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Par
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« Théorie de quasi-applications catégorifiée des champs de Deligne–Mumford dérivés »

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Théorie de quasi-applications catégorifiée des champs de Deligne–Mumford dérivés

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Categorified quasimap theory of derived Deligne–Mumford stacks

Abstract

This thesis extends the results of [MR18] on the categorification of Gromov–Witten invariants to stack targets. This requires constructing a brane action for certain coloured ∞-operads, for which we develop a language for lax morphisms as well as a dendroidal version of monoidal envelopes. We finally obtain an action on a cyclotomic loop stack, given by moduli stacks of stable quasimaps. An application to the categorification of the quantum Lefschetz principle is also provided.

Keywords: derived geometry,operads,Gromov–Witten theory

Théorie de quasi-applications catégorifiée des champs de Deligne–Mumford dérivés

Résumé

Nous étendons les résultats de[MR18] sur la catégorification des invariants de Gromov–Witten aux cibles champêtres. Cela implique de construire une action de membranes pour certaines ∞-opérades colorées, ce pour quoi nous développons un langage pour les morphismes laxes ainsi qu'une version dendroïdale des enveloppes monoïdales. Nous obtenons finalement une action sur un champ de lacets cyclotomique, donnée par des champs de modules de quasi-applications. Nous décrivons également une application à la catégorification du principe de Lefschetz quantique.

Mots clés : géométrie dérivée,opérades,théorie de Gromov–Witten

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SOMMAIRE

INTRODUCTION

Motivation

Topological string theories

The cohomology ring of any smooth projective variety X (and more generally, by [FV21, §2.2], the Chow motive of X) admits a structure of Frobenius algebra, hence it defines a 2-dimensional topological quantum field theory, that is a representation of the 2-dimensional cobordism category, in the category of graded $\mathbb{Z}$-modules (respectively, the category of Chow motives). This field theory, however, is not local, in that it cannot come from an extended 2-dimensional topological quantum field theory (a representation of the 2-dimensional extended cobordism bicategory). Indeed, by [Sch09, Corollary 3.53], the underlying Frobenius algebra of an extended topological quantum field theory must be semi-simple, which here can only happen when X is a union of points.

In order to have any hope of defining an interesting field theory from X, one must thus change the algebraic structure of its cohomology ring. Since the 2-dimensional cobordism category consists of Riemann surfaces, a reasonable proposal is to incorporate structures coming from complex, or more generally algebraic, curves into the definition of this new cohomology product. In fact, this intuition also provides a new way of interpreting the need for locality: the isolated Riemann surfaces of the cobordism category can be replaced by *families* of curves, leading to the notion of topological field theory in families considered in [Tel12], a variant of topological conformal field theories. For this, both the shape category (of cobordisms) and the category of coefficients are replaced by suitably local versions, (pre)sheaves of categories over a category of base spaces.

Specifically, a topological field theory in families is defined as a symmetric monoidal natural transformation ζ between the following indexed categories. On the one hand, the shape is the functor associating with a base space B the (topologically enriched) category whose morphisms are (classifying spaces of) families of stable marked nodal curves over B, the source and target of a morphism being its families of ingoing and outgoing markings (and monoidal structure given by disjoint union). On the other hand, the coefficient family of categories associates with a base space B the differential-graded category whose objects are flat complexes of local systems on B, with complex of morphisms from $\mathcal{V}$ to $\mathcal{W}$ given by $\mathbb{R}\Gamma(B, \mathcal{H}om(\mathcal{V}, \mathcal{W}))$ (and monoidal structure given by the tensor product).

From the requirement of monoidality, the action of ζ on objects is easy to specify: it is determined, on a base B, by a local system $\mathcal{V}$, the image of the family of one point on B, as then the image of the family of m ingoing and n outgoing points is $(\mathcal{V}^\vee)^{\otimes m} \otimes \mathcal{V}^{\otimes n}$. The action on morphisms, however, can only be classified thanks to the existence of a universal family of curves.

Let $\overline{\mathscr{M}}_{g,(m,n)}$ denote the moduli stack of stable curves of genus g with $m+n$ markings (the first m considered ingoing and the last n outgoing for the purposes of this exposition), and let $\overline{\mathscr{C}}_{g,(m,n)} \to \overline{\mathscr{M}}_{g,(m,n)}$ denote its (tautological) universal family of curves. By definition, any family of curves $C \to B$ on a base B is given as the base-change of $\overline{\mathscr{C}}_{g,(m,n)}$ along a uniquely determined morphism $\ulcorner C \urcorner \colon B \to \overline{\mathscr{M}}_{g,(m,n)}$. Then the naturality requirement for ζ implies that the class $\zeta_B(C) \in \mathbb{R}\Gamma(B, \mathcal{H}om(\mathcal{V}^{\otimes m}, \mathcal{V}^{\otimes n}))$ must be isomorphic to the inverse image $\ulcorner C \urcorner^*(\zeta_{\overline{\mathscr{M}}_{g,(m,n)}}(\overline{\mathscr{C}}_{g,(m,n)}))$. In other words, the field theory ζ, once fixed the coefficient system $\mathcal{V}$ on $\overline{\mathscr{M}}_{g,(m,n)}$, is completely determined by the family of "universal" classes

$$\Omega_{g,(m,n)} = \zeta_{\overline{\mathscr{M}}_{g,(m,n)}}(\overline{\mathscr{C}}_{g,(m,n)}) \in \mathbb{R}\Gamma(\overline{\mathscr{M}}_{g,(m,n)}, \mathcal{H}om(\mathcal{V}^{\otimes m}, \mathcal{V}^{\otimes n})). \tag{1}$$

Gromov–Witten classes as a local field theory

This reformulation now suggests a way of building a "local" field theory on the cohomology of X: the aim is to obtain classes in $\mathbb{R}\Gamma(\overline{\mathscr{M}}_{g,(m,n)}, \mathcal{H}om(A_\bullet X^m, A_\bullet X^n))$, which may be viewed as maps $A_\bullet \overline{\mathscr{M}}_{g,(m,n)} \otimes A^\bullet X^m \to A_\bullet X^n$. Ignoring the cohomology, one may imagine them as coming from "virtual" maps $\overline{\mathscr{M}}_{g,(m,n)} \times X^m \to X^n$, where the virtualisation comes from the fact that geometric maps can give both direct or inverse images in cohomology: it suffices to construct a span (whose forward leg enjoys appropriate properness and dimensionality properties). A somewhat tautological way of achieving this would be to consider maps from points of $\overline{\mathscr{M}}_{g,(m,n)}$ to X. In order to ensure the correct properness properties, there is a need to make sure the intermediate moduli stack of maps is proper; for this a stability condition is used. Letting $\beta \in A_1 X$ be a curve class in X, there is a proper Deligne–Mumford stack $\overline{\mathscr{M}}_{g,(m,n)}(X, \beta)$ whose (generalised) points are families of stable maps from prestable curves into X with class β. There is a forgetful morphism $\mathrm{Stab}\colon \overline{\mathscr{M}}_{g,(m,n)}(X,\beta) \to \overline{\mathscr{M}}_{g,(m,n)}$ which throws away the map and stabilises the curve, as well as evaluation morphisms $\mathrm{ev}_i \colon \overline{\mathscr{M}}_{g,(m,n)}(X,\beta) \to X$ evaluating the map at a specified ith marking, which produces a span

$$\begin{array}{ccc} & \coprod_\beta \overline{\mathscr{M}}_{g,(m,n)}(X,\beta) & \\ {\scriptstyle (\mathrm{Stab}, \mathrm{ev}_1, \cdots, \mathrm{ev}_m)} \swarrow & & \searrow {\scriptstyle (\mathrm{ev}_{m+1}, \cdots, \mathrm{ev}_{m+n})} \\ \overline{\mathscr{M}}_{g,(m,n)} \times X^m & & X^n \end{array} \tag{2}$$

as desired.

