaf is the canonical sheaf. Then $\omega_C = \mathcal{O}_C$, and $\Gamma(C, \mathcal{O}_C) \cong k$. We also find that any degree 1 line bundle is of the form $\mathcal{O}_C(x)$ for a point $x \in C$. Hence once again, $(C; x_1, \ldots, x_n)$ is stable if and only if its logarithmic sheaf is of positive degree, that is ample.

Finally, suppose C has genus at least 2. We compute $\deg \omega = g - 1 + h^0(C, \omega) - h^0(C, \mathcal{O}_C) = 2g - 2 \geq 2 > 0$. Thus every curve of genus greater than 1, which is automatically stable without special points, has an ample (logarithmic) dualising sheaf.

In conclusion, the stability condition on prestable curves can be rephrased as an ampleness condition on their logarithmic dualising sheaves: a prestable curve $(C; x_1, \ldots, x_n)$ is stable if and only if $\omega_{C,\log} = \Omega^1_C\left(\sum_i x_i + \text{nodes}\right) = \omega_C(\sum_i x_i)$ is ample.

3.3.1.2 Geometric quotients

Let $W = \operatorname{Spec} R$ be an affine variety with an action of an algebraic group G. We can consider several quotients. The affine quotient W/G is defined as the affine

$$W/G \coloneqq \operatorname{Spec}(R^G), \tag{3.38}$$

with R-algebra of functions the subring of G-invariant elements of R. This quotient is not always interesting, as the only invariants may be trivial. On the other end is the stack quotient $[W/G]$, whose category of points over a scheme S is that of principal G-bundles (or G-torsors) over S endowed with a G-equivariant map to W.

Geometric invariant theory (GIT) allows us to consider a quotient as a (relatively projective) scheme $W /\!\!/ G$, over which a certain sub-quotient (DM, under some assumptions) stack $[W^{ss}/G]$ is actually proper. Considering the graded R-algebra

$$R[z] = R \otimes k[z] = \Gamma(W \times \mathbb{A}^1_k, \mathcal{O}_{W\times\mathbb{A}^1_k}), \tag{3.39}$$

whose graduation is only induced by the polynomial degree of the added variable z, we can rewrite W as $W \simeq \operatorname{Proj}(R[z])$ (indeed the homogeneous ideals will have to be concentrated in degree 0 so as to not contain the irrelevant ideal $zR[z]$; this corresponds geometrically to the tensor product making R into the constant functions on the Proj). Here we have thus introduced the trivial line bunde $W \times \mathbb{A}^1$ over k. The idea of GIT is then to use this new expression of W to take the invariants for a G-action extended to $W \times \mathbb{A}^1$, called a G-linearisation of the bundle.

Fix a character $\vartheta \in \hom(G, \mathbb{G}_m)$. It induces (by multiplication, *i.e.* $GL(\mathbb{A}^1) = \mathbb{A}^1 \backslash \{0\} = \mathbb{G}_m$) a 1-dimensional representation of G on $\mathbb{A}^1$, which we write as $\mathbb{A}^1_\vartheta$. We now have an action of G on both W and $\mathbb{A}^1$, which allows us to "twist" the trivial line bundle with an action, and we write the resulting G-variety as $W \times \mathbb{A}^1_\vartheta$. Explicitly, the action is $g \cdot (x, \lambda) = (g \cdot x, \vartheta(g)^{-1}\lambda)$ for $x \in W$ and $\lambda \in \mathbb{A}^1$: we use the inverse character ϑ^{-1}. The graduation on the algebra $R[z]$ of global sections of the bundle induces a graduation on the invariants $R[z]^G$, and we can set

$$W /\!\!/_\vartheta G \coloneqq \operatorname{Proj}\left(R[z]^G\right), \tag{3.40}$$

which is projective over the affine quotient. The graded algebra defining this **GIT quotient** $W /\!\!/_\vartheta G$ admits a simple reinterpretation in terms of the action on R. Say that an element $f \in R$ is a relative invariant of weight ϑ if, for any $x \in W$ and any $g \in G$, we have $f(g \cdot x) = \vartheta(g) f(x)$. The set of such relative invariants is denoted $R^{G,\vartheta}$. A general homogeneous element of $R[z]$ is written $f \cdot z^n$ with $f \in R$ and $n \in \mathbb{N}$. From the definition of the action, we immediately see that such an element is G-invariant if and only if f is invariant of weight ϑ^n. Hence

$$R[z]^G \simeq \bigoplus_{n\geq 0} R^{G,\vartheta^n}. \tag{3.41}$$

Furthermore we also have $R^{G,\vartheta^n} = \Gamma(W, (W \times \mathbb{A}^1_\vartheta)^{\otimes n})^G$ (where the tensor product is that of line bundles over W).

In addition, the GIT quotient also has a geometric interpretation.

Definition 3.3.1.2.1 (ϑ-stability[Kin94])**.**

1. A point $x \in W$ is ϑ-unstable if for every relative invariant $f \in R^{G,\vartheta^n}$, $n \geq 1$, we have $f(x) = 0$.

2. A point $x \in W$ is ϑ-stable if for every one-parameter subgroup $\mathbb{G}_m \subset G$, the orbit $\mathbb{G}_m \cdot (x, 1)$ in $W \times \mathbb{A}^1_\vartheta$ is closed.

3. A point $x \in W$ is ϑ-semistable if it is not ϑ-unstable.

We denote by $W^{ss,\vartheta}$, $W^{s,\vartheta}$, and $W^{us,\vartheta}$the open loci in W of respectively ϑ-semistable, ϑ-stable, and ϑ-unstable points. Then $W/\!\!/_\vartheta G$ is isomorphic to the quotient of $W^{ss,\vartheta}$ by the equivalence relation that $x \sim y$ if and only if the closures of their G-orbits intersect non-trivially in $W^{ss,\vartheta}$.

Example 3.3.1.2.2. Let $R = k[x_0, \dots, x_n]$, so $W = \mathbb{A}^{n+1}$, and $G = \mathbb{G}_m$ act on W by global rescaling. If $x \in W$, its G-orbit is $\{\lambda x \mid \lambda \in \mathbb{G}_m\} \simeq \mathbb{A}^1 \setminus \{0\}$, a line passing through the origin and x with the origin removed. The orbit of 0 is $\{0\}$, which is a closed point. Any other orbits have their closures intersect in $\{0\}$ only.

Let $\vartheta = \mathbb{1}_{\mathbb{G}_m}$ be the identity character. A function $f \in k[x_1, \dots, x_n]$ is invariant of weight ϑ^ℓ if and only if it is homogeneous of degree ℓ. From this we see that $\mathbb{A}^{n+1}/\!\!/_{\mathbb{1}_{\mathbb{G}_m}}\mathbb{G}_m = \mathbb{P}^n$. In fact, it follows from the description that the only unstable point is 0, with the points of $\mathbb{A}^{n+1} \setminus \{0\}$ being semistable, from which we recover the usual description of the points of $\mathbb{P}^n$.

Furthermore, the orbit of $(0, 1)$ in $\mathbb{A}^{n+1} \times \mathbb{A}^1$ is the punctured line $\{0\} \times (\mathbb{A}^1 \setminus \{0\})$ which is open, while for a semistable point $a = (a_0, \dots, a_n) \in \mathbb{A}^{n+1} \setminus \{0\}$ the orbit of $(a, 1)$ is the closed subscheme defined by $(\lambda x_0 - a_0, \dots, \lambda x_n - a_n)$ in $\mathrm{Spec}(k[x_0, \dots, x_n][\lambda])$, so the stable points are exactly the semistable.

3.3.2 Definition of quasimaps

3.3.2.1 Quasimaps to a GIT quotient

We suppose from now on that $W^s = W^{ss} \neq \emptyset$, that is the stable and semistable loci coincide, and that they are nonsingular and are acted upon freely by G, so $W/\!\!/_\vartheta G$ coincides with $[W^{s,\vartheta}/G]$. By definition, a morphism $[u]$ from a scheme S to the stack quotient $[W/G]$ consists of a principal G-bundle $P \to S$ and a G-equivariant map $\widetilde{u} \colon P \to S$.

Remark 3.3.2.1.1. The map $[u]$ can be equivalently given by a section u of the associated bundle $P \times_G W \to S$. Indeed, let $\widetilde{u}$ be a map as above. Since the G-orbits of P are parametrised by the points of S, for a given point $s \in S$ the fibre over s will be sent (equivariantly) to a G-orbit in W, hence, by construction of the associated bundle, a single point over s is selected, giving a section.

Conversely, let $u \colon S \to P \times_G W$. For $p \in P$ lying above $s \in S$, write $u(x) = [p', w]$ (which is equal to $[gp', g^{-1}w]$ for all $g \in G$). Since the fibre P_s is a G-torsor, there is a unique $g \in G$ such that $p = gp'$. Then we set $\widetilde{u}(p) = g^{-1}w$, which is well-defined independently of the choice of representatives (p', w). Notice also that this map is G-equivariant.

Construction 3.3.2.1.2 (Degree of a quasimap from a curve)**.** Suppose now the base scheme is a *curve* C. Let $L \in \mathrm{Pic}^G(W)$ be a G-equivariant line bundle on W. There is an induced line bundle $P \times_G L \to P \times_G W$, and a line bundle $L^u \coloneqq u^*(P \times_G L) \to C$. Then we define the **degree** $\beta \in \mathrm{Pic}^G(W)^\vee$ by

$$\begin{aligned} \beta \colon \mathrm{Pic}^G(W) &\to \mathbb{Z} \\ L &\mapsto \beta(L) \coloneqq \deg_C L^u = \deg_C\big(u^*(P \times_G L)\big). \end{aligned} \tag{3.42}$$

The induced line bundle L^u can also be reinterpreted through the (defining) isomorphism $\mathrm{Pic}^G(W) \cong \mathrm{Pic}([W/G])$, given by $\mathcal{L} \mapsto [\mathcal{L}/G]$. Then, writing $[u]$ for the corresponding map to the quotient stack $[W/G]$, we have $L^u = [u]^*[L/G]$.

We set

$$\mathcal{L}_\vartheta \coloneqq \left(W \times \mathbb{A}^1_\vartheta\right)^u \coloneqq u^* \left(P \times_G (W \times \mathbb{A}^1_\vartheta)\right). \tag{3.43}$$

Definition 3.3.2.1.3 ((Prestable) Quasimap)**.** A **quasimap** of genus g, with n marked points, of class $\beta \in \mathrm{Pic}^G(W)^\vee$, to $W/\!\!/_\vartheta G$, is a prestable curve $(C; x_1, \ldots, x_n)$ of genus g with a principal G-bundle $P \to C$ and a section u of $P \times_G W \to C$ of class β such that the generic point η_i of each irreducible component $C_i$of C is sent by u to the stable locus W^s of the fibre W_{η_i}; in other words there are at most finitely many points sent to the unstable locus $W \setminus W^{ss}$.

A quasimap is said to be **prestable** if its basepoints are disjoint from the special (*i.e.* nodal and marked) points of the underlying prestable curve.

Let $(C, (x_i)_i, P, u)$ be a prestable quasimap to $W/\!\!/_\vartheta G$, and let $x \in C$. We define the **length** at x to be the order of contact $\ell(x)$ of the image $u(C)$ with the unstable locus $P \times_G W^{us}$ at $u(x)$. Explicitly, let $\mathcal{I}$ denote the ideal sheaf defining the closed subscheme $P \times_G W^{us} \subset P \times_G W$; then

$$\ell(x) \coloneqq \mathrm{length}_x\big(\mathrm{coker}(u^\sharp \colon u^*\mathcal{I} \to \mathcal{O}_C)\big). \tag{3.44}$$

Property 3.3.2.1.4. *[CKM14, p. 42] For any $x \in C$, we have $\beta(W \times \mathbb{A}^1_\vartheta) \geq \ell(x) \geq 0$, whith $\ell(x) > 0$ if and only if x is a basepoint (and in particular, $\beta(W \times \mathbb{A}^1_\vartheta) > 0$ if $\beta \neq 0$).*

3.3.2.2 Stability condition

Definition 3.3.2.2.1 (ε-stability)**.** Let $\varepsilon \in \mathbb{Q}_{>0}$. A prestable quasimap $(C, (x_i)_i, P, u)$ to $W/\!\!/_\vartheta G$ is **ε-stable** if

1. the $\mathbb{Q}$-line bundle $\omega_{C,\log} \otimes \mathcal{L}_\vartheta^{\otimes\varepsilon} \in \mathrm{Pic}(C) \otimes_{\mathbb{Z}} \mathbb{Q}$ is ample,
2. for every point x of the curve, $\varepsilon \cdot \ell(x) \leq 1$.

We define in the obvious way the notions of isomorphisms of stable quasimaps $(C, (x_i)_i, P, u) \to (C', (x_i')_i, P', u')$, consisting of isomorphisms $f\colon C \xrightarrow{\simeq} C'$ and $\varphi\colon P \xrightarrow{\simeq} f^*P'$ preserving the marked points and the section, and the notions of families of quasimaps over a base scheme.

Remark 3.3.2.2.2. Observe that a quasimap to $W/\!\!/_\vartheta G$ is ε-stable if and only if it is $\frac{\varepsilon}{m}$-stable as a quasimap to $W/\!\!/_{m\vartheta} G$. More generally, we may take ϑ to be instead a *rational* character, which allows to replace the choice of ε by that of the coefficient of $\vartheta \in \hom(G, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Explicitly, the ε-stability conditions impose the following properties:

- Let Σ be a rational component of the source curve C. If $\mathcal{L}_\vartheta|_\Sigma$ is trivial (*i.e.* of degree 0) then $\omega_{\log}$ must be ample and Σ stable, that is it must have at least three special points. Otherwise the number of special points on Σ and the degree of $\mathcal{L}_\vartheta|_\Sigma$ will compensate for stability; more precisely if $\mathcal{L}_\vartheta|_\Sigma$ has negative degree $-d$ then Σ must have $\varepsilon d + 3$ special points, and if Σ only has i special points (for $i = 0, 1, 2$) then $\mathcal{L}_\vartheta$ must be of degree $\frac{3-i}{\varepsilon}$.
- Similarly, if Σ is a component of genus 1 with i marked points, we must have $i \geq -\varepsilon \deg \mathcal{L}_\vartheta|_\Sigma + 1$. More generally, for Σ of genus h, the stability condition gives $i \geq -\varepsilon \deg \mathcal{L}_\vartheta|_\Sigma - 2h + 3$.

This can be made clearer in the two extreme cases.

Remark 3.3.2.2.3 (Stable maps and stable quasimaps).

For $\varepsilon \leq \beta(W \times \mathbb{A}^1_\vartheta)^{-1}$**,** condition 2 imposes no additional condition by property 3.3.2.1.4. Concomitantly, since on any component of genus h the degree of $\mathcal{L}_\vartheta$ cannot be greater than $\beta(W \times \mathbb{A}^1_\vartheta) \leq \varepsilon^{-1}$, we have at best the need for $2 - 2h$ special points. In particular there can be no rational tail (rational components with only one special point). We refer to this as the "$\varepsilon \to 0^+$ chamber", and call the 0^+-stable quasimaps (which are thus ε-stable for any ε) simply **stable quasimaps**.

For $\varepsilon > 1$**,** condition 2 allows no basepoint (since $\ell(x)$ is an integer), so the quasimap datum defines an actual morphism $[u]$ to $W/\!\!/_\vartheta G$, of cycle class $[u]_*[C] \in A_1(W/\!\!/_\vartheta G)$ which vanishes if and only if β does. Furthermore, from the discussion of section 3.3.1.1, we see that the quasimap is ε-stable if and only if it is a stable map. We also write this chamber as the "$\varepsilon = +\infty$ chamber".

In fact, by similar arguments, the condition of ε-stability remains identical in the chamber $\frac{1}{d} \geq \varepsilon > \frac{1}{d+1}$, for $d \in \mathbb{N}^*$.