There comes however one issue when passing back to cohomology: although the morphism Stab is proper, it is far from being smooth and in fact is not even equidimensional; the boundary components (parameterising maps from nodal possibly unstable curves) added to the smooth interior to compactify it can be of a greater dimension. As a result, the stack $\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)$ does not have a fundamental class to compute degrees against. To overcome this issue, following the suggestion of [Kon95], [BF97] constructed a *virtual fundamental class* $[\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)]^{\mathrm{vir}} \in A_\bullet \overline{\mathscr{M}}_{g,(m,n)}(X,\beta)$, living in the degree expected from the enumerative interpretation of the moduli problem modulated by $\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)$. This construction uses in a nontrivial way the tools of deformation theory, that is the cotangent complex $\mathbb{L}_{\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)}$ and the additional datum of a "perfect obstruction theory" $\mathcal{E} \to \mathbb{L}_{\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)}$, in order to access cohomological corrections to the lack of smoothness, and allows one to pass from the span of equation (2) to a morphism in homology, by replacing pushforward along Stab by a "virtual" version, twisted by product with the virtual class.

Categorification and geometrisation

As was already noticed in [Kon95], the virtual class can actually be defined at a higher level than Chow homology. Indeed, through the Chern character, the virtual class lifts to a G-theory class, called the *virtual structure sheaf* $\left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)}\right]$. In this way, Gromov–Witten classes can be defined in G-theory as well, again by replacing the pushforward along Stab by its virtual version twisted by tensoring with the virtual sheaf.

By definition, the G-theory groups are a decategorification of the ("derived") ∞-categories $\mathfrak{Coh}^b$ of bounded coherent complexes. Toën and Manin independently proposed that the G-theoretic Gromov–classes may also be a decategorification of a structure living at the level of ∞-categories of bounded coherent sheaves. One may even go further and suggest that the structure of Gromov–Witten classes should exist at the geometric level, in the spans of equation (2) themselves. The first question raised for this problem is that of what the algebraic structure to capture is in the first place.

Of course, the classes $\Omega_{g,(m,n)}$ constituting the family of equation (1) do not exist independently of each other. Since composition in the category of curves is given by gluing curves together at their marked points, functoriality implies that these classes must be compatible with the gluing operations. A system of such cohomology classes on the moduli stacks of stable curves compatible with gluing is know as a **cohomological field theory**, and can be reformulated in a purely algebraic way, using the language of operads and operadic structure.

Indeed, one may interpret a curve with (n, m) markings as an operation with n inputs and m outputs and the gluing of curves as giving the partial composition operation, producing a properad (enriched in stacks), or in fact a wheeled properad due to the higher-genus curves and self-gluings. Further, we may notice that the distinction between "ingoing" and "outgoing" points is purely artificial, so that the actual structure best describing the curves is a *modular operad* $\overline{\mathscr{M}} = (\overline{\mathscr{M}}_{g,n})_{\underline{g,n}+1}$. Applying the products-preserving homology functor $A_\bullet$ produces a modular operad $A_\bullet\overline{\mathscr{M}}$ in graded $\mathbb{Q}$-modules, whose algebras are directly seen (up to duality) to coincide exactly with cohomological field theories.

A by-product of this reformulation is that the algebraic structure underlying cohomological field theories can be interpreted in different contexts, using the more fundamental operad $\overline{\mathscr{M}}$. For example, we may call a **categorical field theory** an algebra (in stable ∞-categories) over the ∞-operad $\mathfrak{Coh}^b(\overline{\mathscr{M}})$. More generally, we will say that a **geometric field theory** is an algebra (or more generally a lax algebra) over $\overline{\mathscr{M}}$ itself in the ∞-bicategory $\mathfrak{Span}(\mathfrak{Stk})$ of stacks and spans between them.

The second question is that of whether the spans equation (2) do define a geometric field theory on the target X, or if at least passing to bounded coherent sheaves can define a categorical field theory. But this question is in fact the wrong one, as the spans studied are the wrong one: so as to really generalise the cohomological field theory of Gromov–Witten classes, the spans need to be twisted by an analogue of the virtual class. However, as its name indicates, this class is only a virtual object existing as a G-theory or a cohomology class, but it is not an actual geometric object; in addition it is not defined only from the moduli stack $\overline{\mathscr{M}}_{g,(m,n)}(X, \beta)$ but with the help of the added perfect obstruction theory $\mathcal{E} \to \mathbb{L}_{\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)}$. Thus, while one could imagine adapting the definition of the virtual structure sheaf in $\mathfrak{Coh}^b(\overline{\mathscr{M}}_{g,n}(X, \beta))$ to define a "virtual" sheaf lifting the virtual sheaf class, such a thing is not possible at the geometric level (and in fact the sheaf complex one may define in this way is not the right lift of the virtual sheaf as it is devoid of geometric content).

Brane actions and (derived) geometric Gromov–witten invariants

Derived moduli stacks of maps

A solution to the problem of finding a geometric interpretation to the virtual structure sheaves was already proposed in [Kon95], where Kontsevich proposed his "hidden smoothness" philosophy, asserting that homological constructs such as those used to define virtual sheaves should be seen as shadows of better-behaved higher structures: stacks augmented with structure sheaves of differential graded algebras, which would accomodate spectra of "derived" algebras. Indeed, from this point of view the perfect obstruction theory $\mathcal{E} \to \mathbb{L}_{\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)}$ can be seen as specifying a inclusion of "derived vector bundles" $\mathbb{V}\big(\mathbb{L}^{\vee}_{\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)}\big) \hookrightarrow \mathbb{V}(\mathcal{E}^{\vee})$, and the formula defining the virtual structure sheaf can be directed lifted to one giving the derived intersection of a certain subcone of $\mathbb{V}\big(\mathbb{L}^{\vee}_{\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)}\big)$ with the zero section of $\mathbb{V}(\mathcal{E}^{\vee})$. But this construction, while it does turn the virtual structure sheaf into the (G-theory class of the) actual structure sheaf of a derived stack thickening $\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)$, fails to give any further geometric insight into this derived thickening.

Kontsevich's proposal was further followed by [CK02] which employed the language of dg-manifolds to construct derived moduli stacks of maps between algebraic stacks, which as derived mapping stacks contain the information of higher Ext groups, and in particular a derived moduli stack of stable maps $\mathbb{R}\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)$, which is derived local complete intersections and thus "smooth enough", and whose cotangent complex recovers exactly the perfect obstruction theory used to define virtual classes.