Example 3.3.2.2.4 (Quasimaps to a projective variety,following [CJR17] and [CKM14]). We study the case where the target space is a subvariety$Z \subset \mathbb{P}^m$, defined by r homogeneous polynomials $P_1, \ldots, P_r$. To express it as a GIT quotient, consider its affine cone $W = A(Z) = \mathrm{Spec}(k[t_0, \ldots, t_m]/(P_1, \ldots, P_r))$, where the P_is are homogeneous of respective degrees d_i. We define the diagonal action of $k^\times = \mathbb{G}_m$ on $\mathbb{A}^{m+1}$, which passes to $A(Z)$. We pick the positive character $\vartheta = \mathbb{1}_{\mathbb{G}_m} \in \hom(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}$. We recover the case of example 3.3.1.2.2.

A quasimap to Z is then determined from the invertible sheaf $\mathcal{L}_\vartheta$. A map from a curve C to $\mathbb{P}^m$ with basepoints is indeed known to be equivalent to the data of a line bundle $\mathcal{L}$ on C with $m+1$ sections $(s_0, \dots, s_m) \in \Gamma(C, \mathcal{L}^{\oplus m+1})$ which only vanish simultaneously at the basepoints. Such a section gives a map (with basepoints) to Z if its components respect the defining equations: $\forall 1 \leq i \leq r, P_i(s_0, \dots, s_m) = 0 \in \Gamma(C, \mathcal{L}^{\otimes d_i})$. This is indeed equivalent to a section u of a bundle with fibers isomorphic to $A(Z) \subset \mathbb{A}^{m+1}$, by taking $P = C \times \mathbb{G}_m \to C$ to be the trivial $k^\times$-fibration, and then the associated bundle $P \times_G A(Z)$ (since the vanishing of the P_is is unchanged by the diagonal action). We recover $\mathcal{L}$ as $\mathcal{L}_\vartheta$.

The unstable locus is $\{0\}$, so the length $\ell(x)$ at a point x is simply the order of vanishing of the section $(s_0, \dots, s_n)$.

3.3.3 Properties of the moduli space of ε-stable quasimaps

3.3.3.1 Relation to other moduli spaces

It is clear from the definition of stable quasimaps that their moduli stack should be a substack of the internal mapping stack from the moduli stack of prestable curves. More precisely, recall the moduli stack $\mathfrak{M}_{g,n}$ of prestable curves of genus g with n marked points. There is also a moduli stack $\mathfrak{M}_{g,n}([W/G], \beta)$ of maps from prestable curves parameterised by $\mathfrak{M}_{g,n}$ to the stack quotient $[G/W]$ whose underlying section of an associated G-bundle has class $\beta \in \mathrm{Pic}([W/G])^\vee$.

Let $\mathfrak{QMap}_{g,n}(W /\!\!/_\vartheta G, \beta)$ be the moduli stack parameterising all (not necessarily stable, or even prestable) quasimaps of genus g with n marked points of class β. All such quasimaps are in particular morphisms to the stack quotient, and from boundedness results of [CKM14] it follows that $\mathfrak{QMap}_{g,n}(W /\!\!/_\vartheta G, \beta) \subset \mathfrak{M}_{g,n}([W/G], \beta)$ is an open substack, which is furthermore of finite type over $\mathfrak{M}_{g,n}$.

Let now $\mathfrak{Q}^\varepsilon_{g,n}(W /\!\!/_\vartheta G, \beta)$ be the moduli stack for ε-stable quasimaps.

Lemma 3.3.3.1.1. *[CKM14, Proposition 7.1.5] The automorphism group of an ε-stable quasimap is finite.*

Theorem 3.3.3.1.2. *[CKM14, Theorem 7.1.6] The stack $\mathfrak{Q}^\varepsilon_{g,n}(W /\!\!/_\vartheta G, \beta)$ is a separated DM stack of finite type, with a natural proper morphism over the affine quotient* $\mathrm{Spec}(R^G) = W/G$.

Let $\mathfrak{Bun}^G_{g,n} \xrightarrow{\gamma} \mathfrak{M}_{g,n}$ denote the relative moduli stack of principal G-bundles on the fibers of the universal curve $\mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$, which can be constructed[Wan11] as the relative mapping stack

$$\mathfrak{Bun}^G_{g,n} \coloneqq \mathfrak{Hom}_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, \mathfrak{B}G \times \mathfrak{M}_{g,n}), \tag{3.45}$$

where $\mathfrak{B}G$ is the classifying stack $[\mathrm{Spec}\, k/G]$ (for the trivial G-action on $\mathrm{Spec}\, k$). It is a smooth Artin stack, locally of finite type over $\mathrm{Spec}\, k$. It has a universal curve with a universal G-bundle $\mathfrak{P}^G_{g,n} \to \mathfrak{C}^G_{g,n}$; the relative curve $\mathfrak{C}^G_{g,n}$ is the pullback of $\mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$ along the forgetful functor $\gamma\colon \mathfrak{Bun}^G_{g,n} \to \mathfrak{M}_{g,n}$.

There is a tautological forgetful morphism

$$\mu\colon \mathfrak{Q}^\varepsilon_{g,n}(W/\!/_\vartheta G, \beta) \to \mathfrak{Bun}^G_{g,n} \tag{3.46}$$

The universal curve $\pi\colon \mathfrak{C}^\varepsilon_{g,n} \to \mathfrak{Q}^\varepsilon_{g,n}(W/\!/_\vartheta G, \beta)$ is the pullback of $\mathfrak{C}^G_{g,n}$ along μ, or equivalently the pullback of $\mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$ along $\gamma \circ \mu$. It has universal G-bundle $\mathfrak{P}^\varepsilon_{g,n}$ which is the pullback of $\mathfrak{P}^G_{g,n}$. We also have the associated bundle $\rho\colon \mathfrak{P}^\varepsilon_{g,n} \times_G W \to \mathfrak{C}^\varepsilon_{g,n}$ and its universal section u.

3.3.3.2 Perfect obstruction theory

The above discussion naturally defines the complex

$$E^\bullet_\mu \coloneqq \left(\mathbb{R}\pi_*(\mathbb{R}\mathcal{H}om(\mathbb{L}^\bullet_u, \mathcal{O}_{\mathfrak{C}^\varepsilon_{g,n}})[1])\right)^\vee \tag{3.47}$$

Since u is a section of ρ, the composition $\rho \circ u = \mathbb{1}_{\mathfrak{C}^\varepsilon_{g,n}}$ gives by proposition 1.1.1.1.1 1 the distinguished triangle

$$u^*\mathbb{L}^\bullet_\rho \to \mathbb{L}^\bullet_{\mathbb{1}_{\mathfrak{C}^\varepsilon_{g,n}}} = 0 \to \mathbb{L}^\bullet_u \to u^*\mathbb{L}^\bullet_\rho[1] \tag{3.48}$$

hence an isomorphism $\mathbb{L}^\bullet_u \simeq u^*\mathbb{L}^\bullet_\rho[1]$ and thus

$$E^\bullet_\mu \simeq \left(\mathbb{R}\pi_*(\mathbb{R}\mathcal{H}om(u^*\mathbb{L}^\bullet_\rho, \mathcal{O}_{\mathfrak{C}^\varepsilon_{g,n}}))\right)^\vee \simeq \left(\mathbb{R}\pi_*(u^*\mathbb{T}^\bullet_\rho)\right)^\vee. \tag{3.49}$$

Theorem 3.3.3.2.1. *The morphism $E^\bullet_\mu \to \mathbb{L}^\bullet_\mu$ is a relative obstruction theory, which is perfect if W only has local complete intersection singularities.*

Proof. [CKM14, Theorem 7.1.6, Theorem 4.5.2] □

Remark 3.3.3.2.2. We can construct a comparison morphism $\overline{\mathcal{M}_{g,n}}(\mathbb{P}^N, d) \to \mathfrak{Q}^\varepsilon_{g,n}(\mathbb{P}^N, d)$, which contracts unstable components to basepoints of corresponding degree.

The comparison morphism is virtually birational[MOP11, Theorem 3], that is its induced morphism on Chow groups sends virtual fundamental class to virtual fundamental class.

Chapter 4

The lax action

4.1 Brane actions on the moduli spaces of stable maps

4.1.1 Derived enhancement of the moduli space of stable maps

4.1.1.1 ∞-operad of moduli stacks and derived enhancement

Construction 4.1.1.1.1 (Derived moduli space of stable maps)**.** By analogy with corollary 3.2.1.1.4, we consider the derived mapping stack $\mathbb{R}\mathcal{M}ap_{\mathfrak{M}_{g,n,\beta}}(\mathfrak{C}_{g,n,\beta}, X \times \mathfrak{M}_{g,n,\beta})$, where we recall that the universal curve $\mathfrak{C}_{g,n,\beta} \to \mathfrak{M}_{g,n,\beta}$ coincides with the forgetful map $\mathfrak{M}_{g,n+1,\beta} \to \mathfrak{M}_{g,n,\beta}$. The truncation $\mathfrak{Hom}_{\mathfrak{M}_{g,n,\beta}}(\mathfrak{C}_{g,n,\beta}, X \times \mathfrak{M}_{g,n,\beta})$ contains the Zariski open substack $\overline{\mathcal{M}_{g,n}}(X, \beta)$.

Using corollary 1.2.2.2.12, we then define the **derived moduli space of stable maps** $\mathbb{R}\overline{\mathcal{M}_{g,n}}(X, \beta)$ as the corresponding Zariski open substack of the derived mapping stack:

$$\begin{tikzcd}
\overline{\mathcal{M}_{g,n}}(X, \beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner h"] & \mathfrak{Hom}_{\mathfrak{M}_{g,n,\beta}}(\mathfrak{C}_{g,n,\beta}, X \times \mathfrak{M}_{g,n,\beta}) \arrow[d] \\
\mathbb{R}\overline{\mathcal{M}_{g,n}}(X, \beta) \arrow[r] & \mathbb{R}\mathcal{M}ap_{\mathfrak{M}_{g,n,\beta}}(\mathfrak{C}_{g,n,\beta}, X \times \mathfrak{M}_{g,n,\beta})
\end{tikzcd} . \tag{4.1}$$

It is a derived enhancement of $\overline{\mathcal{M}_{g,n}}(X, \beta)$.

Theorem 4.1.1.1.2. *[STV15, discussion after Definition 2.6] The derived stack $\mathbb{R}\overline{\mathcal{M}_{g,n}}(X, \beta)$ is quasi–smooth proper derived Deligne–Mumford stack.*

By the results of section 1.3.2, this derived enhancement induces a virtual structure sheaf $\left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}_{g,n}}(X,\beta)}\right]$ on its truncation $\overline{\mathcal{M}_{g,n}}(X, \beta)$, which by remark 1.3.1.2.4 can be constructed either as the inverse pushforward of the class of the structure sheaf $\mathcal{O}_{\mathbb{R}\overline{\mathcal{M}_{g,n}}(X,\beta)}$ or from the induced perfect obstruction theory.

Lemma 4.1.1.1.3. *The collection $\mathfrak{M}_0 = (\mathfrak{M}_{0,n,\beta})_{n,\beta}$ forms a $NE(X)$-graded (monochromatic) ∞-operad in derived stacks.*

Proof. Since the stacks are smooth, they are flat, and their homotopy products (corresponding locally to derived tensor products) are equivalent to their truncations, the classical products. This shows that the morphisms defining the classical operad are also compatible with the requirements for the structure of ∞-operad. □

By the same arguments, there is also an ∞-operad in derived stacks $\overline{\mathcal{M}_0} \times_{\mathbb{\Gamma}} \mathbb{\Gamma}^{NE(X)}$ as well as a morphism of graded ∞-operads $\mathfrak{M}_0 \to \overline{\mathcal{M}_0} \times_{\mathbb{\Gamma}} \mathbb{\Gamma}^{NE(X)}$.

Proposition 4.1.1.1.4. *The morphisms $\coprod_\beta \mathfrak{M}_{0,n,\beta} \to \overline{\mathcal{M}_{0,n}}$ give a lax morphism of categorical ∞-operads $\overline{\mathcal{M}_0}^{\otimes} \leftarrow \mathfrak{M}_0^{\otimes} = \mathfrak{M}_0^{\otimes}$ (or, equivalently, a morphism of $NE(X)$-graded ∞-operads $\overline{\mathcal{M}_0}^{\otimes} \times_{\mathbb{\Gamma}} \mathbb{\Gamma}^{NE(X)} \leftarrow \mathfrak{M}_0^{\otimes} = \mathfrak{M}_0^{\otimes}$).*

Proof. Once again, the derived Artin stacks in question are actually smooth classical stacks. As both structures of the bicategory of spans and of operad only require taking limits (and not colimits), it is enough to exhibit a lax morphism of operads in a bicategory. We must study the diagram (2.15) adapted to the context of the operads of (pre)stable curves, with $\mathfrak{O}^{\otimes} = \overline{\mathcal{M}_0}^{\otimes}$ and $\mathfrak{P}^{\otimes} = \mathfrak{M}_0^{\otimes}$:

$$\begin{tikzcd}
 & & & Z_N = \mathfrak{M}_{0,n_1}\times\mathfrak{M}_{0,n_2} \arrow[dl] \arrow[dr] & & & Z_N \times_{\mathfrak{M}} Z_E \arrow[lll] \arrow[dd] \arrow[ddddllllll, dotted, "h"'] \\
 & & \overline{\mathcal{M}_{0,n_1}} \times \overline{\mathcal{M}_{0,n_2}} & c_{n_1,n_2} & \mathfrak{M}_{0,n_1} \times \mathfrak{M}_{0,n_2} & & \\
Z_W = \overline{\mathcal{M}_{0,n_1}\times\mathcal{M}_{0,n_2}} \arrow[urr] \arrow[drr] & & & & & & Z_E = \mathfrak{M}_{0,n_1}\times\mathfrak{M}_{0,n_2} \arrow[ull] \arrow[dll] \\
 & & \overline{\mathcal{M}_{0,n_1+n_2-2}} & & \mathfrak{M}_{0,n_1+n_2-2} & & \\
Z_W \times_{\mathcal{M}} Z_S \arrow[uu] \arrow[rrr] & & & Z_S = \mathfrak{M}_{0,n_1+n_2-2} \arrow[ul] \arrow[ur] & & &
\end{tikzcd} . \tag{4.2}$$

The "north–east" homotopy fibre product (which is a fibre product since the stacks are smooth) is $\mathfrak{M}_{0,n_1} \times \mathfrak{M}_{0,n_2}$, while the "south–west" one is $Z^{\mathrm{pre}} := \overline{\mathcal{M}_{0,n_1}} \times \overline{\mathcal{M}_{0,n_2}} \times_{\overline{\mathcal{M}_{0,n_1+n_2-2}}} \mathfrak{M}_{0,n_1+n_2-2}$, a prestable (that is, replacing $X_{0,n,\beta}$ by $\mathfrak{M}_{0,n,\beta}$) version of the fibre products $\coprod_\beta Z_\beta$ that appeared in the discussion of section 3.2.2.2.

A map c_{n_1,n_2} from $\mathfrak{M}_{0,n_1} \times \mathfrak{M}_{0,n_2}$ into the (homotopy) fibre product is actually furnished by the commutative square exhibiting $\mathfrak{M}_0^{\otimes} \to \overline{\mathcal{M}_0}^{\otimes}$ as a morphism of operads (the "mirror image" of the central square in (4.2)), and it is not an equivalence in $\mathfrak{dSt}$. □

4.1.1.2 Covering of the virtual fibres

As $\mathfrak{M}_0^{\otimes}$ is an ∞-operad in derived stacks, in particular, the stack $\coprod_\beta \mathfrak{M}_{0,2,\beta}$, being the object of unary operations $\mathfrak{M}_0(1) = \coprod_\beta \mathfrak{M}_0(1,\beta)$, has a structure of monoid object in the cartesian monoidal category $\mathfrak{St}$ induced by the composition $\mathfrak{M}_0(1) \times \mathfrak{M}_0(1) \to \mathfrak{M}_0(1)$. In addition, the operadic compositions (*i.e.* the gluing of curves) makes the moduli stacks $\coprod_\beta \mathfrak{M}_{0,n,\beta}$ into left and right $\mathfrak{M}_0(1)$-modules. Once again, by smoothness, these structures pass to the inclusion into the ∞-category $\mathfrak{dSt}$.