The foundations of derived geometry, incorporating the purely homotopical or higher-categorical aspects of homological algebra, have been developed by Toën–Vezzosi [TV08] and Lurie [Lur04; Lur19] in the last two decades, as an extension of algebraic geometry where moduli stacks can have their points evaluated not just on rings but on "derived rings" (modelled, for example, by commutative differential-graded algebras up to quasi-isomorphism or more generally simplicial algebras up to weak homotopy equivalence) in order to have good intersections and take their values in ∞-groupoids rather than sets or groupoids so as to guarantee good quotients. In this language, derived moduli stacks of stable maps were used in [STV15] and [MR18] (as well as [PY20] in non-archimedean geometry); in particular [MR18] showed that the (G-theory class) of the structure sheaf $\mathcal{O}_{\mathbb{R}\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)}$ restricts to $\Big[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)}\Big]$.

Operadic construction of the geometric field theory

The discussion so far furnishes us with a derived stack $\mathbb{R}\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)$ thickening the moduli stack $\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)$ and recovering its virtual sheaf. In fact, through the projection formula, one can see that integrating classes on $\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)$ with respect to its virtual class comes down exactly to integrating them on $\mathbb{R}\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)$. In other words, an integral transform along a span as in equation (2) twisted by the integral kernel $\Big[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathscr{M}}_{g,(m,n)}(X,\beta)}\Big]$ can be replaced by a simple push-pull along the span of derived stacks

$$\begin{array}{ccc} & \coprod_\beta \mathbb{R}\overline{\mathscr{M}}_{g,n+1}(X,\beta) & \\ {\scriptstyle (\mathrm{Stab},\mathrm{ev}_1,\cdots,\mathrm{ev}_n)} \swarrow & & \searrow {\scriptstyle \mathrm{ev}_{n+1}} \\ \overline{\mathscr{M}}_{g,n+1} \times X^n & & X. \end{array} \tag{3}$$

It now becomes sensible to consider whether these spans directly define a geometric field theory on the target X, and this is indeed what the main result of [MR18] shows.

Theorem 0.0.0.0.1 ([MR18])**.** *Let X be a smooth projective scheme. Then X has a lax $\overline{\mathscr{M}}_0$-action, informally given by the spans of equation* (3).

Since the action constructed now takes place in the ∞-category of derived stacks, it can no longer be constructed by hand, as an infinite number of coherences need to be taken into account. It was proposed by [Toë13] to obtain it as a corollary of a general operadic phenomenon called the brane action, applied here to a more general operad $\mathfrak{M}_0^{\mathrm{sch}}$ of moduli stacks of *pre*stable curves (of genus 0), which serve as the source for stable maps.

The construction of the brane actions is based on the notion of *extensions* of operations in an ∞-operad. Say that a (for now monochromatic) ∞-operad $\mathscr{O}$ is *unital* if its space $\mathscr{O}(0)$ of nullary morphisms is contractible. In that case, it makes sense to define an extension of an operation $\sigma \in \mathscr{O}(n)$ to be another multimorphism $\widetilde{\sigma} \in \mathscr{O}(n+1)$ of higher arity which gives back σ when composed on an input with the essentially unique nullary operation. In the case of the operad $\mathfrak{M}_0^{\mathrm{sch}}$ of prestable curves, an extension of a marked curve is a choice of additional marking on it.

If one assumes in addition that $\mathscr{O}$ is reduced, that is its space of unary operations is also contractible, consisting essentially only of the identity operation, then the space of extensions of the identity is identified with $\mathscr{O}(2)$.

Theorem 0.0.0.0.2 ([Toë13][MR18])**.** *Let $\mathscr{O}$ be a reduced ∞-operad in a hypercomplete $(\infty,1)$-topos $\mathfrak{T}$. Then $\mathscr{O}(2)$ carries a lax $\mathscr{O}$-action in cocorrespondences, which is strong if and only if $\mathscr{O}$ is coherent.*

If X is any object of $\mathfrak{T}$, composing the brane action with the contravariant internal hom ∞-functor represented by X gives a new (lax) $\mathscr{O}$-action, this time in correspondences, on $X^{\mathscr{O}(2)}$. In the case of the operad $\mathfrak{M}_0^{\mathrm{sch}}$, the stack $\mathfrak{M}_0^{\mathrm{sch}}(2)$ is the moduli stack $\mathfrak{M}_{0,3}^{\mathrm{sch}}$ of rational curves with 3 marked points, which is reduced to a point, so that the induced brane action is on the target X itself; however the spans encoding the action are still given by (derived) mapping stacks from $\mathfrak{M}_{0,n+1}^{\mathrm{sch}}$, which contains the derived moduli stacks of stable maps. Then, to pass from an $\mathfrak{M}_0^{\mathrm{sch}}$-action to an $\overline{\mathscr{M}}_0$-action, one extends the morphism of operads along the stabilisation morphism $\mathfrak{M}_0^{\mathrm{sch}} \to \overline{\mathscr{M}}_0$.

Stable quasimaps and orbifold Gromov–Witten invariants

Stacky curves for Gromov–Witten invariants of smooth Deligne–Mumford stacks

When the target X is a smooth Deligne–Mumford stack (an orbifold), Abramovich–Graber–Vistoli realised in [AGV08] that Gromov–Witten theory must be modified in a significant way: the source curves must also be allowed to carry stack structures at their markings. That is, the marked points of a curve must now be étale gerbes, of the form $\mathcal{B}\,\mu_r$ where μ_r is the group of rth roots of unity. In order for the gluing of curves to make sense, the nodal points of a curve must also have a stack structure, obtained from quotient by a balanced action of some μ_r on the two branches of the node.

A first consequence of this is that the Gromov–Witten invariants of a stack X can no longer be defined on the cohomology of X itself but on its *orbifold cohomology*, the cohomology of (a cyclotomic variant of) its inertia stack. This is because the evaluation morphisms for maps from a stacky curve do not land in X but instead, since the markings to evaluate on are μ_r-banded gerbes, they land in a stack parameterising μ_r-gerbes in X.

Recall that the inertia stack $\mathscr{I}X$ can be defined as the 2-fibre product $X \times_{X \times X} X$ and is a moduli stack for automorphisms of points of X. Separating the automorphisms by their order, one obtains a decomposition of $\mathscr{I}X$ as a disjoint union $\coprod_{r \geqslant 1} t_0 \mathcal{M}or^{\mathrm{rep}}(\mathcal{B}\,\mathbb{Z}/(r), X)$ of (at this stage, classical, which is represented by the presence of the truncation ∞-functor t_0) stacks of representable (*i.e.*, inducing monomorphisms on the isotropy groups) morphisms from the trivial cyclic $\mathbb{Z}/(r)$-gerbe into X, that is of automorphisms of order r of points of X. By analogy, the stack

$$\mathscr{I}_\mu X := \coprod_{r \geqslant 1} t_0 \mathcal{M}or^{\mathrm{rep}}(\mathcal{B}\,\mu_r, X) \tag{4}$$

in which the evaluation of maps from stacky curves at their markings tautologically lands is called the **cyclotomic inertia stack** of X. On each indexed component of $\mathscr{I}_\mu X$ there is a canonical action of the group stack $\mathcal{B}\,\mu_r$. The stack obtained by quotienting out these actions is called the **rigidified** inertia stack of X and denoted $\overline{\mathscr{I}}_\mu X$; it is the stack on whose cohomology the Gromov–Witten action is eventually constructed in [AGV08].

Brane actions for monochromatically-unital coloured operads

The main project of this thesis is to turn to the question of defining Gromov–Witten (or more generally quasimap) invariants at the geometric level for stacky targets.

The presence of stack structures on the source curves for orbifold Gromov–Witten theory has another, this time operadic, consequence. When gluing together two curves along a pair of marked points, one needs to ensure that the markings are compatible, that is that they carry gerbes of the same order. This means that the composition in the operad of stacky curves $\mathfrak{M}$ cannot be defined indiscriminately for all inputs and outputs, but must rather be encoded as the composition for a *coloured* operad, whose colours are the orders of the gerbes.

Because of this, theorem 0.0.0.0.2 can not be applied to the operad $\mathfrak{M}$, and a coloured version is needed instead. But while the operad considered is indeed non monochromatic, the full coloured structure cannot be allowed to contribute to the brane action. Indeed, one expects to recover from the extensions the universal curve, which in this case is the substack of $\mathfrak{M}_{g,n+1}$ parameterising only those stacky curves whose last point is of order 1, *i.e.* is schematic: the only marked points which can be safely added or removed are schematic ones.

From the operadic point of view, this corresponds to the fact that the operad $\mathfrak{M}$ is not actually unital. The monochromatic definition of unitality given above extends straightforwardly to a coloured operad $\mathscr{O}$ by requiring that the space of nullary operations targetting each colour of $\mathscr{O}$ all be contractible. For stacky curves, it is natural to define the objects of nullary operations with target r to be the classifying 2-stacks $\mathcal{B}^2\,\mu_r$ parameterising μ_r-gerbes. The automorphism groups of their unique points are the cyclotomic groups μ_r, so outside of the schematic case they do not satisfy the unitality condition.

Thus we will formulate our "coloured" brane action not for unital ∞-operads but for ∞-operads endowed with a choice a single colour required to satisfy the unitality condition. For those operads, the only sensible notion of extension is that which extends only by the distinguished colour, so although the brane action now gives rise to a morphism of coloured operads, its construction is close to the monochromatic case, and we obtain our first main result:

Theorem 0.0.0.0.3 (theorem 2.2.2.0.1). *Let* $\mathscr{C}$ *be an* ∞*-operad with a distinguished unital colour* C_0*. There is a lax morphism of* $(\infty, 2)$*-operads* $\mathscr{C} \to \mathfrak{Cospan}(\infty\text{--}\mathfrak{Grpd})^{\amalg}$ *whose value at a colour* C *is the* ∞*-groupoid* $\mathrm{Ext}(\mathrm{id}_C)$ *of extensions of* id_C *by* C_0.

As can be seen from the statement of the theorem, the morphism to construct is of a very ∞-bicategorical nature. In [Toë13], it was constructed using model categories and strictification arguments, while in [MR18] the Grothendieck construction was used to reduce the formulation to an $(\infty, 1)$-categorical one, at the cost of some simplicity of the definitions.

Here we wish to give at least a formulation, if not fully a construction, which fits in the general framework of lax bicategorical structures. This requires two main steps: first to define the notion of lax morphisms of $(\infty, 2)$-categorical structures, and then to work with a model of $(\infty, 2)$-operads compatible with this framework. This first step will be the object of chapter 1, which presents work, inspired by Lack's lax morphisms classifiers [Lac02] and methods of formal higher category theory developed by Riehl–Verity [**riehl20:_elemen**], not yet fully completed but still giving the ideas of the construction of lax morphisms.

For the second point, we have chosen to adopt the language of abstract Segal conditions developed by [CH21] under the name of *algebraic patterns*. It allows both the flexibility of encoding general operadic structures and the rigidity which ensures the monadicity results required to access the framework of lax morphisms. Using this language, we adapt the construction of [MR18] to one for Segal (almost) dendroidal objects; in particular we give a new construction of the monoidal envelope of an ∞-operad in this model.

The quasimap geometric field theory

We now have described the necessary tools to apply the brane action to the operad of stacky curves as in [MR18] to obtain the lax Gromov–Witten geometric field theory. But before doing this, we should note that, in the orbifold setting, the Gromov–Witten stability condition is no longer the only one possible.

Quasimap theory, developed in [CKM14; CCK15], is a theory for stacky targets with the added flexibility of a choice (parameterised by $\mathbb{Q}$) of stability condition. This stability condition was originally formulated only for GIT quotients, but the "Beyond GIT" program of [Hal18] makes it possible to extend these ideas to more general tame Artin stacks endowed with a polarising line bundle. For a polarisation (an ample line bundle) $\mathcal{L}_0$ on X and a positive rational number $\varepsilon \in \mathbb{Q}$, which we may view as forming together a *rational polarisation* $\mathcal{L} := \varepsilon \otimes \mathcal{L}_0 \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Pic}(X)$, there are moduli stacks of $\mathcal{L}$-quasistable maps $\mathcal{Q}^{\mathcal{L}}_{g,n}(X, \beta)$ which when $\varepsilon > 2$ recover $\overline{\mathscr{M}}_{g,n}(X^{\mathcal{L}_0-\mathrm{stable}})$. We deduce immediately derived thickenings $\mathbb{R}\mathcal{Q}^{\mathcal{L}}_{g,n}(X, \beta)$.

We can now state our main result, on the existence of the quasimap geometric field theories for polarised Deligne–Mumford stacks. Recall that the classical orbifold quasimap invariants of X are defined on the cyclotomic inertia stack defined in equation (4) by analogy with inertia stacks. There is an obvious thickening to a **cyclotomic (derived) loop stack** which we denote $\mathscr{L}_{\mu}X$, wherein the truncated mapping stacks are replaced by derived mapping stacks, and its rigidified version.

Theorem 0.0.0.0.4 (theorem 4.2.2.2.7). *Let* X *be a quasi-projective derived* 1*-Deligne–Mumford stack, let* $\mathcal{L}_0 \in \mathrm{Pic}(X)$ *be a polarisation on* X *and let* $\varepsilon \in \mathbb{Q}_{>0}$*. There is a lax* $\overline{\mathscr{M}}_0$*-algebra structure (in correspondences)* *on the rigidified cyclotomic loop derived stack $\overline{\mathscr{L}}_\mu X^{\mathcal{L}\text{-st}}$ of the $\mathcal{L}$-stable locus $X^{\mathcal{L}\text{-st}}$, informally given by the spans*

$$\begin{array}{ccc} & \coprod_{\beta \in \mathrm{Eff}(X,\mathcal{L})} \mathbb{R}\mathcal{Q}^{\mathcal{L}}_{0,n+1}(X,\beta) & \\ \swarrow & & \searrow \\ \overline{\mathscr{M}}_{0,n+1} \times (\overline{\mathscr{L}}_\mu X^{\mathcal{L}\text{-st}})^n & & \overline{\mathscr{L}}_\mu X^{\mathcal{L}\text{-st}}. \end{array} \tag{5}$$

The important observation is that the cyclotomic loop stack does not need to be inputted by hand here, but instead appears as a direct consequence of the brane action and the operadic structure on the moduli stacks of curves.