The moduli stack $\mathbb{R}X_2 := \coprod_\beta \mathbb{R}\overline{\mathcal{M}_{0,2}}(X,\beta)$ then inherits a structure of monoid object with $\mathbb{R}X_2 \times_X \mathbb{R}X_2 \to \mathbb{R}X_2$, and similarly the $\mathbb{R}X_n := \coprod_\beta \mathbb{R}\overline{\mathcal{M}_{0,n}}(X,\beta)$ are left and right modules over it. We shall also write $\mathbb{R}X_{n,\beta} \coloneqq \mathbb{R}\overline{\mathcal{M}_{0,n}}(X,\beta)$ (note that we do not specify the genus as it is fixed to zero).

We can then consider the two-sided bar complex of these modules:

$$\begin{aligned}&\left[\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \mathbb{R}X_{n_1} \underset{X}{\times} \mathbb{R}X_2 \underset{X}{\times} \mathbb{R}X_2 \underset{X}{\times} \mathbb{R}X_{n_2} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathbb{R}X_{n_1} \underset{X}{\times} \mathbb{R}X_2 \underset{X}{\times} \mathbb{R}X_{n_2} \substack{\longrightarrow\\\longrightarrow} \mathbb{R}X_{n_1} \underset{X}{\times} \mathbb{R}X_{n_2}\right]\\ &= \left[\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \coprod_{\substack{\beta\in NE(X),\\ \beta_0+\beta_1+\beta_2=\beta}} \mathbb{R}X_{n_1,\beta_0} \underset{X}{\times} \mathbb{R}X_{2,\beta_1} \underset{X}{\times} \mathbb{R}X_{n_2,\beta_2} \substack{\longrightarrow\\\longrightarrow} \coprod_{\substack{\beta\in NE(X),\\ \beta_0+\beta_1=\beta}} \mathbb{R}X_{n_1,\beta_0} \underset{X}{\times} \mathbb{R}X_{n_2,\beta_1}\right].\end{aligned} \tag{4.3}$$

As in paragraph 3.2.2.2.2, define $\mathbb{R}Z_\beta$ as the (homotopy) pullback

$$\begin{tikzcd} \mathbb{R}Z_\beta \arrow[r, "\lrcorner_h"'] \arrow[d] & \mathbb{R}\overline{\mathcal{M}_{0,n_1+n_2-2}}(X,\beta) = \mathbb{R}X_{n_1+n_2-2,\beta} \arrow[d] \\ \overline{\mathcal{M}_{0,n_1}} \times \overline{\mathcal{M}_{0,n_2}} \arrow[r] & \overline{\mathcal{M}_{0,n_1+n_2-2}} \end{tikzcd} . \tag{4.4}$$

Put $n \coloneqq n_1 + n_2 - 2$. Note that the notation is consistent as $t_0(\mathbb{R}Z_\beta) = (\overline{\mathcal{M}_{0,n_1}} \times \overline{\mathcal{M}_{0,n_2}}) \times_{\overline{\mathcal{M}_{0,n}}} t_0(\mathbb{R}X_{n,\beta}) = (\overline{\mathcal{M}_{0,n_1}} \times \overline{\mathcal{M}_{0,n_2}}) \times_{\overline{\mathcal{M}_{0,n}}} X_{n,\beta} = Z_\beta$. Then the semi-simplicial object obtained in (4.3) is naturally augmented by a map to $\coprod_\beta \mathbb{R}Z_\beta$.

Fix now a $\beta \in NE(X)$ and consider the (homotopy) pullback along the open inclusion $\mathbb{R}\overline{\mathcal{M}_{0,n_1+n_2-2}}(X,\beta) = \mathbb{R}X_{n_1+n_2-2,\beta} \subset \mathbb{R}X_{n_1+n_2-2} = \coprod_{\beta'} \mathbb{R}\overline{\mathcal{M}_{0,n_1+n_2-2}}(X,\beta')$, which is informally presented as:

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \coprod_{\beta_0+\beta_1+\beta_2=\beta} \mathbb{R}X_{n_1,\beta_0} \underset{X}{\times} \mathbb{R}X_{2,\beta_1} \underset{X}{\times} \mathbb{R}X_{n_2,\beta_2} \substack{\longrightarrow\\\longrightarrow} \coprod_{\beta_0+\beta_1=\beta} \mathbb{R}X_{n_1,\beta_0} \underset{X}{\times} \mathbb{R}X_{n_2,\beta_1} . \tag{4.5}$$

I order to save space, we write

$$\mathbb{R}X_{n,[\beta]|r} = \coprod_{\sum_{i=0}^{r+1}\beta_i=\beta} \mathbb{R}X_{n_1,\beta_0} \underset{X}{\times} \mathbb{R}X_{2,\beta_1} \underset{X}{\times} \cdots \underset{X}{\times} \mathbb{R}X_{2,\beta_r} \underset{X}{\times} \mathbb{R}X_{n_{r+1},\beta_{r+1}}, \tag{4.6}$$

so that (4.5) provides a semi-simplicial set $\mathbb{R}X_{n,[\beta]|\bullet}$ augmented to $\mathbb{R}Z_\beta$.

Theorem 4.1.1.2.1. *[MR18b, Theorem 5.3.11] The morphism $\varinjlim \mathbb{R}X_{n,[\beta]|\bullet} \to \mathbb{R}Z_\beta$ is an equivalence.*

Property 4.1.1.2.2. *[MR18b, Theorem 5.4.2, (2), (3), (5)]*

- $\left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}_{g_1,n_1}}(X,\beta_1)\times\overline{\mathcal{M}_{g_2,n_2}}(X,\beta_2)}\right] = \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}_{g_1,n_1}}(X,\beta_1)}\right] \boxtimes \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}_{g_2,n_2}}(X,\beta_2)}\right]$
- $\left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}_{g_1,n_1}}(X,\beta_1)\times_X\overline{\mathcal{M}_{g_2,n_2}}(X,\beta_2)}\right] = \Delta^! \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}_{g_1,n_1}}(X,\beta_1)\times\overline{\mathcal{M}_{g_2,n_2}}(X,\beta_2)}\right]$, *where $\Delta\colon X \to X \times X$ is the diagonal morphism and $\Delta^!$ as in paragraph 3.2.2.2.4 (replacing $X_{g,n,\beta}$ by its derived enhancement).*

- *With notation as in* (3.26) *(replacing* Z_β *and* $X_{n,\beta}$ *by their derived enhancements), we have (note that the genus is implicitly zero)*

$$\sum_{r\geq 0}(-1)^r\Psi_{r,\beta,*}\sum_{\sum_i \beta_i=\beta}\left[\mathcal{O}^{\mathrm{vir}}_{X_{n_1+1,\beta_0}\times_X X_{2,\beta_1}\times_X\cdots\times_X X_{n_2+1,\beta_r}}\right]=\Phi^!\left[\mathcal{O}^{\mathrm{vir}}_{X_{n_1+n_2,\beta}}\right]. \tag{4.7}$$

Proof. The product formula is simply the Künneth formula. The other two are obtained by base-change along the diagrams defining the fibred products, and for (4.7) from the fact that $\mathbb{R}Z_\beta$ is the colimit of $\mathbb{R}X_{n_1+n_2,[\beta]|\bullet}$, so that the G-class of its structure sheaf is the alternating sum of those of the $\mathbb{R}X_{n_1+n_2,[\beta]|r}$. □

Remark 4.1.1.2.3. These properties are part of the axioms for an orientation in G-theory. In fact, [MR18b, Theorem 5.4.2] shows that the virtual sheaf satisfies all the orientation axioms.

Corollary 4.1.1.2.4. *The formula* (3.29) *holds in* $G_0(t_0(\mathbb{R}Z_\beta))=G_0(Z_\beta)$. □

Remark 4.1.1.2.5. The above discussion adds weight to the idea of remark 3.2.2.2.6: the simplicial object $\mathbb{R}X_{n,[\beta]|\bullet}$ may be seen as an effective hypercovering of $\mathbb{R}Z_\beta$, consisting of the higher intersections of divisors. Then the K-theoretic virtual sheaf will remember how the divisors are glued together along this covering, while the intersection theoretic virtual class only sees the discrete cover $\mathbb{R}X_{n,[\beta]|0}\twoheadrightarrow\mathbb{R}Z_\beta$.

4.1.2 From the brane action to the Gromov–Witten action

4.1.2.1 Stable sub-action

Property 4.1.2.1.1. *The* $NE(X)$*-graded* ∞*-operad* $\mathfrak{M}_0^\otimes$ *is not coherent.*

Proof. Combining eq. (2.30) and construction 2.2.2.2.2, it follows that an operad in derived stacks is coherent if and only if, for every n, m, every derived stack Z and every pair of "operations" $\sigma\colon Z\to\mathcal{O}_n$, $\tau\colon Z\to\mathcal{O}_m$, classifying $C_\sigma=Z\times_{\mathcal{O}_n}\mathcal{O}_{n+1}$, $C_\tau=Z\times_{\mathcal{O}_m}\mathcal{O}_{m+1}$, the induced maps $C_\sigma\amalg_{Z\times\mathcal{O}_2}C_\tau\to C_{\sigma\circ_i\tau}$ are equivalences. For the graded case we only need to check coherence at the level of the underlying non-graded ∞-operad.

In our case, let C_σ, C_τ be two prestable curves with respectively $n+1$ and $m+1$ marked points. The curve $C_{\sigma\circ\tau}$ is obtained by gluing marked points, so it is the pushout (of underived schemes) $C_\sigma\amalg_{\operatorname{Spec}k}C_\tau$. On the other hand, $\mathfrak{M}_0(2)=\mathfrak{M}_{0,3,\beta=0}$ is contractible, so we must compare with the (homotopy) pushout of derived stacks $C_\sigma\amalg^h_{\operatorname{Spec}k}C_\tau$. The universal property of the homotopy pushout gives a canonical arrow $\theta\colon C_\sigma\amalg^h_{\operatorname{Spec}k}C_\tau\to C_\sigma\amalg_{\operatorname{Spec}k}C_\tau$, which is generally not an equivalence as the inclusion of schemes into stacks, and thus into derived stacks, does not commute with pushouts. □

Although it is not coherent, $\mathfrak{M}_0^\otimes$ is still reduced, so it induces a lax brane action on $\mathfrak{M}_0(0,0)=\mathfrak{M}_{0,3,0}$. By corollary 2.2.2.2.7, applying $\mathbb{R}\mathcal{M}ap(-,X)$ gives a lax $\mathfrak{M}_0^\otimes$-algebra structure on X.

Proposition 4.1.2.1.2. *[MR18a, Corollary 3.1.8] After applying $\mathbb{R}\mathcal{M}ap(-, X)$, the morphism θ becomes an equivalence, and there is a (non-lax) $\mathfrak{M}_0^{\otimes}$-algebra structure on X (in correspondences), given by the correspondences*

$$\begin{tikzcd} & \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}_{0,n+1,\beta}}(\mathfrak{M}_{0,n+2,\beta}, X \times \mathfrak{M}_{0,n+1,\beta}) \arrow[dl] \arrow[dr] & \\ X^n \times \mathfrak{M}_{0,n+1,\beta} & & X \end{tikzcd} . \tag{4.8}$$

We wish to restrict these correspondences to the *stable* derived moduli stack $\mathbb{R}\overline{\mathcal{M}_{0,n}}(X, \beta) \subset \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}_{0,n,\beta}}(\mathfrak{M}_{0,n,\beta}, X \times \mathfrak{M}_{0,n,\beta})$, as it is the derived stack responsible for the virtual phenomena of section 4.1.1.2.

Recall from (2.38) that the brane action on X is classified by the cocartesian fibration in spaces

$$B^{\mathfrak{d}\mathfrak{S}\mathfrak{t}}(\mathfrak{M}_o, X) \to \int^{co} \mathfrak{T}\mathfrak{w}(\mathfrak{E}\mathfrak{n}\mathfrak{v}(\mathfrak{M}_o))^{\otimes} \times_{\mathfrak{d}\mathfrak{S}\mathfrak{t}^{op}} \mathfrak{F}\mathfrak{u}\mathfrak{n}([1], \mathfrak{d}\mathfrak{S}\mathfrak{t})^{op}. \tag{4.9}$$

We will thus formulate the stable sub-action as a sub-∞-category $B^{\mathfrak{d}\mathfrak{S}\mathfrak{t}}(\mathfrak{M}_o, X)^{stbl} \subset B^{\mathfrak{d}\mathfrak{S}\mathfrak{t}}(\mathfrak{M}_o, X)$, such that the fibre of the restriction of the above fibration over

$$(\sigma = (\sigma_1, \dots, \sigma_n) \colon Z \to \mathfrak{T}\mathfrak{w}(\mathfrak{E}\mathfrak{n}\mathfrak{v}(\mathfrak{M}_0))(n), u \colon Y \to Z) \tag{4.10}$$

is the subspace of the mapping space consisting of morphisms such that for each i, the map $Y \to \mathbb{R}\mathcal{M}ap_{/Z}\left(C_{\sigma_i} = Z \times_{\mathfrak{M}_{0,n_i,\beta_i}} \mathfrak{M}_{0,n_i+1,\beta_i}, X \times Z\right)$ factors through the open substack $\mathbb{R}\mathcal{M}ap^{stbl}_{/Z}(C_{\sigma_i}, X \times Z)$ classifying those families of maps sending the fundamental classes of the fibres of C_{σ_i} to $\beta_i \in A_1 X$, *ergo* stable maps.

Theorem 4.1.2.1.3. *[MR18a, Proposition 3.2.1] The morphism*

$$B^{\mathfrak{d}\mathfrak{S}\mathfrak{t}}(\mathfrak{M}_o, X)^{stbl} \to \int^{co} \mathfrak{T}\mathfrak{w}(\mathfrak{E}\mathfrak{n}\mathfrak{v}(\mathfrak{M}_o))^{\otimes} \times_{\mathfrak{d}\mathfrak{S}\mathfrak{t}^{op}} \mathfrak{F}\mathfrak{u}\mathfrak{n}([1], \mathfrak{d}\mathfrak{S}\mathfrak{t})^{op} \tag{4.11}$$

is a cocartesian fibration in spaces, defining a map of $NE(X)$-graded ∞-operads in correspondences in derived stacks $\mathfrak{M}_0^{\otimes} \to \left((\mathfrak{T}_{/-})^{\times}\right)^{corr} \times_{\Gamma} \Gamma^{NE(X)}$.

4.1.2.2 Lax action on the stable moduli spaces

Finally, we must see that the composition of the brane action on X with the lax morphism of categorical ∞-operads $\overline{\mathcal{M}_0}^{\otimes} \leftarrow \mathfrak{M}_0^{\otimes} = \mathfrak{M}_0^{\otimes}$ gives a lax action of $\overline{\mathcal{M}_0}^{\otimes}$ on X.