The categorified quantum Lefschetz principle

Finally, we describe an application of the derived moduli stacks of stable maps.

The main lesson gleaned from [MR18] and followed in this work is that phenomena observed at the virtual level can be improved and formulated as completely geometric statements thanks to derived geometry.The quantum Lefschetz principle is a tool (due to Kim–Kresch–Pantev [KKP03] and Joshua [Jos10]) computing the virtual sheaf of the moduli stack of stable maps to the zero locus $Z(s)$ of a section s of a vector bundle $E \to X$ from that of the ambient scheme X.

Consider the universal diagram around the universal curve

$$\begin{array}{ccc} & \mathfrak{C}_{g,n} \underset{\mathfrak{M}_{g,n}}{\times} \mathcal{M}or_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}) & \\ \swarrow^{\rho} & & \searrow^{\mathrm{ev}} \\ \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n}) & & X \end{array}. \tag{6}$$

The virtual sheaf of $\overline{\mathscr{M}}_{g,n}(Z)$ is obtained by tensoring that of $\overline{\mathscr{M}}_{g,n}(X)$ with the Euler class of a pullback-pushforward $\mathfrak{E}_{g,n} = \mathbb{R}^0 \rho_* \operatorname{ev}^* E$ of E on $\overline{\mathscr{M}}_{g,n}(X)$ along this correspondence:

Theorem 0.0.0.0.5 ([KKP03; Jos10]). *For any $\gamma \in A_1 Z$ such that $i_*\gamma = \beta$, let $u_\gamma \colon \overline{\mathscr{M}}_{0,n}(Z,\gamma) \hookrightarrow \overline{\mathscr{M}}_{0,n}(X,\beta)$ denote the closed immersion. Suppose E is convex, that is $\mathbb{R}^1 p_*(C, \mu^* E) = 0$ for any stable map $\mu\colon C \to X$ from a rational (*i.e. *genus-0) stable curve $C \xrightarrow{p} S$ (so that the cone $\mathbb{R}^0 \rho_* \operatorname{ev}^* E$ is a vector bundle). Then*

$$\sum_{i_*\gamma = \beta} u_{\gamma,*} \left[\overline{\mathscr{M}}_{0,n}(Z,\gamma)\right]^{\mathrm{vir}} = \left[\overline{\mathscr{M}}_{0,n}(X,\beta)\right]^{\mathrm{vir}} \smile c_{\mathrm{top}}(\mathbb{R}^0 \rho_* \operatorname{ev}^* E) \in A_\bullet\left(\overline{\mathscr{M}}_{0,n}(X,\beta)\right), \tag{7}$$

and

$$\sum_{i_*\gamma = \beta} u_{\gamma,*} \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathscr{M}}_{0,n}(Z,\gamma)}\right] = \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathscr{M}}_{0,n}(X,\beta)}\right] \otimes \lambda_{-1}(\mathbb{R}^0 \rho_* \operatorname{ev}^* E) \in G_0\left(\overline{\mathscr{M}}_{0,n}(X,\beta)\right). \tag{8}$$

This is valid only under several assumptions:

(i) X should be smooth and the section s regular,
(ii) the genus is restricted to $g = 0$,
(iii) and finally E must not have nonvanishing first cohomology on curves.

Euler classes are usually understood as indicating intersections (here, in a virtual sense) with the zero section, and we indeed prove the following

Theorem 0.0.0.0.6 (Geometric quantum Lefschetz principle, [Ker20] (*cf.* corollary 3.2.2.3.4))**.** *the derived enhancement* $\mathbb{R}\overline{\mathscr{M}}_{g,n}(Z(s))$ *is the (derived) zero locus of the induced section* $\mathbb{R}\mathfrak{s}_{g,n}$ *of the derived bundle* $\mathbb{R}\mathfrak{E}_{g,n} \to \mathbb{R}\overline{\mathscr{M}}_{g,n}(X)$*, in the sense that the following square is cartesian:*

$$\begin{array}{ccc} \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z(s)) & \longrightarrow & \mathbb{R}\overline{\mathscr{M}}_{g,n}(X) \\ \downarrow & \lrcorner & \downarrow \mathbb{R}\mathfrak{s}_{g,n} \\ \mathbb{R}\overline{\mathscr{M}}_{g,n}(X) & \xrightarrow{0} & \mathbb{R}\mathfrak{E}_{g,n}. \end{array} \tag{9}$$

Along the way we provide an understanding of the aforementioned bundle $\mathbb{R}\mathfrak{E}_{g,n}$ as the derived moduli $\mathbb{R}\overline{\mathscr{M}}_{g,n}(E)$ of stable maps to E. Working with derived stacks allows us to remove all of the assumptions from the theorem.

From the description of $\mathbb{R}\overline{\mathscr{M}}_{g,n}(Z(s))$ given by equation (9), one may compute its $\mathcal{O}_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X)}$-algebra of functions using the standard resolution by the Koszul complex (a categorification of the Euler class, also incorporating the section); this gives a categorified form of the quantum Lefschetz formula. However, this computation depends on the choice of resolution while the formula should not. To better understand this refined Euler class, I gave a completely intrinsic formula for structure sheaves of derived zero loci. More precisely, if $\sigma\colon \mathcal{E}_{g,n} \to \mathcal{O}_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X)}$ is the algebraic counterpart of the induced section of the derived bundle $\mathbb{R}\mathfrak{E}_{g,n}$ whose sheaf of sections is $\mathcal{E}_{g,n}^{\vee}$, we exhibit in subsection 3.2.2.2 the structure sheaf of $\mathbb{R}\overline{\mathscr{M}}_{g,n}(Z(s))$ as the quotient of $\mathcal{S}ym(\operatorname{cofib}\sigma)$ relative to a canonical $\mathcal{O}_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X)}[t]$-algebra structure.

For the passage back to G-theory, and the check that the categorified formula does decategorify to the classical one, we extend in subsection 3.1.3 the derived geometric interpretation of virtual classes to Manolache's virtual pullbacks [Man12a; Qu18], a relative version of them. From this I was able to gain new insight into the quantum Lefschetz theorem by concluding that the classical hypotheses are in fact necessary to make the decategorification possible and get back the classical quantum Lefschetz principle.

Outline of the thesis

We now summarise the information contained in this introduction in the order in which it appears in the rest of the text. This thesis is separated into two parts, themselves made up each of a first more general chapter and second one more closely related to the results announced above.

In chapter 1, we begin by laying down the language that will be used to talk about lax morphisms of $(\infty,2)$-operads but also about all higher-categorical structures. In that respect, section 1.1 consists of reminders on formal category theory which will form the base language for our work, and does not contain any original work. Then, in section 1.2, we initiate the development of the notions of lax morphisms of (lax) algebras over $(\infty,2)$-monads. The main results on the construction of lax morphisms classifiers rely on a lemma which is yet to be proved, but the universal property of the 2-categories of lax algebras given in theorem 1.2.2.1.5 is, to the best of the author's knowledge, new. This chapter deals with a different level of generality than the rest of the text and can safely be skipped.

Then, chapter 2 is dedicated to the construction of brane actions. We begin in subsection 2.1.1 by reminders on the recently developed theory of algebraic patterns, and follow in subsection 2.1.2 by describing a "plus construction" for them which we apply to the construction of monoidal envelopes of Segal objects for appropriate patterns, which leads to a new proof in theorem 2.1.2.2.14 of the adjunction between ∞-operads and symmetric monoidal $(\infty, 1)$-categories, and even between categorical ∞-operads and symmetric monoidal $(\infty, 2)$-categories, this time in the language of dendroidal Segal objects.

In part II we switch gears toward algebraic geometry in order to apply the previous constructions to moduli stacks of stable curves. In chapter 3 we initiate the study by discussing general (derived) moduli stacks of morphisms. In section 3.1 we first review the essentials of derived geometry and show in subsection 3.1.3 how to recover virtual pullbacks, the relative version of virtual classes, from pullbacks of derived thickenings. Then we specialise in section 3.2 to stacks of morphisms. After quickly reviewing their basic properties in subsection 3.2.1, we prove in subsection 3.2.2 a general version of the quantum Lefschetz principle, for maps whose target is the zero locus of a section of a vector bundle.

Finally, in chapter 3, we bring together the geometric and operadic aspects to construct the quasimap geometric field theory of a polarised orbifold. In section 4.1 we go through the definitions of the source stacky curves and the stable loci induced by the polarisation on the target, to define the quasimap moduli stacks in subsection 4.2.1.1. This enables us to use the brane action on the operad of curves in subsection 4.2.2 to construct an action encoding the quasimap field theory.

Some notations and conventions

- In a (higher) category, or more generally an enriched ∞-category, $\mathfrak{C}$, the hom-object between two objects C and D shall be denoted $\mathfrak{C}(C, D)$, unless stated otherwise.
- If the enrichment comes from a cartesian closed monoidal ∞-category, the internal homs shall instead be denoted as exponential objects D^C.
- In particular, the ∞-category of (strong, in the ∞-bicategorical case) ∞-functors $\mathfrak{C} \to \mathfrak{D}$ is always written $\mathfrak{D}^{\mathfrak{C}}$. This includes the $(\infty, 1)$-category of presheaves on an $(\infty, 1)$-category $\mathfrak{C}$, which is denoted $\infty\text{--}\mathfrak{Grpd}^{\mathfrak{C}^{op}}$.
- A sub-(∞, n)-category $\mathfrak{B} \hookrightarrow \mathfrak{C}$ is said to be **wide** if it contains all objects, and **locally full** if the inclusion functors on hom $(\infty, n-1)$-categories are full. In particular, a wide and locally full sub-(∞, n)-category is determined simply by a choice of 1-morphisms.
- For any natural integer n, the discrete category with n objects will also be denoted n. In the cases $n \in \{0, 1\}$, we will also denote $\emptyset = 0$ the initial category, and $* = 1$ the terminal one. More generally, a terminal object in any ∞-category will typically be denoted $*$ by default.
- For any integer $n \geqslant -1$, the category freely generated by the connected linear graph with n objects (and $n-1$ arrows) shall be denoted $\mathbb{n}$ (it is also known as $[n-1]$). In particular, the category $\mathbb{2}$ is the generic (or "walking") arrow, and for any ∞-category $\mathfrak{C}$, its ∞-category of arrows is $\mathfrak{Arr}(\mathfrak{C}) = \mathfrak{C}^{\mathbb{2}}$.
- We call **lax extensions** what is usually referred to as right (Kan) extensions, and similarly **oplax extensions** what is usually known as left (Kan) extensions. If $\mathscr{K} \colon \mathfrak{C} \to \mathfrak{D}$ and $\mathscr{F} \colon \mathfrak{C} \to \mathfrak{E}$, the lax extension of $\mathscr{F}$ along $\mathscr{K}$ is denoted $\mathrm{Lex}_{\mathscr{K}} \mathscr{F}$ while the oplax extension of $\mathscr{F}$ along $\mathscr{K}$ is denoted $\mathrm{Opex}_{\mathscr{K}} \mathscr{F}$.
- If $\mathfrak{C}$ is an (∞, n)-category and $k < n$ is an integer, we let $\iota_k \mathfrak{C}$ denote the k-**core** of $\mathfrak{C}$, the maximal sub-(∞, k)-category it contains.

Part I

Operad theory and brane actions

CHAPTER

1

LAX MORPHISMS OF $(\infty, 2)$-CATEGORICAL ALGEBRAIC STRUCTURES

In chapter 2, we will describe one of our main results, the construction of the brane action, which is a certain lax morphism of $(\infty, 2)$-operads. While it can, through the use of universal(*i.e.* cartesian) arrows, be recast in $(\infty, 1)$-categorical terms as in [MR18], it is more convenient and conceptually clearer to formulate it and manipulate as a truly $(\infty, 2)$-categorical lax object. To obtain suitable algebraic properties, we can realise that lax morphisms of $(\infty, 2)$-algebras are an example of a general notion of lax morphism for $(\infty, 2)$-categorical algebraic structures.

In section 1.2, we will develop a general formalism of lax morphisms for algebras over $(\infty, 2)$-monads, explaining, after [Laco2], that they can be understood in terms of strong morphisms through a "lax morphisms classifier". its construction will require methods of formal category theory such as weighted limits.

To facilitate this, we first explain in section 1.1 some of the main tenets and tools of formal category theory. In fact, be it for the study of ∞-operads or of derived algebraic geometry, the language of higher category theory has a crucial role in this work, and in this section we will more generally explain how to use formal category theory to do *higher* category theory, in a model-independent way.

The reader who does not wish to be reintroduced to higher category theory may safely skip section 1.1 and refer back to it as needed. Similarly, it is possible to skip section 1.2 if one takes the characterisation of proposition 2.1.1.3.5 as a definition of lax morphisms of $(\infty, 2)$-categorical operadic structures.

1.1 Formal higher category theory

To deal with the infinite towers of coherences that appear when working with higher categories, one is forced to resort to homotopy theory to properly define the relevant structures in a way compatible with the principle of equivalence. Indeed, the tools of abstract and concrete homotopy theory give access to explicit models which can be used to actually construct ∞-categories, and to a language to deal with localisations and enforce equivalence-invariance.

The choice of a model for higher categories usually leads to analytic reasoning and constructions in this model, which combine the hindrances of not being obviously compatible with changes of models and of requiring non-categorical arguments specific to the models to prove results which ought to be purely categorical. A different approach, pioneered by [RV21], is to work with a *synthetic* theory of higher categories.