Theorem 4.1.2.2.1. *[MR18a, Theorem 3.3.1] There is a lax map of categorical ∞-operads in derived stacks $\overline{\mathcal{M}_0}^{\otimes} \to \mathfrak{S}\mathfrak{p}\mathfrak{a}\mathfrak{n}(\mathfrak{T}^{\times}_{/-})$, informally sending each family of curves $\sigma \colon Z \to \overline{\mathcal{M}_{0,n}}$ to the relative correspondence*

$$\begin{tikzcd} X^n \times Z \arrow[dr] & \coprod_\beta \mathbb{R}\overline{\mathcal{M}_{0,n}}(X, \beta) \times_{\overline{\mathcal{M}_{0,n}}} Z \arrow[l] \arrow[r] \arrow[d] & X \times Z \arrow[dl] \\ & Z & \end{tikzcd} . \tag{4.12}$$

Explicitly, the lax character of the morphism is given by the following version of (2.15):

$$
\begin{tikzcd}
 & & & \coprod\limits_{\substack{\beta\in NE(X)\\ \beta_1+\beta_2=\beta}} \mathbb{R}\overline{\mathcal{M}_{0,n_1}}(X,\beta_1)\times\mathbb{R}\overline{\mathcal{M}_{0,n_2}}(X,\beta_2) \arrow[dll, "\mathrm{Stab}\times\mathrm{Stab}"'] \arrow[drr, "\mathrm{ev}\times\mathrm{ev}"] & & & \coprod\limits_{\substack{\beta\in NE(X)\\ \beta_1+\beta_2=\beta}} \mathbb{R}\overline{\mathcal{M}_{0,n_1}}(X,\beta_1)\underset{X}{\times}\mathbb{R}\overline{\mathcal{M}_{0,n_2}}(X,\beta_2) \arrow[lll] \arrow[dd] \arrow[ddddllllll, dotted, bend right, "c_{n_1,n_2}"'] \arrow[d, phantom, "{}_h\llcorner"] \\
 & \overline{\mathcal{M}_{0,n_1}}\times\overline{\mathcal{M}_{0,n_2}} & & & & X^{n_1}\times X^{n_2} & \\
\overline{\mathcal{M}_{0,n_1}}\times\overline{\mathcal{M}_{0,n_2}} \arrow[ur, "\mathbb{1}"] \arrow[drr, "\gamma_{n_1,n_2}"] & & & & & & X^{n_1-1}\times X\times X^{n_2-1} \arrow[ul, "\mathbb{1}_{n_1}\times\Delta_X\times\mathbb{1}_{n_2}"'] \arrow[dl, "\mathbb{1}_{n_1}\times *\times\mathbb{1}_{n_2}"] \\
 & & \overline{\mathcal{M}_{0,n_1+n_2-2}} & & & X^{n_1+n_2-2} & \\
\coprod\limits_{\beta}\mathbb{R}Z_\beta \arrow[uu] \arrow[rrr] \arrow[ur, phantom, "\urcorner^h"] & & & \coprod\limits_{\beta}\mathbb{R}\overline{\mathcal{M}_{0,n_1+n_2-2}}(X,\beta) \arrow[ul, "\mathrm{Stab}"] \arrow[urr, "\mathrm{ev}"'] & & &
\end{tikzcd}
.
\tag{4.13}
$$

The morphism c_{n_1,n_2} is the coproduct of the canonical morphisms $\mathbb{R}X_{1,\beta} \to Z_\beta = \varinjlim \mathbb{R}X_{\bullet,\beta}$ given by theorem 4.1.1.2.1, which is not an equivalence. Hence we deduce that the lax character of the action is the reason for the additional terms appearing in the G-theoretic quantum product of (3.22).

Remark 4.1.2.2.2. By [MR18a, Proposition 3.3.3], the Gromov–Witten action also admits a *NE*(X)-graded refinement.

4.2 Categorification of Gromov–Witten invariants

Appendices

Appendix A

Higher category theory

A.1 Quasi-categories

A.1.1 Properties of quasi-categories

A.1.1.1 Quasi-categories and ∞-functors

Definition A.1.1.1.1 (Quasi-category)**.** A **quasi-category** is a simplicial set $\mathfrak{C}$ such that any inner horn $\Lambda^n_i \to \mathfrak{C}, 0 < i < n$ in $\mathfrak{C}$ admits an extension to a simplex $\Delta^n \to \mathfrak{C}$ along the inclusion:

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\forall"] \arrow[d, hook] & \mathfrak{C} \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd} . \tag{A.1}$$

The (simplicially enriched) category $\mathfrak{QCat}$ of quasi-categories is the full subcategory of $\mathfrak{sSet}$ spanned by the quasi-categories; in other words a morphism of quasi-categories, called an ∞**-functor**, is simply a map of simplicial sets between quasi-categories.

Proposition A.1.1.1.2. *Let $\mathfrak{C}$ be a quasi-category, and I be any simplicial set. The mapping simplicial set $\mathrm{Map}(I, \mathfrak{C})$ is a quasi-category. In particular, given any pair of quasi-categories, the ∞-functors between them form a quasi-category.*

Example A.1.1.1.3.

- [Lur09, Proposition 1.1.2.2] If $\mathfrak{C}$ is a category, its **nerve** $N(\mathfrak{C})_\bullet$ defined by $\hom(\Delta^n, N(\mathfrak{C})_\bullet) := \hom_{\mathfrak{Cat}}([n], \mathfrak{C})$ is a quasi-category; in fact the required extensions are all unique.
- A Kan complex is a quasi-category, which we also refer to as ∞-groupoid (or space). The quasi-category of spaces (defined later) is denoted $\mathfrak{S}$.

A vertex, that is a 0-simplex, in a quasi-category $\mathfrak{C}$ is called an **object** of $\mathfrak{C}$. An edge, that is a 1-simplex, is called a **morphism**. An inner 2-horn $\Lambda^2_1 \to \mathfrak{C}$ is identified with a pair of composable morphisms. By [Lur09, Corollary 2.3.2.2, Remark 2.3.2.3], although composition of a string of morphisms in a quasi-category is not uniquely defined, it is well-defined up to a contractible space of choices.

A quasi-category $\mathfrak{C} \in \mathfrak{QCat}$ has an associated homotopy category $\mathrm{Ho}\,\mathfrak{C} \in \mathfrak{Cat}$. It can be more easily constructed through a different model for ∞-categories. We will see in subsubsection A.1.2.2 that there is an adjunction (in fact a Quillen equivalence) $\mathbb{C}\colon \mathfrak{sCat} \leftrightarrows \mathfrak{Cat}_\Delta\colon \mathrm{N}_\Delta$ between simplicial sets and simplicially enriched category, such that[Lur09, Proposition 1.1.5.10] for any simplicially enriched category $\mathfrak{M}$ which is locally fibrant (*i.e.* whose mapping simplicial sets are Kan complexes) the coherent nerve N_Δ is a quasi-category. Thus to any quasi-category corresponds a simplically enriched category $\mathbb{C}[\mathfrak{C}]$ and, for any pair of objects $X, Y \in \mathfrak{C}$, there is a simplicial set $\mathrm{Map}(X, Y)$, called the **mapping space** from X to Y in $\mathfrak{C}$. We can simply define $\mathrm{Ho}\,\mathfrak{C} = \pi_0\mathbb{C}[\mathfrak{C}]$, that is $\hom_{\mathrm{Ho}\,\mathfrak{C}}(X, Y) = \pi_0 \mathrm{Map}_{\mathfrak{C}}(X, Y)$. We may also define a category $\mathfrak{hC}$ enriched in the homotopy category of spaces $\mathrm{Ho}\,\mathfrak{S}$, the category of homotopy types, by taking the hom-sets to be the homotopy types of the mapping spaces: $\hom_{\mathfrak{hC}}(X, Y) = [\mathrm{Map}(X, Y)]$.

An ∞-functor is said to be a **categorical equivalence** if the simplicial functors it induces by $\mathbb{C}$ is a Dwyer–Kan equivalence of simplicially enriched categories, that is essentially surjective on the homotopy categories and inducing weak homotopy equivalences on the mapping spaces.

Definition A.1.1.1.4 (Equivalences)**.** A morphism f in a quasi-category $\mathfrak{C}$ is an **equivalence** if its image in $\mathfrak{hC}$ is an isomorphism.

Example A.1.1.1.5 (Simplicial localisation). Let $\mathfrak{C}$ be a quasi-category and $\mathcal{W} \subset \mathfrak{C}_1$ be a set of morphisms of $\mathfrak{C}$. A **localisation** of $\mathfrak{C}$ at $\mathcal{W}$ is a quasi-category $\mathfrak{C}[\mathcal{W}^{-1}]$ with an ∞-functor $\mathcal{L}\colon \mathfrak{C} \to \mathfrak{C}[\mathcal{W}^{-1}]$ sending the morphisms in $\mathcal{W}$ to equivalences in $\mathfrak{C}[\mathcal{W}^{-1}]$, such that for any quasi-category $\mathfrak{D}$, the induced map $\mathfrak{Fun}(\mathfrak{C}[\mathcal{W}^{-1}], \mathfrak{D}) \to \mathfrak{Fun}(\mathfrak{C}, \mathfrak{D})$ is a categorical equivalence on the full sub-quasi-category of functors $\mathfrak{C} \to \mathfrak{D}$ which send morphisms in $\mathcal{W}$ to equivalences. A localisation is determined up to equivalence.

Example A.1.1.1.6 (∞-localisation of model categories). Let $\mathfrak{M}$ be a category with a model structure whose set of weak equivalences is $\mathcal{W}$ (more generally, we only need the datum of the relative category $(\mathfrak{M}, \mathcal{W})$). Its ∞-localisation is the quasi-category $\mathrm{N}(\mathfrak{M})[\mathcal{W}^{-1}]$, with homotopy category the homotopy category of the model structure on $\mathfrak{M}$. The construction can also be adapted in a straightforward manner to coherent nerves of simplicial model categories, in which case an existence result is given from an explicit construction. This gives a good way of obtaining quasi-categories from more easily understood relative categories.

A.1.1.2 Joins of simplicial sets

In order to have a theory of limits in quasi-categories, we will develop a notion of cones over a diagram. To that end, we need to have a way of freely adding universal vertices to a simplicial set, which is realised by the join operation.

Construction A.1.1.2.1 (Day convolution)**.** Let $(\mathfrak{C}, \otimes, \mathbf{1})$ be a monoidal category. We define an external tensor product $- \boxtimes -\colon \mathfrak{Set}^{\mathfrak{C}} \times \mathfrak{Set}^{\mathfrak{C}} \to \mathfrak{Set}^{\mathfrak{C}\times\mathfrak{C}}$ by $(\mathcal{F} \boxtimes \mathcal{G})(c_1, c_2) = \mathcal{F}(c_1) \times \mathcal{G}(c_2)$. Then the Day convolution of $\mathcal{F}$ and $\mathcal{G}$ is defined as the left Kan extension $\mathcal{F} \star \mathcal{G}\colon \mathfrak{C} \to \mathfrak{Set}$ of $\mathcal{F} \boxtimes \mathcal{G}\colon \mathfrak{C} \times \mathfrak{C} \to \mathfrak{Set}$ along $- \otimes -\colon \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$.

Equivalently, writing $\mathcal{Y}\colon \mathfrak{C} \to \mathfrak{Set}^{\mathfrak{C}}$ for the Yoneda embedding, the functor $- \star -\colon \mathfrak{Set}^{\mathfrak{C}} \times \mathfrak{Set}^{\mathfrak{C}} \to \mathfrak{Set}^{\mathfrak{C}}$ is the left Kan extension of $\mathcal{Y} \circ (- \boxtimes -)\colon \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C} \to \mathfrak{Set}^{\mathfrak{C}}$ along $\mathcal{Y} \times \mathcal{Y}\colon \mathfrak{C} \times \mathfrak{C} \to \mathfrak{Set}^{\mathfrak{C}} \times \mathfrak{Set}^{\mathfrak{C}}$.

Example A.1.1.2.2 (Simplicial sets). Let Δ_+ denote the augmented simplex category, with the added initial object $[-1] = \emptyset$, and let $\mathfrak{sSet}_+ = \mathfrak{Set}^{\Delta_+^{\mathrm{op}}}$ denote the category of augmented simplicial sets. The sum $[n] \oplus [m] \coloneqq [n + m + 1]$ endows Δ_+ with a monoidal structure, and hence $\mathfrak{sSet}_+$ with the Day monoidal structure $\star$.

We have[Rie14, §17.1] that for any augmented simplicial sets $X_\bullet$ and $Y_\bullet$, the augmentation of the Day product is $(X_\bullet \star Y_\bullet)_{-1} = X_{-1} \times Y_{-1}$. In particular, if X and Y are ordinary simplicial sets given the trivial augmentation, then so is their Day product. It follows that there is an induced monoidal product on $\mathfrak{sSet}$, also denoted $\star$, and called the **join** of simplicial sets.

Explicitly, we have $(X_\bullet \star Y_\bullet)_n = X_n \cup Y_n \cup \bigcup_{i=0}^{n-1} X_i \times Y_{n-1-i}$.

Property A.1.1.2.3.

1. *[Lur09, Proposition 1.2.8.3] The join of two quasi-categories is again a quasi-category.*

2. *If $\mathfrak{C}$ and $\mathfrak{D}$ are two categories, then $\mathrm{N}(\mathfrak{C}) \star \mathrm{N}(\mathfrak{D}) = \mathrm{N}(\mathfrak{C} \star \mathfrak{D})$ where the join of categories $\mathfrak{C} \star \mathfrak{D}$ is the category given by the disjoint union of the two categories $\mathfrak{C}$ and $\mathfrak{D}$, adding a unique arrow from any object of the first category $\mathfrak{C}$ to any of the second $\mathfrak{D}$.*

Remark A.1.1.2.4. In fact the sum $[n] \oplus [m]$ of objects of Δ is their join of categories (seen as ordered sets).

Example A.1.1.2.5 (Cone categories). Let $\mathcal{F}\colon \mathfrak{I} \to \mathfrak{C}$ be an ∞-functor between quasi-categories. The **cone over-category** $\mathfrak{C}_{\triangleleft/\mathcal{F}}$ is defined as the simplicial set whose set of n-simplexes is the subset $\hom_{\mathcal{F}}(\Delta^n \star \mathfrak{I}, \mathfrak{C})$ of morphisms whose restriction to $\mathfrak{I}$ equals $\mathcal{F}$ (it is a quasi-category by[Lur09, Proposition 1.2.9.3]). It verifies the universal property[Lur09, Proposition 1.2.9.2] that, for any simplicial set X, $\hom(X, \mathfrak{C}_{/\mathcal{F}}) = \hom_{\mathcal{F}}(X \star \mathfrak{I}; \mathfrak{C})$. Replacing $X \star \mathfrak{I}$ by $\mathfrak{I} \star X$, we similarly define the **cocone under-category** $\mathfrak{C}_{\mathcal{F}/\triangleright}$.

In particular, we denote $\mathfrak{I}^{\triangleleft} = \Delta^0 \star \mathfrak{I}$ the quasi-category of left cones of $\mathfrak{I}$, and $\mathfrak{I}^{\triangleright} = \mathfrak{I} \star \Delta^0$ the quasi-category of right cones. We call cone point the vertex coming from Δ^0. By definition, an object in $\mathfrak{C}_{/\mathcal{F}}$ (resp. in $\mathfrak{C}_{\mathcal{F}/}$) is an ∞-functor $\mathfrak{I}^{\triangleleft} \to \mathfrak{C}$ (resp. $\mathfrak{I}^{\triangleright} \to \mathfrak{C}$) extending $\mathcal{F}$, that is a left (resp. right) cone over (resp. under) the diagram $\mathcal{F}$ in $\mathfrak{C}$.

Remark A.1.1.2.6 (Terminology). Despite the name it often receives, the quasi-category $\mathfrak{C}_{\triangleleft/\mathcal{F}}$ expresses a different notion from the usual slice (or comma) categories $(\mathfrak{C} \downarrow \mathcal{F})$ of objects of a category $\mathfrak{C}$ over the image of $\mathcal{F}$. Indeed, a *cone* with cone point S is a natural transformations from the constant functor $\underline{*}$ (with value the singleton $*$) to $h_S^{\mathcal{F}} \coloneqq \hom(S, \mathcal{F}-)$ and so the category of cones would be expressed as $(\underline{*} \downarrow h_\bullet^{\mathcal{F}})$.

Example A.1.1.2.7 (Twisted arrow category). Consider the functor $\epsilon\colon \Delta \to \Delta$ given on objects (ordered sets seen as categories) by $[n] \rightsquigarrow [n] \star [n]^{\mathrm{op}} = [2n + 1]$ (where $\star$ is the join of categories). For any simplicial set $\mathfrak{C}_\bullet$, its simplicial set of twisted arrows is $\epsilon^*\mathfrak{C}_\bullet = \mathfrak{C}_\bullet \circ \epsilon$. If $\mathfrak{C}$ is a quasi-category, $\mathfrak{Tw}(\mathfrak{C}) := \epsilon^*\mathfrak{C}$ is called the **twisted arrows quasi-category** of $\mathfrak{C}$.