There the need for choice of model is relegated to one for the model of the $(\infty,2)$-categories of ∞-categories in which to work, the most-relevant one called an ∞-cosmos. Then, once fixed this notion of ∞-cosmos as a model, one is free to use the methods of formal category theory in it to study higher categories as one would for classical categorical structures. It is this approach that we have chosen to follow, and in this section we recall its main features.

1.1.1 Elements of ∞-cosmology: the 2-category theory of ∞-categories

1.1.1.1 ∞-cosmoi

An ∞-cosmos is roughly a category of fibrant objects enriched in the Joyal model structure for quasicategories, whose weak equivalences are determined by the model enrichment, and satisfying good completeness properties.

Definition 1.1.1.1.1 (∞-cosmos)**.** An **∞-cosmos** is an $\mathfrak{sSet}$-enriched category whose hom-simplicial sets are quasicategories, equipped with a choice of a class of arrows called **isofibrations**, denoted as $\twoheadrightarrow$ and satisfying certain closure properties, and which is cotensored (or powered) over $\mathfrak{sSet}$ and admits $\mathfrak{sSet}$-enriched small products, pullbacks along isofibrations, and limits of countable towers of isofibrations.

An arrow $f\colon K \to K'$ in an ∞-cosmos $\mathfrak{K}$ is an **equivalence** if for every object $L \in \mathfrak{K}$, the map $\mathfrak{K}(L,K) \xrightarrow{\mathfrak{K}(L,f)} \mathfrak{K}(L,K')$ is an equivalence of quasicategories.

A **cosmological functor** from an ∞-cosmos $\mathfrak{K}$ to an ∞-cosmos $\mathfrak{L}$ is an $\mathfrak{sSet}$-enriched functor $\mathfrak{K} \to \mathfrak{L}$ which preserves the isofibrations and the $\mathfrak{sSet}$-limits specified in the definition of ∞-cosmos.

Example 1.1.1.1.2 (∞-cosmos of $(\infty,1)$-categories). By [RV21, Proposition 1.2.10] the category of quasicategories, with its cartesian closed self-enrichment, defines an ∞-cosmos $\mathfrak{QCat}$, with the classes of isofibrations and of weak equivalences coinciding with the ones defined analytically for quasicategories.

We shall say that an ∞-cosmos $\mathfrak{K}$ is an ∞-cosmos of $(\infty,1)$-categories if its canonical cosmological functor $\mathfrak{K}(*,-)\colon \mathfrak{K} \to \mathfrak{QCat}$ is a cosmological biequivalence (by [RV21, Proposition 10.2.1], this is equivalent to the apparently weaker condition that $\mathfrak{K}$ simply be cosmologically biequivalent to $\mathfrak{QCat}$). We shall generally denote this functor as N and call it the **nerve** functor.

Example 1.1.1.1.3 (Sliced ∞-cosmoi). For any object $K \in \mathfrak{K}$, there is a **sliced ∞-cosmos** $\mathfrak{K}_{/K}$ whose objects are the isofibrations in $\mathfrak{K}$ with codomain K.

In particular, the objects of $\mathfrak{K}_{/K\times L}$, the spans of isofibrations $K \xleftarrow{p} E \xrightarrow{q} L$ defining an isofibration $E \xrightarrow{(p,q)} K \times L$, are called **two-sided isofibrations**.

Example 1.1.1.1.4 (Discrete objects). An object $K \in \mathfrak{K}$ is said to be **discrete** if the isofibration $K^{\mathbb{2}[\to^{-1}]} \twoheadrightarrow K^{\mathbb{2}}$ (where $\mathbb{2}[\to^{-1}]$, the localisation of $\mathbb{2}$ along its unique arrow, is the walking isomorphism) is a weak equivalence. By [RV21, Lemma 1.2.27], K is discrete if and only if, for any $L \in \mathfrak{K}$, the quasicategory $\mathfrak{K}(L, K)$ is a Kan complex.

By [RV21, p. 6.1.6], the full subcategory $\mathfrak{Disc}(\mathfrak{K}) \subset \mathfrak{K}$ on the discrete objects inherits a structure of ∞-cosmos.

By local application of the functor Ho: $\mathfrak{sSet} \to \mathfrak{Cat}$ left-adjoint to the nerve and taking a simplicial set to its homotopy category, any ∞-cosmos $\mathfrak{K}$ has a **homotopy** 2**-category** denoted $\operatorname{Ho}\mathfrak{K}$, so that $(\operatorname{Ho}\mathfrak{K})(K, L) = \operatorname{Ho}(\mathfrak{K}(K, L))$ for any objects K, L of $\mathfrak{K}$.

Definition 1.1.1.1.5 (Smothering functors and weak universal properties)**.** A functor $\mathscr{F}: \mathfrak{C} \to \mathfrak{D}$ is **smothering** if it is surjective on objects, full, and conservative.

Many usual constructions of (higher) category theory, such as adjunctions, limits, or Kan extensions, can be characterised by weak universal properties. However, thanks to the tools of formal category theory, these are all subsumed by the calculus of profunctors, for which only the example of comma ∞-categories will be (crucially) needed.

Example 1.1.1.1.6 (Simplicial cotensors). Let K be an object of an ∞-cosmos $\mathfrak{K}$, and let S be a simplicial set. The simplicial cotensor (or power) K^S of K by S, whose existence is required in $\mathfrak{K}$ from the completeness axioms of ∞-cosmoi, is determined by the universal property that for any other object L there is an isomorphism $\mathfrak{K}(L, K^S) \simeq \mathfrak{K}(L, K)^S$ natural in L, where $\mathfrak{K}(L, K)^S$ is the cotensor of simplicial sets (given, by cartesian closure, by their internal hom). The identity arrow $\mathrm{id}_{K^S} \in \mathfrak{K}(K^S, K^S)$ corresponds to an element of $\mathfrak{K}(K^S, K)^S$, a diagram of shape S in the quasicategory $\mathfrak{K}(K^S, K)$. By a variant of [RV21, Proposition 3.2.5], the functor $\operatorname{Ho}\big(\mathfrak{K}(L, K^S)\big) \to \operatorname{Ho}\big(\mathfrak{K}(L, K)\big)^S$ induced by this construction is smothering.

In particular, letting $S = N\mathbb{2}$, we find the corresponding result for arrow categories.