From the definition, $\hom(\Delta^n, \mathfrak{Tw}(\mathfrak{C})) = \hom(\Delta^n * (\Delta^n)^{\mathrm{op}}, \mathfrak{C})$. Explicitly, we have $\mathfrak{Tw}(\mathfrak{C})_n = \mathfrak{C}_{2n+1}$ and the faces $\breve{d}$ and degeneracies $\breve{s}$ are given in terms of the faces d and degeneracies s of $\mathfrak{C}$ by $\breve{d}_i(x) = d_{n-i}d_{n+1+i}$ and $\breve{s}_j(x) = s_{n-j}s_{n+1+j}(x)$. The objects of $\mathfrak{Tw}(\mathfrak{C})$ are arrows of $\mathfrak{C}$, and a morphism from f to g in $\mathfrak{Tw}(\mathfrak{C})$ is a sequence $\cdot \xrightarrow{h_2} \cdot \xrightarrow{g} \cdot \xrightarrow{h_1} \cdot$ whose composite is homotopic to f, that is a factorisation (up to homotopy) of f through g, which we usually represent as a (homotopy) commutative square.

The inclusions $[n], [n]^{\mathrm{op}} \subset [n] \star [n]^{\mathrm{op}}$ determine morphisms $\mathfrak{Tw}(C) \to \mathfrak{C}, \mathfrak{Tw}(C) \to \mathfrak{C}^{\mathrm{op}}$ and thus a morphism $\mathfrak{Tw}(C) \to \mathfrak{C} \times \mathfrak{C}^{\mathrm{op}}$. By [Lur12, Proposition 5.2.1.3], $\mathfrak{Tw}(\mathfrak{C})$ is a quasi-category if $\mathfrak{C}$ is one (in fact $\mathfrak{Tw}(\mathfrak{C}) \to \mathfrak{C} \times \mathfrak{C}^{\mathrm{op}}$ is an inner fibration).

A.1.2 Comparison with other models for $(\infty, 1)$-categories

A.1.2.1 Model-categorical tools for higher categories

A.1.2.1.1 Reedy model structures

Definition A.1.2.1.1.1 (Reedy category)**.** A **Reedy category** is a category $\mathfrak{C}$ with a function $d\colon \operatorname{obj}\mathfrak{C} \to \mathbb{N}$ and the data of two wide subcategories $\mathfrak{C}^{\nearrow}$ and $\mathfrak{C}^{\searrow}$ whose non-identity morphisms respectively (strictly) raise and lower degree, and such that every arrow factors as the composite of an arrow of $\mathfrak{C}^{\searrow}$ followed by one of $\mathfrak{C}^{\nearrow}$.

Remark A.1.2.1.1.2.

1. A Reedy category must be skeletal with no non-trivial automorphism.
2. The canonical factorisation of a map by $\mathfrak{C}^{\searrow}$ and $\mathfrak{C}^{\nearrow}$ is also the unique factorisation through an object of minimal degree.

Theorem A.1.2.1.1.3 (Reedy model structure)**.** *[Rie14, Theorem 14.2.7], [Lur09, Proposition A.2.9.19] Let $\mathfrak{M}$ be a model category, and let $\mathfrak{I}$ be a Reedy category. There exists a model structure, called the **Reedy model structure**, on $\mathfrak{Fun}(\mathfrak{I}, \mathfrak{M})$ whose weak equivalences are the pointwise weak equivalences.*

We now describe the fibrations and cofibrations of this model structure. We first introduce the following notations: for any $d \in \mathbb{N}$, we write $\mathfrak{I}_{<d}$ for the full subcategory whose objects have degree lesser than d, and $\mathfrak{I}^{\nearrow}_{<d}$ for the subcategory whose morphisms are those of $\mathfrak{I}^{\nearrow}$.

Definition A.1.2.1.1.4 (Latching and matching)**.** Let $\mathcal{D}\colon \mathfrak{I} \to \mathfrak{M}$ be a diagram.

For any $i \in \mathfrak{I}$, we define the i-**latching object** $L^i\mathcal{D}$ as the colimit of the composite functor $(\mathfrak{I}^{\nearrow}_{<\deg i})_{/i} \to \mathfrak{I} \xrightarrow{\mathcal{D}} \mathfrak{M}$.

The i-**matching object** $M_i\mathcal{D}$ is similarly defined as the limit of $(\mathfrak{I}^{\searrow}_{<\deg i})_{i/} \to \mathfrak{I} \xrightarrow{\mathcal{D}} \mathfrak{M}$.

This is equivalently the colimit (resp. limit) of $\mathcal{D}$ weighted by the subfunctor of $\hom(-, i)$ (resp. of $\hom(i, -)$) generated by the maps whose Reedy factorisation has the minimal degree object of degree lesser than $\deg(i)$.

Construction A.1.2.1.1.5. For any $i \in \mathfrak{I}$, the object $\mathcal{D}(i) \in \mathfrak{M}$ tautologically receives morphisms from the images of all objects of $(\mathfrak{I}^{\nearrow}_{<\deg i})_{/i}$ with the appropriate functorial properties defining a cocone under the functor above, so there is a canonical morphism $\ell_i \colon L^i\mathcal{D} \to \mathcal{D}(i)$, called the i-latching map. There is, for the same reasons, a canonical i-matching map $m_i \colon \mathcal{D}(i) \to M_i\mathcal{D}$.

Let $f \colon \mathcal{D} \to \mathcal{E}$ be a morphism in $\mathfrak{Fun}(\mathfrak{I}, \mathfrak{M})$. The **relative i-latching and matching maps** are defined by the commutative squares

$$\begin{array}{ccccccc} L^i\mathcal{D} & \longrightarrow & \mathcal{D}(i) & & \mathcal{D}(i) & \longrightarrow & M_i\mathcal{D} \\ \downarrow & & \downarrow & \text{and} & \downarrow & & \downarrow \\ L^i\mathcal{E} & \longrightarrow & \mathcal{E}(i) & & \mathcal{E}(i) & \longrightarrow & M_i\mathcal{E} \end{array} \tag{A.2}$$

which give canonical arrows $L^i\mathcal{D} \amalg_{L^i\mathcal{E}} \mathcal{D}(i) \to \mathcal{E}(i)$ and $\mathcal{D}(i) \to \mathcal{E}(i) \times_{M_i\mathcal{E}} M_i\mathcal{D}$.

Definition A.1.2.1.1.6 (Reedy (co)fibrations)**.** The arrow f is a **Reedy cofibration** if for every $i \in \mathfrak{I}$ the relative i-latching map is a cofibration in $\mathfrak{M}$, and it is a **Reedy fibration** if every relative matching map is a fibration in $\mathfrak{M}$.

The content of theorem A.1.2.1.1.3 is then that the Reedy (co)fibrations are the (co)fibrations in the model category $\mathfrak{Fun}(\mathfrak{I}, \mathfrak{M})$.

A.1.2.1.2 Left Bousfield localisation

Definition A.1.2.1.2.1 (Left Bousfield localisation)**.** Let $(\mathfrak{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})$ be a closed model category. A **left Bousfield localisation** of the model structure $(\mathcal{W}, \mathcal{F}, \mathcal{C})$ is a model structure $(\mathcal{W}', \mathcal{F}', \mathcal{C}')$ such that $\mathcal{C}' = \mathcal{C}$ and $\mathcal{W}' \supseteq \mathcal{W}$ (and hence $\mathcal{F}' \subseteq \mathcal{F}$ is determined in the only way posible).

Suppose now $\mathfrak{M}$ is a left proper cofibrantly generated model category. If the left Bousfield localisation exists, then its weak equivalences can be described more geometrically as those that are "local" with respect to a certain class of cofibrations.

Definition A.1.2.1.2.2 (Local equivalences)**.** Let $\mathfrak{M}$ be a simplicial model category. Let $\mathbb{R}\mathrm{Map} \colon \mathfrak{M}^{\mathrm{op}} \times \mathfrak{M} \to \mathfrak{sSet}$ denote the derived mapping space bifunctor (modelled for example by using cofibrant and fibrant replacements). Let S be a collection of morphisms of $\mathfrak{M}$.

1. An object $M \in \mathfrak{M}$ is S**-local** if, for every $f \in S$, the induced map of simplicial sets $\mathbb{R}\mathrm{Map}(f, Z)$ is weak homotopy equivalence of simplicial sets.

2. A morphism f in $\mathfrak{M}$ is an S**-local equivalence** if, for every S-local object $M \in \mathfrak{M}$, the induced map of simplicial sets $\mathbb{R}\mathrm{Map}(f, M)$ is a weak homotopy equivalence.

It is clear that any weak equivalence is an S-local equivalence. Then we can define a left Bousfield localisation at the class of S-local equivalences.

The result is that any left Bousfield localisation is of this form for a certain collection S of morphisms; in fact any choice of generating acyclic cofibrations of the localised structure[Lur09, Proposition A.3.7.4]. This even provides an existence result.

Proposition A.1.2.1.2.3 (Existence of localisation)**.** *[Rie14, Digression 12.3.3], [Lur09, Proposition A.3.7.3] Suppose $\mathfrak{M}$ is a left proper combinatorial (i.e.presentable and cofibrantly generated) simplicial model category, and S be any set of cofibrations. Then the left Bousfield localisation at the class of S-local equivalences exists and is a left proper combinatorial simplicial model category. Furthermore, the fibrant objects are the S-local objects which are fibrant in the original model structure.*

Example A.1.2.1.2.4 (Joyal model structure for quasi-categories)*.* Given a model category with a fixed class of cofibrations, the trivial fibrations (and thus the model structure) are entirely determined by the choice of fibrant objects in the category. Hence we can define (if it does exist) a model structure on $\mathfrak{sSet}$ by requiring the cofibrations to be those from the Kan–Quillen model structure, that is the injections of simplicial sets, and the fibrant (and thus fibrant–cofibrant) objects to be the quasi-categories.

Recall that a categorical equivalence is a morphism of simplicial sets inducing a weak equivalence (in the Bergner model structure) of the associated simplicially enriched category. The model structure described above, called the **Joyal model structure** for quasi-categories, is a left Bousfield localisation of the Kan–Quillen model structure on simplicial sets at the class of categorical equivalences.

A.1.2.2 Simplicial categories and the homotopy coherent nerve

Definition A.1.2.2.1 (Simplicial category)**.** To set notations, we call **simplicially enriched category** a category enriched in the closed monoidal category $(\mathfrak{sSet}, \times, \Delta^0)$ of simplicial sets. We recall that a simplicial object in the category $\mathfrak{Cat}$, that is a functor $\mathfrak{C}_\bullet \colon \Delta^{\mathrm{op}} \to \mathfrak{Cat}$, gives a simplicially enriched category $\underline{\mathfrak{C}}$ if and only if it has constant object sets, that is $\operatorname{obj} \mathfrak{C}_i = \operatorname{obj} \mathfrak{C}_j$ for all $i, j \in \Delta$. The correspondence is given by $\hom_{\underline{\mathfrak{C}}}(X, Y)_\bullet = \hom_{\mathfrak{C}_\bullet}(X, Y)$.

To avoid confusion, we will not use the phrase "simplicial category" to refer to either simplicially enriched categories or simplicial objects in $\mathfrak{Cat}$.

Construction A.1.2.2.2 (Homotopy coherence)**.** Let $\mathfrak{Q}$ denote the category of reflexive directed graphs. Then the forgetful functor $\mathcal{U} \colon \mathfrak{Cat} \to \mathfrak{Q}$ (which forgets composition) admits a left adjoint $\mathcal{F} \colon \mathfrak{Q} \to \mathfrak{Cat}$, forming the free category on a given graph. The adjunction induces a comonad $\mathcal{C} = \mathcal{F}\mathcal{U}$ on $\mathfrak{Cat}$, and thus a simplicial resolution functor $[\mathcal{C}]_\bullet = (\mathcal{C}^{\bullet+1}) \colon \mathfrak{Cat} \to \mathfrak{sCat}$. Note that, since both $\mathcal{U}$ and $\mathcal{F}$ induce the identity on objects, this is also the case for all components of $[\mathcal{C}]_\bullet$, which can thus be interpreted as a functor $\underline{[\mathcal{C}]_\bullet(-)}$ to simplicially enriched categories.

Let $\mathfrak{M}$ be a simplicially enriched category, and let $\mathfrak{I}$ be an indexing category. We define a **homotopy coherent diagram** of shape $\mathfrak{I}$ in $\mathfrak{M}$ to be a functor $\underline{[\mathcal{C}]_\bullet \mathfrak{I}} \to \mathfrak{M}$.

For $[n] \in \mathbb{\Delta}$ (and $\Delta^n \in \mathfrak{sSet}$), set $\mathbb{C}\Delta^n = \underline{[\mathcal{C}]_\bullet[n]}$, where $[n]$ is seen as a category with its ordering. This defines a cosimplicial simplicially enriched category $\mathbb{C}\Delta^\bullet \colon \mathbb{\Delta} \to \mathfrak{sCat}$.

Since, by the density theorem, a simplicial set $X_\bullet$ can be expressed as a colimit, we set $\mathbb{C}X_\bullet = \varinjlim_{\Delta^n \to X_\bullet} \mathbb{C}\Delta^n$. This is equivalent to defining the functor $\mathbb{C} \colon \mathfrak{sSet} \to \mathfrak{sCat}$ as the left Kan extension of $\mathbb{C}\Delta^\bullet = \underline{[\mathcal{C}]_\bullet(-)}\Big|_{\mathbb{\Delta}} : \mathbb{\Delta} \to \mathfrak{sCat}$ along the Yoneda embedding.

We now define tautologically the right adjoint $N_\Delta \colon \mathfrak{sCat} \to \mathfrak{sSet}$ of $\mathbb{C}$ by $N_\Delta(\mathfrak{M})_n = \hom_{\mathfrak{sSet}}(\Delta^n, N_\Delta(\mathfrak{M})) \coloneqq \hom_{\mathfrak{sCat}}(\mathbb{C}\Delta^n, \mathfrak{M})$, so that n-simplices of $N_\Delta(\mathfrak{M})$ are the strings of n "homotopy composable" morphisms in $\mathfrak{M}$. We call this functor the **homotopy coherent nerve**.

Lemma A.1.2.2.3 (Bergner model structure)**.** *[Rie14, Theorem 16.1.2] There exists a cofibrantly generated model structure on $\mathfrak{sCat}$ whose weak equivalences are the simplicially enriched functors inducing essentially surjective functors on the homotopy categories and weak equivalences of mapping spaces, and whose fibrant objects are the simplicially enriched categories whose mapping spaces are Kan complexes.*

Proposition A.1.2.2.4. *The cosimplicial simplicially enriched category $\mathbb{C}[\Delta^\bullet]$ is a cofibrant replacement in the Reedy model structure on $\mathfrak{Fun}(\mathbb{\Delta}, \mathfrak{sCat})$ for $[\bullet] \colon \mathbb{\Delta} \to \mathfrak{sCat}$ which sees $[n]$ as a discrete simplicially enriched category.*

Theorem A.1.2.2.5. *[Lur09, p. 2.2] The adjunction $\mathbb{C} \dashv N_\Delta$ induces a Quillen equivalence between the Joyal model structure (having quasicategories as fibrant objects) and the Bergner model structure (having locally Kan simplicially enriched categories as fibrant objects).*

Example A.1.2.2.6 (Dwyer–Kan localisation). Let $\mathfrak{M}$ be a simplicial model category. Then $N_\Delta(\mathfrak{M}_{cf}) \simeq N_\Delta(\mathfrak{M})[\mathcal{W}^{-1}]_\infty$: the ∞-categorical localisation of the relative category $(\mathfrak{M}, \mathcal{W})$ is given by the homotopy coherent nerve of the full subcategory on fibrant–cofibrant objects.

Example A.1.2.2.7 (The category of spaces). Consider the category $\mathfrak{sSet}$ (cartesian closed, as thus self-enriched) with its standard Kan model structure, whose weak equivalences are the weak homotopy equivalences of simplicial sets. We can then define the ∞-category of spaces, or equivalently of ∞-groupoids, as $\mathfrak{S} = N_\Delta(\mathfrak{sSet})[\mathcal{W}^{-1}]_\infty$.

A.1.2.3 Segal conditions for higher categories

Endow $\mathfrak{sSet}$ with its usual Kan–Quillen model structure, and endow the category of bisimplicial sets $\mathfrak{s}^2\mathfrak{Set} = \mathfrak{ssSet} = \mathfrak{sSet}^{\mathbb{\Delta}^{op}}$ with the corresponding Reedy model structure (for the canonical Reedy category structure on $\mathbb{\Delta}^{op}$).

Definition A.1.2.3.1 (Segal space)**.** A bisimplicial set $X_{\bullet,\bullet}$ is a **Segal space** if for every $m, n \geq 1$, the maps $X_{n+m,\bullet} \to X_{n,\bullet} \times^h_{X_{0,\bullet}} X_{m,\bullet}$ are weak equivalences of simplicial sets. Equivalently, for every $k \geq 2$, the Segal map $X_{k,\bullet} \to X_{1,\bullet} \times^h_{X_{0,\bullet}} \cdots \times^h_{X_{0,\bullet}} X_{1,\bullet}$ is a weak equivalence of simplicial sets.

If $X_{\bullet,\bullet}$ is Reedy fibrant, the homotopy products can be taken to be the actual products.

Construction A.1.2.3.2 (Category theory in a Segal space)**.** Let $X_{\bullet,\bullet}$ be a Segal space. The simplicial set $X_{0,\bullet}$ is called the space of objects of $X_{\bullet,\bullet}$, while $X_{1,\bullet}$ is the space of arrows and $X_{k,\bullet}$ for $k > 1$ is the space of sequences of k composable morphisms. If $x, y \in X_{0,0}$ are two objects, the mapping space from x to y is $\{x\} \times_{X_{0,\bullet}} X_{1,\bullet} \times_{X_{0,\bullet}} \{y\}$. Hence the homotopy category is seen easily by taking π_0 of the mapping spaces. We call $X^{\circ}_{1,\bullet}$ the subspace of $X_{1,\bullet}$ given by the components sent to isomorphisms in the homotopy category. The degeneracy map $s\colon X_{0,\bullet} \to X_{1,\bullet}$, which sends x to its identity morphism, factors through $X^{\circ}_{1,\bullet}$.

Definition A.1.2.3.3 (Complete Segal space)**.** A **complete Segal space** is a Reedy fibrant Segal space such that the map $X_{0,\bullet} \to X^{\circ}_{1,\bullet}$ is a weak equivalence of simplicial sets.

Definition A.1.2.3.4 (Segal category)**.** A **Segal category** is a bisimplicial set $X_{\bullet,\bullet}$ such that $X_{0,\bullet}$ is a discrete simplicial set and for each $k \geq 2$ the Segal map $X_{k,\bullet} \to X_{1,\bullet} \times_{X_{0,\bullet}} \cdots \times_{X_{0,\bullet}} X_{1,\bullet}$ is a weak equivalence of simplicial sets.

Theorem A.1.2.3.5. *[Ber10]*

- *[Ber10, Theorem 4.4] There is a model structure on $\mathfrak{s}^2\mathfrak{Set}$ whose equivalences are the Dwyer–Kan equivalences, cofibrations the monomorphisms, and fibrant objects the complete Segal spaces.*
- *[Ber10, Theorem 5.3] There is a model structure on the full subcategory of $\mathfrak{s}^2\mathfrak{Set}$ spanned by bisimplicial sets with discrete simplicial set of 0-simplices, whose cofibrations are the monomorphisms, and whose fibrant objects are the Segal categories.*
- *[Ber10, Theorem 7.1, Theorem] These two model structures are Quillen equivalent, and they are also Quillen equivalent to the Joyal model structure on $\mathfrak{sSet}$.*

Remark A.1.2.3.6. By a theorem of Toën, any theory of ∞-category satisfying a certain list of axioms must give a model category Quillen equivalent to complete Segal spaces.

Construction A.1.2.3.7 (n-fold complete Segal spaces)**.** Although the use of a model-categorical presentation of $\mathfrak{S}$ is needed to *define* ∞-categories as complete Segal spaces, the construction could in fact be performed for simplicial objects in the ∞-category $\mathfrak{S}$; in fact a complete Segal object in an ∞-category $\mathfrak{C}$ gives a notion of category object in $\mathfrak{C}$.

Consider now a simplicial object in $\mathfrak{Cat}_\infty$, that is an ∞-functor $\mathfrak{X}_\bullet\colon \Delta^{op} \to \mathfrak{Cat}_\infty$. Then we say that $\mathfrak{X}_\bullet$ is a **2-fold complete Segal space**, provided that it satisfies the following consditions:

- the ∞-category $\mathfrak{X}_0$ is an ∞-groupoid;
- for any $n, m \geq 0$, the ∞-category $\mathfrak{X}_{n+m}$ is equivalent to the homotopy fibred product $\mathfrak{X}_n \times_{\mathfrak{X}_0} \mathfrak{X}_m$;
- the factorisation $\mathfrak{X}_0 \to \mathfrak{X}_1^{\circ} \hookrightarrow \mathfrak{X}_1$ induces an equivalence $\mathfrak{X}_0 \simeq \mathfrak{X}_1^{\circ}$, where $\mathfrak{X}_1^{\circ}$ is the 2-full sub-∞-category of $\mathfrak{X}_1$ containing only the invertible 1-morphisms of $\mathfrak{X}_{\bullet,\bullet}$.

This construction can be generalised inductively to define n-fold complete Segal spaces, giving a model for (∞, n)-categories.

A.2 Presheaves on ∞-categories

From now on we consider ∞-categories as given independently of any model chosen; in particular any 1-category defines a particular case of ∞-category and will be considered as such without taking a nerve. This means that results coming from any model for higher categories can be used for our ∞-categories.

In particular (to set notation) there is an ∞-category $\mathfrak{Cat}_\infty$ of ∞-categories, and a full sub-∞-category $\mathfrak{S}$ of spaces or ∞-groupoids. Given any pair of parallel ∞-functors $\mathcal{F}, \mathcal{G} \colon \mathfrak{C} \rightrightarrows \mathfrak{D}$, there is a mapping space $\mathrm{Map}_{\mathfrak{Cat}_\infty}(\mathcal{F}, \mathcal{G}) \in \mathfrak{S}$ which is the underlying space of the ∞-category $\mathfrak{Fun}(\mathcal{F}, \mathcal{G})$.

A.2.1 ∞-functors and (homotopy) limits

A.2.1.1 Universal objects and (co)limits

The following transports directly from classical category theory to the higher context.

Let $\mathfrak{C}$ be an ∞-category. An object $Z \in \mathfrak{C}$ is said to be **initial** (respectively **final**) if for any object $C \in \mathfrak{C}$ the space $\mathrm{Map}(Z, C)$ (resp. $\mathrm{Map}(C, Z)$) is contractible.

Definition A.2.1.1.1 (Limit)**.** Let $\mathcal{F} \colon \mathfrak{I} \to \mathfrak{C}$ be an ∞-functor. A **limit** of $\mathcal{F}$ is an initial object of $\mathfrak{C}_{\triangleleft/\mathcal{F}}$ (a universal cone over $\mathcal{F}$). A **colimit** of $\mathcal{F}$ is a final object of $\mathfrak{C}_{\mathcal{F}/\triangleright}$ (a universal cocone under $\mathcal{F}$). We let $\varprojlim_{\mathfrak{C}} \mathcal{F}$ denote the limit of $\mathcal{F}$ and $\varinjlim_{\mathfrak{C}} \mathcal{F}$ denote its colimit.

We see that a limit of $\mathcal{F}$ is given by a functor $\mathfrak{I}^{\triangleleft} \to \mathfrak{C}$ whose restriction is $\mathcal{F}$; we will say that $\mathcal{F}$ has an extension to a colimit diagram.

If the ∞-categories are obtained from model categories, the homotopy (co)limits actually have the universal property of their ∞-categorical enhancements. This is shown by Lurie in the language of quasi-categories.

Theorem A.2.1.1.2. *[Lur09, Theorem 4.2.4.1] Let $\mathcal{F} \colon \mathfrak{I} \to \mathfrak{C}$ be a simplicial functor between simplicial model categories. Let C be an object of $\mathfrak{C}$. Then $C = \mathbb{R}\varinjlim \mathcal{F}$ if and only if $N_\Delta(\mathcal{F}) \colon N_\Delta(\mathfrak{I}) \to N_\Delta(\mathfrak{C})$ admits an extension to a colimit diagram $N_\Delta(\mathfrak{I})^{\triangleright} \to N_\Delta(\mathfrak{C})$.*

Example A.2.1.1.3 (Ends and coends)*.* Let $\mathfrak{C}, \mathfrak{D}$ be ∞-categories, and let $\mathcal{F}$ be an ∞-functor $\mathfrak{C} \times \mathfrak{C}^{\mathrm{op}} \to \mathfrak{D}$. The **end** and **coend** of $\mathcal{F}$ are the limit and colimit of its composition with the canonical functor $\mathfrak{Tw}(\mathfrak{C}) \to \mathfrak{C} \times \mathfrak{C}^{\mathrm{op}}$, which we denote as $\varprojlim_{\mathfrak{Tw}(\mathfrak{C})} \mathcal{F}$ and $\varinjlim_{\mathfrak{Tw}(\mathfrak{C})} \mathcal{F}$.

We can give an alternate characterisation of limits, as verifying a universal property, using the Grothendieck construction developed in subsubsection A.2.1.2.

Construction A.2.1.1.4 (Adjoint functors)**.** Let $\mathcal{F} \colon \mathfrak{C} \to \mathfrak{D}$ be a functor; it be equivalently be seen as a morphism in the ∞-category $\mathfrak{Cat}_\infty$, so as an ∞-functor $[1] \to \mathfrak{Cat}_\infty$, where $[1]$ is the interval ∞-category (*e.g.* Δ^1 in the model of quasi-categories). Then it also corresponds to a cocartesian fibration $\int \mathcal{F} \to [1]$. If it is also a bifibration, we say that $\mathcal{F}$ is a left adjoint, whose right adjoint is given by the diagram $[1] \to \mathfrak{Cat}_\infty$ obtained from the cartesian fibration $\int \mathcal{F} \to [1]$.

Hence the data of an adjunction between $\mathfrak{C}$ and $\mathfrak{D}$ is a bifibration $\mathfrak{A} \to [1]$ with equivalences $\mathfrak{A}_0 \simeq \mathfrak{C}$ and $\mathfrak{A}_1 \simeq \mathfrak{D}$.

Property A.2.1.1.5 (Behaviour of adjoints)**.**

1. *[Lur09, Lemma 5.2.2.10] An adjunction $\mathcal{F}\colon \mathfrak{C} \rightleftarrows \mathfrak{D}\colon \mathcal{G}$ between ∞-categories induces a $\mathfrak{S}$-adjunction $\mathrm{h}\mathcal{F}\colon \mathrm{h}\mathfrak{C} \rightleftarrows \mathrm{h}\mathfrak{D}\colon \mathrm{h}\mathcal{G}$ between their $\mathfrak{S}$-enriched homotopy categories (and thus also an adjunction between the homotopy categories).*

2. *[Lur09, Proposition 5.2.3.5] Let $\mathcal{F}\colon \mathfrak{C} \rightleftarrows \mathfrak{D}\colon \mathcal{G}$ be an adjunction of ∞-functors. Then $\mathcal{F}$ preserves all the colimits of $\mathfrak{C}$ and $\mathcal{G}$ preserves all the limits of $\mathfrak{D}$.*

Definition A.2.1.1.6 (Kan extensions)**.** Let $\mathfrak{C}$ be an ∞-category, and let $\mathcal{K}\colon \mathfrak{I} \to \mathfrak{J}$ be an ∞-functor, inducing $\mathcal{K}^*\colon \mathfrak{Fun}(\mathfrak{J}, \mathfrak{C}) \to \mathfrak{Fun}(\mathfrak{I}, \mathfrak{C})$. An ∞-functor of **left Kan extension** along $\mathcal{K}$, denoted $\mathrm{Lan}_{\mathcal{K}}$, (resp. of **right Kan extension**, denoted $\mathrm{Ran}_{\mathcal{K}}$) is a left (resp. right) adjoint to $\mathcal{K}^*$.

If $\mathfrak{J} = *$ is the terminal ∞-category, and $\mathcal{K}$ is the unique functor, the if $\mathfrak{C}$ is cocomplete (resp. complete), we have $\mathrm{Lan}_{\mathcal{K}} = \varinjlim\colon \mathfrak{Fun}(\mathfrak{I}, \mathfrak{C}) \to \mathfrak{C}$ (resp. $\mathrm{Ran}_{\mathcal{K}} = \varprojlim\colon \mathfrak{Fun}(\mathfrak{I}, \mathfrak{C}) \to \mathfrak{C}$).

A.2.1.2 Cartesian fibrations and the Grothendieck construction

Remark A.2.1.2.1 (Tensors and cotensors on $\mathfrak{Cat}_\infty$)**.** Here we simply give as motivation a generalisation of the phenomenon from enriched categories, without delving into the theory of enriched ∞-categories. Recall (see *e.g* [Rie14, §3.7, 4.1]) that a $\mathfrak{V}$-enriched category $\mathfrak{C}$ is said to be **tensored** and **cotensored** if there are $\mathfrak{V}$-adjunctions

$$\underline{\mathrm{hom}}_{\mathfrak{C}}(v \otimes c, d) \cong \underline{\mathrm{hom}}_{\mathfrak{V}}(v, \underline{\mathrm{hom}}_{\mathfrak{C}}(c, d)) \cong \underline{\mathrm{hom}}(c, d^v). \tag{A.3}$$

In that case, given functors $\mathcal{F}\colon \mathfrak{I}^{\mathrm{op}} \to \mathfrak{V}$ and $\mathcal{G}\colon \mathfrak{I} \to \mathfrak{C}$, we can define their **functor tensor product** $\mathcal{F} \otimes_{\mathfrak{I}} \mathcal{G}$ as the coend of $\mathcal{F}- \otimes \mathcal{G}-$. Similarly, the functor cotensor product of $\mathcal{F}\colon \mathfrak{I} \to \mathfrak{V}$ and $\mathcal{G}\colon \mathfrak{I} \to \mathfrak{C}$ is the end of $(\mathcal{G}-)^{\mathcal{F}-}$.

The ∞-category $\mathfrak{Cat}_\infty$ has a closed monoidal product given by its cartesian product. Let $\mathfrak{C}$ be an ∞-category, and consider the ∞-category $\mathfrak{Cat}_{\infty,/\mathfrak{C}}$ (for example $\mathfrak{C} = *$, and $\mathfrak{Cat}_{\infty,/\mathfrak{C}} = \mathfrak{Cat}_\infty$). This is an ∞-category enriched over $\mathfrak{Cat}_\infty$, and in fact tensored and cotensored, with tensors $\mathfrak{D} \times \mathfrak{E} \to \mathfrak{C}$ and cotensors $\mathfrak{Fun}(\mathfrak{D}, \mathfrak{F})$ (for $\mathfrak{D} \in \mathfrak{Cat}_\infty$ and $\mathfrak{E}, \mathfrak{F} \in \mathfrak{Cat}_{\infty,/\mathfrak{C}}$).

Let $\mathfrak{C}$ be an ∞-category. The assignments $c \rightsquigarrow \mathfrak{C}_{/c}$ and $c \rightsquigarrow \mathfrak{C}_{c/}$ define ∞-functors $\mathfrak{C}_{/\bullet}\colon \mathfrak{C} \to \mathfrak{Cat}_{\infty,/\mathfrak{C}}$ and $\mathfrak{C}_{\bullet/}\colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{Cat}_{\infty,/\mathfrak{C}}$.

Construction A.2.1.2.2 (Grothendieck construction)**.** Let $\mathfrak{C}$ be an ∞-category and let $\mathcal{F}\colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{Cat}_\infty$ be an ∞-functor. The Grothendieck construction for $\mathcal{F}$ is the functor tensor product

$$\int_{\mathfrak{C}} \mathcal{F} = \varinjlim_{\mathfrak{Tw}(\mathfrak{C})} (\mathfrak{C}_{/\bullet} \times \mathcal{F}) \in \mathfrak{Cat}_{\infty,/\mathfrak{C}}. \tag{A.4}$$

If $\mathcal{F}\colon \mathfrak{C} \to \mathfrak{Cat}_\infty$, the Grothendieck construction for $\mathcal{F}$ is the functor tensor product

$$\int_{\mathfrak{C}}^{\mathrm{co}} \mathcal{F} = \varinjlim_{\mathfrak{Tw}(\mathfrak{C})} (\mathfrak{C}_{\bullet/} \times \mathcal{F}) \in \mathfrak{Cat}_{\infty,/\mathfrak{C}}. \tag{A.5}$$

Remark A.2.1.2.3 (Interpretation). Following [GHN15], the functor tensor products appearing in the definition of the Grothendieck construction, colimits of $\mathcal{F}$ weighted by $\mathfrak{C}_{/\bullet}$ (when $\mathcal{F}$ is contravariant) and $\mathfrak{C}_{\bullet/}$ (when $\mathcal{F}$ is covariant), can be seen as respectively oplax and lax colimits of $\mathcal{F}$. Indeed[Maz15b], while a colimit of a (covariant) functor $\mathcal{F}$ could be interpreted as the ∞-category obtained from the union of the $\mathcal{F}(C), C \in \mathfrak{C}$ by adding equivalences $S \simeq (\mathcal{F}\phi)(S)$ for every $S \in \mathcal{F}(C)$ and every $\phi\colon C \to C'$, a lax colimit takes into account the bicategorical aspect by only adding (non-invertible) morphisms $\phi_*\colon S \to (\mathcal{F}\phi)S$.

Definition A.2.1.2.4 ((Co)Cartesian fibrations)**.** Let $\mathcal{P}\colon \mathfrak{F} \to \mathfrak{C}$ be an ∞-functor.

- A morphism $\phi\colon \xi \to \psi$ in $\mathfrak{F}$, lifting $\mathcal{P}\xi = X \xrightarrow{\mathcal{P}\phi = f} Y = \mathcal{P}\psi$ in $\mathfrak{C}$, is $\mathcal{P}$**-cartesian** if the canonical map $\mathfrak{F}_{/\xi} \to \mathfrak{F}_{/\psi} \times_{\mathfrak{C}_{/Y}} \mathfrak{C}_{/X}$ it induces by postcomposition is an equivalence. We also call (ξ, ϕ) an inverse image of ψ by f, written $f^*\psi \xrightarrow{f_*} \psi$.

  Dually, ϕ is $\mathcal{P}$-cocartesian if the map $\mathfrak{F}_{\psi/} \to \mathfrak{F}_{\xi/} \times_{\mathfrak{C}_{X/}} \mathfrak{C}_{Y/}$ induced by precomposition is an equivalence. We say that ψ is a direct image of ξ by f, written $f_*\xi$.

- The ∞-functor $\mathcal{P}$ is a **cartesian fibration** if every morphism of $\mathfrak{C}$ admits an inverse image for every object of $\mathfrak{F}$ lifting its target.

  Dually, $\mathcal{P}$ is a **cocartesian fibration** if every morphism of $\mathfrak{C}$ induces a direct image for every object of $\mathfrak{F}$ lifting its source.

Remark A.2.1.2.5 (Equivalence with the quasi-categorical definition). An **inner fibration** is a map of simplicial sets having the right lifting property against all inner horn inclusions. It follows that if the target of an inner fibration is a quasi-category, so is its source (an inner fibration over a quasi-categories can be seen as a "bundle of quasi-categories").

Let $p\colon F \to C$ be an inner fibration of simplicial sets. An edge $\phi \in F_1$, seen rather as $\phi\colon \Delta^1 \to F$, is p-cartesian if $F_{\triangleleft/\phi} \to F_{/\phi(1)} \times_{C_{/p(f(1))}} C_{\triangleleft/p(f)}$ is a trivial inner fibration. As before, we say that p is a cartesian fibration of simplicial sets if any edge of C and any vertex of F lifting its target give rise to a cartesian lift. We define dually the cocartesian fibrations of simplicial sets.

If an inner fibration between quasi-categories presents a (co)cartesian fibration, then it is a (co)cartesian fibration of simplicial sets[Maz15a, Corollary 3.4].

Theorem A.2.1.2.6 (Straightening and unstraightening)**.** *[GHN15, Theorem 7.4][Lur09, Theorem 3.2.0.1]*

- *For any $\mathcal{F}\colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{Cat}_\infty$, the ∞-category $\int_{\mathfrak{C}} \mathcal{F}$ is cartesian over $\mathfrak{C}$.*

  For any $\mathcal{F}\colon \mathfrak{C} \to \mathfrak{Cat}_\infty$, the ∞-category $\int_{\mathfrak{C}}^{\mathrm{co}} \mathcal{F}$ is cocartesian over $\mathfrak{C}$.

- *The ∞-functor $\int_{\mathfrak{C}} \colon \mathfrak{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Cat}_\infty) \to \mathfrak{Cat}^{\mathrm{cart}}_{\infty,/\mathfrak{C}}$ admits a left adjoint, called the* ***straightening*** *functor, which induces an equivalence of ∞-categories.*

  Similarly $\int^{\mathrm{co}}_{\mathfrak{C}} \colon \mathfrak{Fun}(\mathfrak{C}, \mathfrak{Cat}_\infty) \to \mathfrak{Cat}^{\mathrm{cocart}}_{\infty,/\mathfrak{C}}$ admits a quasi-inverse.

A.2.1.3 The Yoneda embedding

Definition A.2.1.3.1 (Prestack)**.** Let $\mathfrak{C}$ be an ∞-category. The ∞-category of **presheaves of spaces** or **prestacks** on $\mathfrak{C}$ is $\mathfrak{PSh}(\mathfrak{C}) \coloneqq \mathfrak{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{S})$.

Construction A.2.1.3.2. Let $\mathfrak{C}$ be an ∞-category. Since $\mathfrak{Tw}(\mathfrak{C}) \to \mathfrak{C}^{\mathrm{op}} \times \mathfrak{C}$ is a cartesian fibration in spaces, we can consider the corresponding presheaf $\mathfrak{C}^{\mathrm{op}} \times \mathfrak{C} \to \mathfrak{S}$. Using the inernal homs of the cartesian product on $\mathfrak{Cat}_\infty$, this is equivalent to an ∞-functor $\mathcal{Y} \colon \mathfrak{C} \to \mathfrak{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{S}) = \mathfrak{PSh}(\mathfrak{C})$, called the **Yoneda embedding**.

Proposition A.2.1.3.3. *[Lur12, Proposition 5.2.1.11],[Lur09, Proposition 5.1.3.1] For any ∞-category $\mathfrak{C}$, the Yoneda embedding is fully faithful.*

Theorem A.2.1.3.4. *[Lur09, Theorem 5.1.5.6] Let $\mathfrak{C}$ be a small ∞-category. For any small ∞-category $\mathfrak{D}$ which admits small colimits, restriction along the Yoneda embedding of $\mathfrak{C}$ induces an equivalence of categories $\mathfrak{Fun}^{\mathrm{colim}}(\mathfrak{PSh}(\mathfrak{C}), \mathfrak{D}) \xrightarrow{\sim} \mathfrak{Fun}(\mathfrak{C}, \mathfrak{D})$, where $\mathfrak{Fun}^{\mathrm{colim}}$ indicates the full sub-∞-category of ∞-functors preserving small colimits.*

Corollary A.2.1.3.5 (Density theorem for ∞-categories)**.** *[Lur09, Corollary 5.1.5.8] The Yoneda embedding is dense, that is its image $\mathcal{Y}(\mathfrak{C})$ generates $\mathfrak{PSh}(\mathfrak{C})$ under small colimits.*

Theorem A.2.1.3.6 (Yoneda lemma)**.** *[Lur09, Lemma 5.5.2.1] Let $\mathfrak{C}$ be an ∞-category, A an object of $\mathfrak{C}$ and $\mathcal{F}$ a presheaf (of spaces) on $\mathfrak{C}$. Let $\mathcal{Y} \colon \mathfrak{C} \hookrightarrow \mathfrak{PSh}(\mathfrak{C})$ denote the Yoneda embedding for $\mathfrak{C}$ and $\widehat{\mathcal{Y}} \colon \mathfrak{PSh}(\mathfrak{C}) \hookrightarrow \mathfrak{PSh}(\mathfrak{PSh}(\mathfrak{C})^{\mathrm{op}})$ the co-Yoneda embedding on $\mathfrak{PSh}(\mathfrak{C})$. Then the ∞-functor $\widehat{\mathcal{Y}}(\mathcal{F}) \circ \mathcal{Y}$ is equivalent to $\mathcal{F}$.*

A.2.2 Topologies and sheaves

A.2.2.1 Sieves and local equivalences

Categories of presheaves are often determined by their exactness properties. In order to study them in further details, we need to impose certain finiteness conditions.

Definition A.2.2.1.1.
- An ∞-category $\mathfrak{C}$ is (countably) **accessible** if it admits all countably filtered colimits and is generated under small countably filtered colimits by an essentially small full sub-∞-category consisting of compact objects. Equivalently[Lur09, Proposition 5.4.2.2], $\mathfrak{C}$ can be realised as the ∞-category of ind-objects of a small ∞-category.
- An ∞-category is **presentable** if it is accessible and admits all small colimits.

Theorem A.2.2.1.2. *[Lur09, Theorem 5.5.1.1] An accessible ∞-category is presentable if and only if it is the ∞-category of presheaves of a small ∞-category.*

Definition A.2.2.1.3 (Stack ∞-topos)**. Saturation:** Let $\mathfrak{C}$ be an ∞-category with small colimits. A collection of morphisms S is **strongly saturated** if it is stable under pushouts, has the 2-out-of-3 property, and the full subcategory $\mathfrak{Fun}([1], \mathfrak{C})$ spanned by S is stable under small colimits. The smallest subclass S° of S which is strongly saturated (by [Lur09, Reamrk 5.5.4.7]) is said to **generate** S.

Topological localisation: Let $\mathfrak{C}$ be a presentable ∞-catgory, and S a strongly saturated class of morphisms. We say that S is **topological** if it is generated by a subclass S° consisting of monomorphisms and it is stable by base-change (pullbacks). We then say that $\mathfrak{C} \to \mathfrak{C}[S^{-1}]$ is a **topological localisation** of $\mathfrak{C}$.

∞-Topos: A **stack ∞-topos** is a topological localisation of an ∞-category of presheaves on some ∞-category.

Lemma A.2.2.1.4. *[Lur09, Proposition 5.5.4.20] Let $\mathfrak{C}$ be a presentable ∞-category and S a small set of morphisms is $\mathfrak{C}$. Then the localisation $\mathcal{L}\colon \mathfrak{C} \to \mathfrak{C}[S^{-1}]$ making morphisms in S into equivalences is reflective, i.e. has a fully faithful right adjoint $\mathcal{I}$ exhibiting $\mathfrak{C}[S^{-1}]$ as a full subcategory of $\mathfrak{C}$.*

Proposition A.2.2.1.5. *[Lur09, Proposition 5.5.4.2] Let $\mathfrak{C}$ be a presentable ∞-category, let S be a small set of morphisms in $\mathfrak{C}$. Then an object C of $\mathfrak{C}$ is in the image of $\mathcal{I} \circ \mathcal{L}$ (i.e. in the sub-∞-category $\mathfrak{C}[S^{-1}]$) if and only if it is S-local (as in definition A.1.2.1.2.2). Furthermore, every S-local equivalence is in S.*

We will now see that any stack ∞-topos can be constructed as a category of *sheaves* on an ∞-site. We define a **Grothendieck topology** on an ∞-category $\mathfrak{C}$ as a Grothendieck topology on its homotopy category $\operatorname{Ho}\mathfrak{C}$. An ∞**-site** is a pair $(\mathfrak{C}, \tau)$ of an ∞-category $\mathfrak{C}$ with a Grothendieck topology τ. As in classical category theory, we have the following correspondence.

Lemma A.2.2.1.6. *[Lur09, Proposition 6.2.2.5] Let $\mathfrak{C}$ be a small ∞-category, and denote the Yoneda embedding by $\mathcal{Y}\colon \mathfrak{C} \hookrightarrow \mathfrak{PSh}(\mathfrak{C})$. For every object $C \in \mathfrak{C}$, there is a bijection between the set of subobjects of $\mathcal{Y}(C)$ and the set of all sieves on C.*

Definition A.2.2.1.7 (Sheaf)**.** Let $(\mathfrak{C}, \tau)$ be an ∞-site. A **τ-local equivalence** is a monomorphism of presheaves on $\mathfrak{C}$ whose associated sieve is τ-covering. The **category of τ-sheaves** on $\mathfrak{C}$ is the localisation of $\mathfrak{PSh}(\mathfrak{C})$ at the τ-local equivalences. In other words, **τ-sheaves** are the τ-local objects. We denote this reflective subcategory as $\mathfrak{Sh}_{\tau}(\mathfrak{C})$.

Proposition A.2.2.1.8. *[Lur09, Lemma 6.2.2.7] The class of τ-local equivalences is topological, so $\mathfrak{Sh}_{\tau}(\mathfrak{C})$ is a stack ∞-topos.*

Theorem A.2.2.1.9. *[Lur09, Proposition 6.2.2.9] Let $\mathfrak{C}$ be a small ∞-category. There is a bijective correspondence between equivalence classes of topological localisations of $\mathfrak{PSh}(\mathfrak{C})$ and Grothendieck topologies on $\mathfrak{C}$, so that any stack ∞-topos is equivalent to the ∞-topos of sheaves on an ∞-site.*

A.2.2.2 Descent and hypercovers

Construction A.2.2.2.1 (Coskeleton)**.** Let $\Delta_{\leq n}$ be the full subcategory of Δ whose objects are $[0], \dots, [n]$.The inclusion functor $\iota_n \colon \Delta_{\leq n} \to \Delta$ induces for any ∞-category $\mathfrak{C}$ a restriction functor $\iota_n^* \colon \mathfrak{s}\mathfrak{C} \to \mathfrak{s}_{\leq n}\mathfrak{C} \coloneqq \mathfrak{Fun}(\Delta_{\leq n}{}^{\mathrm{op}}, \mathfrak{C})$. Left and right Kan extension provide left and right adjoints to this functor[Rie14, Example 1.1.9], and the composite monads on $\mathfrak{s}\mathfrak{C}$ are called respectively n**-skeleton** and n**-coskeleton** and denoted sk_n and cosk_n.

Definition A.2.2.2.2 (Hypercovering)**.** Let $(\mathfrak{C}, \tau)$ be an ∞-site. Let $C \in \mathfrak{C}$, and write $y_C = \mathcal{Y}(C) \in \mathfrak{PSh}(\mathfrak{C})$ for its representable presheaf. An augmented simplicial object $\mathcal{F}_{\bullet+}$ of $\mathfrak{PSh}(\mathfrak{C})$ on y_C, seen as a morphism of simplicial presheaves $\mathcal{F}_\bullet \to y_{C,\bullet}$ (where $y_{C,\bullet}$ is the constant simplicial object associated to y_C) is a τ**-hypercovering** of C if for every $[n] \in \Delta$ the map $\mathcal{F}_n \to (\mathrm{cosk}_{n-1} \mathcal{F}_\bullet)_n$ (where $\mathcal{F}_{-1}$ is the augmentation y_C) is a τ-covering.

A hypercovering $\mathcal{F}_{\bullet+}$ of C is **effective** if its totalisation $\varinjlim_n \mathcal{F}_n$ is y_C.

Example A.2.2.2.3 (Čech nerve of a covering)*.* Let $p \colon U \to V$ be a τ covering in an ∞-site $(\mathfrak{C}, \tau)$. The ∞-category $\mathfrak{C}_{/V}$ admits finite products of copies p (by the axioms of a Grothendieck topology) as fiber products in $\mathfrak{C}$ of copies U over V, providing a cotensor for the category of finite sets. Restricting along the inclusion of Δ in the category of finite sets, we obtain a simplicial object $\mathcal{N}_\bullet(p)$, called the **Čech nerve** of p, characterised by $\mathcal{N}_i(p) = U \times_V \cdots \times_V U$, and which we view through the Yoneda embedding as a simplicial presheaf. Since the simplicial object $\mathcal{N}_\bullet(p)$ is determined by its first degrees, it is 0-coskeletal, which implies that it is a τ-hypercovering since $\mathcal{N}_1(p) = U \to \mathcal{N}_0(p) = V$ is τ-covering.

By [Lur09, Lemma 6.5.3.9], all Čech nerves (in fact all coskeletal hypercoverings) are effective hypercoverings.

Definition A.2.2.2.4 (Descent)**.** Let $(\mathfrak{C}, \tau)$ be an ∞-site. Let H denote the collection of effective τ-hypercovers, and $C \subset H$ the collection of Čech nerves of τ-coverings. A presheaf $\mathcal{F}$ on $\mathfrak{C}$ is said to have τ**-hyperdescent** if it is H-local (that is

$$\varprojlim_n \mathrm{Map}(\mathcal{X}_n, \mathcal{F}) \simeq \mathrm{Map}\left(\varinjlim_n \mathcal{X}_n, \mathcal{F}\right) = \mathcal{F}(U) \tag{A.6}$$

for any hypercover $\mathcal{X}_\bullet$ of any object U), and it is said to have τ**-descent** if it is C-local.

Corollary A.2.2.2.5. *A presheaf is a τ-sheaf if and only if it has τ-descent.* □

Definition A.2.2.2.6 (Hypercompletion)**.** Let $\mathfrak{T}$ be a stack ∞-topos. A morphism $f \colon X \to Y$ is ∞**-connective** if for any truncated object Z, the induced map $\mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ is an equivalence of spaces.

The ∞-topos $\mathfrak{T}$ is **hypercomplete** if its equivalences are exactly the ∞-connective morphisms.

By [Lur09, Proposition 6.5.2.8], the collection of ∞-connective morphisms of a stack ∞-topos $\mathfrak{T}$ is strongly saturated. We call an object of $\mathfrak{T}$ **hypercomplete** if it is local for the class of ∞-connective morphisms. It follows that the full sub-∞-category spanned by hypercomplete objects is the (reflective) localisation of $\mathfrak{T}$ at the ∞-connective morphisms, called its **hypercompletion** $\mathfrak{T}^{\wedge}$, which is hypercomplete.

Proposition A.2.2.2.7. *[Lur09, Corollary 6.5.3.13] Let $(\mathfrak{C}, \tau)$ be an ∞-site. The hypercompletion $\mathfrak{Sh}_{\tau}(\mathfrak{C})^{\wedge}$ is the ∞-category of presheaves with τ-hyperdescent.*

Theorem A.2.2.2.8. *[TV05, Theorem 3.8.3] Let $\mathfrak{C}$ be a small ∞-category. There is a bijective correspondence between Grothendieck topologies on $\mathfrak{C}$ and hypercomplete left exact localisations of $\mathfrak{PSh}(\mathfrak{C})$.*

Bibliography

[AGV72] Michael Artin, Alexander Grothedieck and Jean-Louis Verdier. *Théorie des topos et cohomologie étale des schémas. SGA 4*. Vol. 1,2. 1972. Chap. I–Vbis. URL: http://fabrice.orgogozo.perso.math.cnrs.fr/SGA4/index.html.

[Ant13] Omar Antolín Camarena. "A Whirlwind Tour of the World of $(\infty, 1)$-categories". In: *ArXiv e-prints* (Mar. 2013). arXiv: 1303.4669 [math.CT].

[Avr99] Luchezar L. Avramov. "Locally Complete Intersection Homomorphisms and a Conjecture of Quillen on the Vanishing of Cotangent Homology". In: *Annals of Mathematics* 150.2 (1999), pp. 455–487. ISSN: 0003486X. arXiv: 9909192. URL: http://www.jstor.org/stable/121087.

[Bar15] Clark Barwick. "On exact ∞-categories and the Theorem of the Heart". In: *Compositio Mathematica* 151.11 (2015), pp. 2160–2186. DOI: 10.1112/S0010437X15007447. arXiv: 1212.5232 [math.KT]. URL: http://www.maths.ed.ac.uk/~cbarwick/papers/k2-1-rev1.pdf.

[Beh14] Kai Behrend. "On the Virtual Fundamental Class". In: *Proceedings of the International Congress of Mathematicians*. ICM 2014. Vol. 2. 2014. URL: https://www.mathunion.org/fileadmin/ICM/Proceedings/ICM2014.2/ICM2014.2.pdf.

[Beh97] Kai Behrend. "Gromov–Witten invariants in algebraic geometry". In: *Inventiones mathematicae* 127.3 (Sept. 1997), pp. 601–617. ISSN: 1432-1297. DOI: 10.1007/s002220050132. arXiv: alg-geom/9601011.

[Ber10] Julia E. Bergner. "A Survey of $(\infty, 1)$-categories". In: John C. Baez and J. Peter May. *Towards Higher Categories*. The IMA Volumes in Mathematics and its Applications 152. Springer-Verlag New York, 2010. DOI: 10.1007/978-1-4419-1524-5. arXiv: 0610239.

[BF97] Kai Behrend and Barbara Fantechi. "The Intrinsic Normal Cone". In: *Inventiones Mathematicae* 128.1 (1997), pp. 45–88. arXiv: alg-geom/9601011.

[CJR17] Emily Clader, Felix Janda and Yongbin Ruan. "Higher-genus quasimap wall-crossing via localization". In: *ArXiv e-prints* (Feb. 2017). arXiv: 1702.03427 [math.AG].

[CK09] Ionuţ Ciocan-Fontanine and Mikhail Kapranov. "Virtual fundamental classes via dg–manifolds". In: *Geometry & Topology* 13 (3 2009). DOI: 10.2140/gt.2009.13.1779. arXiv: math/0703214 [math.AG].

[CKM14] Ionuţ Ciocan-Fontanine, Bumsig Kim and Davesh Maulik. “Stable quasimaps to GIT quotients”. In: *Journal of Geometry and Physics* 75 (2014), pp. 17–47. ISSN: 0393-0440. DOI: 10.1016/j.geomphys.2013.08.019. arXiv: 1106.3724. URL: http://www.sciencedirect.com/science/article/pii/S0393044013001666.

[CM13] Denis-Charles Cisinski and Ieke Moerdijk. “Dendroidal sets and simplicial operads”. In: *Journal of Topology* 6.3 (2013), pp. 705–756. DOI: 10.1112/jtopol/jtt006. arXiv: 1109.1004.

[Cos06] Kevin Costello. “Higher Genus Gromov-Witten Invariants as Genus Zero Invariants of Symmetric Products”. In: *Annals of Mathematics* 164.2 (2006), pp. 561–601. ISSN: 0003486X. DOI: 10.4007/annals.2006.164.561. JSTOR : 20160000.

[Deb16] Olivier Debarre. *Introduction to Mori Theory. Cours de M2*. 2010–2011. 2016. URL: https://www.math.ens.fr/~debarre/M2.pdf.

[DM69] Pierre Deligne and David Mumford. “The irreducibility of the space of curves of given genus”. In: *Publications Mathématiques de l'IHÉS* 36 (1969), pp. 75–109. EuDML : 103899. URL: http://www.numdam.org/item?id=PMIHES_1969__36__75_0.

[EM06] A.D. Elmendorf and M.A. Mandell. “Rings, modules, and algebras in infinite loop space theory”. In: *Advances in Mathematics* 205.1 (2006), pp. 163–228. ISSN: 0001-8708. DOI: https://doi.org/10.1016/j.aim.2005.07.007. arXiv: math/0403403. URL: http://www.sciencedirect.com/science/article/pii/S0001870805001957.

[Ful98] William Fulton. *Intersection Theory*. 2nd ed. Springer-Verlag, 1998. DOI: 10.1007/978-1-4612-1700-8.

[GH15] David Gepner and Rune Haugseng. “Enriched ∞-categories via non-symmetric ∞-operads”. In: *Advances in Mathematics* 279 (2015), pp. 575–716. arXiv: 1312.3178.

[GHN15] David Gepner, Rune Haugseng and Thomas Nikolaus. “Lax colimits and free fibrations in ∞-categories”. In: *Documenta Mathematica* 22 (2015). arXiv: 1501.02161. URL: https://www.math.uni-bielefeld.de/documenta/vol-22/37.html.

[GK98] Ezra Getzler and Mikhail M. Kapranov. “Modular operads”. In: *Compositio Mathematica* 110.1 (Jan. 1998), pp. 65–125. ISSN: 1570-5846. DOI: 10.1023/A:1000245600345. arXiv: dg-ga/9408003 [dg-ga].

[GR17] Dennis Gaitsgory and Nick Rozenblyum. *A Study in Derived Algebraic Geometry*. Vol. 1. 2 vols. American Mathematical Society, 2017. URL: http://www.math.harvard.edu/~gaitsgde/GL/.

[Gro10] Moritz Groth. *A short course on ∞-categories*. 2010. arXiv: 1007.2925 [math.AT].

[Hau17] Rune Haugseng. "Iterated Spans and Classical Topological Field Theories". In: *Mathematische Zeitschrift* (2017). DOI: 10.1007/s00209-017-2005-x. arXiv: 1409.0837.

[HHM16] Gijs Heuts, Vladimir Hinich and Ieke Moerdijk. "On the equivalence between Lurie's model and the dendroidal model for infinity-operads". In: *Advances in Mathematics* 302 (2016), pp. 869–1043. ISSN: 0001-8708. DOI: 10.1016/j.aim.2016.07.021. arXiv: 1305.3658.

[Ill71] Luc Illusie. *Complexe Cotangent et Déformations*. Vol. I. Lecture Notes in Mathematics 239. Springer-Verlag Berlin Heidelberg, 1971. DOI: 10.1007/BFb0059052.

[King94] Alastair D. King. "Moduli of representations of finite dimensional algebras". In: *The Quarterly Journal of Mathematics* 45.4 (1994), pp. 515–530. DOI: 10.1093/qmath/45.4.515. URL: http://dx.doi.org/10.1093/qmath/45.4.515.

[KKP03] Bumsig Kim, Andrew Kresch and Tony Pantev. "Functoriality in intersection theory and a conjecture of Cox, Katz, and Lee". In: *Journal of Pure and Applied Algebra* 179.1 (2003), pp. 127–136. ISSN: 0022-4049. DOI: https://doi.org/10.1016/S0022-4049(02)00293-1. URL: http://www.sciencedirect.com/science/article/pii/S0022404902002931.

[Kol96] János Kollár. *Rational Curves on Algebraic Varieties*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 32. Springer-Verlag Berlin Heidelberg, 1996. DOI: 10.1007/978-3-662-03276-3.

[Lee04] Y. P. Lee. "Quantum K-theory, I: Foundations". English. In: *Duke Mathematical Journal* 121.3 (Feb. 2004), pp. 389–424. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-04-12131-1. arXiv: 0105014.

[LM00] Gérard Laumon and Laurent Moret–Bailly. *Champs algébriques*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 39. Springer-Verlag Berlin Heidelberg, 2000. DOI: 10.1007/978-3-540-24899-6.

[Lur09] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, 2009. URL: http://www.math.harvard.edu/~lurie/papers/HTT.pdf.

[Lur12] Jacob Lurie. *Higher Algebra*. 2012. URL: http://www.math.harvard.edu/~lurie/papers/HA.pdf.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic Operads*. Grundlehren der mathematischen Wissenschaften 346. Springer-Verlag Berlin Heidelberg, 2012. DOI: 10.1007/978-3-642-30362-3. URL: https://www.math.univ-paris13.fr/%7Evallette/Operades.html.

[Man17] Yuri I. Manin. "Mirrors, Functoriality, and Derived Geometry". In: *ArXiv e-prints* (Aug. 2017). arXiv: 1708.02849 [math.AG].

[Maz15a] Aaron Mazel-Gee. “A user’s guide to co/cartesian fibrations”. In: *ArXiv e-prints* (Oct. 2015). arXiv: 1510.02402 [math.CT].

[Maz15b] Aaron Mazel-Gee. “All about the Grothendieck construction”. In: *ArXiv e-prints* (Oct. 2015). arXiv: 1510.03525 [math.AT].

[MOP11] Alina Marian, Dragos Oprea and Rahul Pandharipande. “The moduli space of stable quotients”. In: *Geometry & Topology* 15.3 (2011), pp. 1651–1706. DOI: 10.2140/gt.2011.15.1651. arXiv: 0904.2992.

[MR18a] Etienne Mann and Marco Robalo. “Brane Actions, Categorification Of Gromov–Witten Theory, And Quantum K-Theory”. In: *Geometry & Topology* 22 (3 2018), pp. 1759–1836. DOI: 10.2140/gt.2018.22.1759. arXiv: 1505.02964.

[MR18b] Etienne Mann and Marco Robalo. “Gromov–Witten theory with derived algebraic geometry”. In: *ArXiv e-prints* (2018). arXiv: 1803.09476 [math.AG].

[MT10] Ieke Moerdijk and Bertrand Toën. *Simplicial Methods for Operads and Algebraic Geometry*. Advanced Courses in Mathematics - CRM Barcelona. Birkhäuser Basel, 2010. DOI: 10.1007/978-3-0348-0052-5.

[Nab15] Navid Nabijou. *Virtual Fundamental Classes in Gromov–Witten Theory*. 2015. URL: http://wwwf.imperial.ac.uk/~nn1209/VFCProject.pdf.

[Ols07] Martin Olsson. “Tangent Spaces and Obstruction Theories”. MSRI summer workshop on Deformation Theory and Moduli in Algebraic Geometry. 2007. URL: https://math.berkeley.edu/~molsson/MSRISummer07.pdf.

[Ras14] Samuel David Raskin. “Chiral Principal Series Categories”. Doctoral dissertation. Harvard University, 2014. URL: http://nrs.harvard.edu/urn-3:HUL.InstRepos:12274305.

[Rie14] Emily Riehl. *Categorical Homotopy Theory*. New Mathematical Monographs. Cambridge University Press, 2014. DOI: 10.1017/CBO9781107261457. URL: http://www.math.jhu.edu/~eriehl/cathtpy/.

[Stacks] The Stacks Project authors. *The Stacks Project*. 2005–2018. URL: https://stacks.math.columbia.edu/.

[STV15] Timo Schürg, Bertrand Toën and Gabriele Vezzosi. “Derived algebraic geometry, determinants of perfect complexes, and applications to obstruction theories for maps and complexes”. In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2015.702 (2015), pp. 1–40. arXiv: 1102.1150 [AG].

[Toë12] Bertrand Toën. “Proper local complete intersection morphisms preserve perfect complexes”. In: *ArXiv e-prints* (Oct. 2012). eprint: 1210.2827.

[Toë13] Bertrand Toën. “Operations on derived moduli spaces of branes”. In: *ArXiv e-prints* (2013). arXiv: 1307.0405 [math.AG].

[Toë14] Bertrand Toën. “Derived Algebraic Geometry”. In: *arXiv* (2014). arXiv: `1401.1044`.

[TV05] Bertrand Toën and Gabriele Vezzosi. “Homotopical algebraic geometry I: topos theory”. In: *Advances in Mathematics* 193.2 (2005), pp. 257–372. ISSN: 0001-8708. DOI: `10.1016/j.aim.2004.05.004`. arXiv: `0207028`. URL: `http://www.sciencedirect.com/science/article/pii/S0001870804001513`.

[TV08] Bertrand Toën and Gabriele Vezzosi. “Homotopical algebraic geometry. II. Geometric stacks and applications.” In: *Memoirs of the American Mathematical Society* 193.902 (Jan. 2008), x+224pp. arXiv: `math/0404373v7`. URL: `https://hal.archives-ouvertes.fr/hal-00772955`.

[Vak17] Ravi Vakil. *The Rising Sea. Foundations of Algebraic Geometry*. Nov. 2017. URL: `http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf`.

[Wan11] Jonathan Wang. “The Moduli Stack of G-Bundles”. In: *ArXiv e-prints* (2011). arXiv: `1104.4828 [math.AG]`.

[Wei07] Ittay Weiss. “Dendroidal sets”. Doctoral thesis. Utrecht University, 2007. URL: `https://dspace.library.uu.nl/bitstream/handle/1874/22859/?sequence=7`.

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