Example 1.1.1.1.7 (Comma ∞-categories). Let $\mathfrak{cocorr} = (\cdot \to \cdot \leftarrow \cdot)$ and $\mathscr{W}: \mathfrak{cocorr} \mapsto (\mathbb{1} \xrightarrow{0} \mathbb{2} \xleftarrow{1} \mathbb{1})$ be the shape category and the weight for comma objects. Let $\mathscr{d}: \mathfrak{cocorr} \mapsto (B \xrightarrow{f} A \xleftarrow{g} C)$ be a diagram of shape $\mathfrak{cocorr}$ (an cospan) in $\mathfrak{K}$. An object $f \downarrow g$ endowed with isofibrations $b: f \downarrow g \to B$ and $c: f \downarrow g \to C$ and a 2-cell $\alpha: fb \Rightarrow gc$ (*i.e.* a $\mathscr{W}$-shaped cone $\gamma: \mathscr{W} \Rightarrow \operatorname{Ho}\mathfrak{K}(f \downarrow g, \mathscr{d})$ whose legs are isofibrations) is a **comma object** for the cospan $\mathscr{d}$ if for every $K \in \mathfrak{K}$ the functor

$$\operatorname{Ho}\mathfrak{K}(K, f \downarrow g) \to \mathfrak{Cat}^{\mathfrak{cocorr}}\big(\mathscr{W}, \operatorname{Ho}\mathfrak{K}(K, \mathscr{d})\big) = \operatorname{Ho}\mathfrak{K}(K, f) \downarrow \operatorname{Ho}\mathfrak{K}(K, g) \tag{1.1}$$

(of whiskering by γ) is smothering.

Given any cospan $B \xrightarrow{f} A \xleftarrow{g} C$, its comma object exists in $\mathfrak{K}$, and following [RV21, Proposition 3.4.6] can be constructed as the fibred product $f \downarrow g \simeq (B \times C) \times_{A \times A} A^{\mathbb{2}}$.

Construction 1.1.1.1.8. Let $\mathscr{p}: \mathfrak{e} \to \mathfrak{b}$ be an isofibration. The definition of the comma ∞-category

$\mathscr{p} \downarrow \mathrm{id}_{\mathfrak{b}}$ furnishes a cartesian square as below.

$$\begin{array}{ccccc} & & \mathfrak{e}^2 & & \\ & \swarrow^{ev_1} & \downarrow{\scriptstyle \mathscr{k}} & \searrow^{\mathscr{p}^2} & \\ & & \mathscr{p} \downarrow \mathrm{id}_{\mathfrak{b}} & & \\ & \swarrow_{ev_1} & \lrcorner & \searrow^{forget} & \\ \mathfrak{e} & & & & \mathfrak{b}^2 \\ & \searrow_{\mathscr{p}} & & \swarrow_{ev_2} & \\ & & \mathfrak{b} & & \end{array} \tag{1.2}$$

The outer diagram's commutativity then induces a canonical map $\mathscr{k} \colon \mathfrak{e}^2 \to \mathscr{p} \downarrow \mathrm{id}_{\mathfrak{b}}$.

Definition 1.1.1.1.9 (Cartesian fibration)**.** The isofibration $\mathscr{p}$ is a **cartesian fibration** if $\mathscr{k}$ is a left-adjoint left-(quasi)-inverse, that it has a right adjoint and the counit is invertible.

It is a **cocartesian fibration** if it defines a cartesian fibration in the co-dual ∞-cosmos, that is if $\mathfrak{e}^2 \to \mathrm{id}_{\mathfrak{b}} \downarrow \mathscr{p}$ is a right-adjoint right-inverse.

By [RV21, Proposition 6.3.14], for any $K \in \mathfrak{K}$, there are sub-∞-cosmoi $\mathfrak{Cart}(\mathfrak{K})_{/K}$ and $\mathfrak{Cocart}(\mathfrak{K})_{/K}$ of the sliced ∞-cosmos $\mathfrak{K}_{/K}$, whose objects are respectively the cartesian and cocartesian fibrations over K, and whose 1-arrows are the cartesian maps.

Example 1.1.1.1.10 ([RV21, Corollary 7.4.6]). For any cospan $B \xrightarrow{f} A \xleftarrow{g} C$, the comma projection $(\mathfrak{b}, \mathfrak{c}) \colon f \downarrow g \to B \times C$ is a two-sided fibration.

Lemma 1.1.1.1.11 ([RV21, Theorem 7.1.4, Proposition 7.1.7])**.** *Let $K \xleftarrow{p} E \xrightarrow{q} L$ be a two-sided isofibration. The triangle below-left*

$$\begin{array}{ccc} E & \xrightarrow{(p,q)} & K \times L \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & K & \end{array} \qquad\qquad \begin{array}{ccc} E & \xrightarrow{(p,q)} & K \times L \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle pr_2} & \\ & L & \end{array} \tag{1.3}$$

defines a cartesian fibration in $\mathfrak{Cocart}(\mathfrak{K})_{/K}$ if and only if the triangle above-right defines a cocartesian fibration in $\mathfrak{Cart}(\mathfrak{K})_{/L}$.

More generally, there is an equivalence of ∞-cosmoi $\mathfrak{Cart}(\mathfrak{Cocart}(\mathfrak{K})_{/L})_{/K \times L} \mathfrak{Cocart}(\mathfrak{Cart}(\mathfrak{K})_{/K})_{/K \times L}$.

We denote these two equivalent ∞-cosmoi as ${}_{K\backslash}\mathfrak{Fib}(\mathfrak{K})_{/L}$. The objects of this ∞-cosmos are called **two-sided fibrations** from K to L.

1.1.1.2 The equipment of bimodules

Definition 1.1.1.2.1 (Profunctors)**.** A **profunctor** from K to L is a discrete object in ${}_{K\backslash}\mathfrak{Fib}(\mathfrak{K})_{/L}$.

Proposition 1.1.1.2.2 ([RV21, Lemma 7.4.2])**.** *A two-sided isofibration defines a profunctor if and only if it is*

1. *cocartesian on the left,*
2. *cartesian on the right,*

3. *discrete in* $\mathfrak{K}_{/K\times L}$.

There is a double category of two-sided fibrations, which furthermore has the pleasant property of equipping its vertical 2-category with proarrows. However, the composite of a sequence of profunctors may fail to be a profunctor, so the structure does not restrict to a sub-double category of profunctors. The saving grace is that, though the composite of profunctors does not exist as a profunctor, cells from it can still be formally defined. This means that profunctors are the horizontal arrows in a weaker structure, called a **virtual double category** or fc-multicategory (where fc is the "free category" monad), which has (vertical) composites of vertical arrows, no composites of horizontal arrows, and general cells of the form displayed below-left

$$\begin{array}{ccccccc} A_0 & \xrightarrow{\;\;+\;\;} & A_1 & \xrightarrow{\;\;+\;\;} & \cdots & \xrightarrow{\;\;+\;\;} & A_n \\ \downarrow & & & \Downarrow & & & \downarrow \\ B & & & \xrightarrow{\qquad\quad +\quad\qquad} & & & C \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \| & \Downarrow \mathrm{id}_f & \| \\ A & \xrightarrow[f]{} & B \end{array} \tag{1.4}$$

(including the case $n = 0$) and identities as above-right, along with appropriate associative and unital pasting operations.

Definition 1.1.1.2.3 (Virtual proarrow equipment)**.** Let $\mathfrak{K}$ be a virtual double category.

- A unary cell

$$\begin{array}{ccc} X_0 & \xrightarrow{e} & X_1 \\ p\downarrow & \Downarrow \xi & \downarrow q \\ A & \xrightarrow[f]{} & B \end{array} \tag{1.5}$$

  is **cartesian** if for any cell

$$\begin{array}{ccccc} Y_0 & \xrightarrow{g_1} & \cdots & \xrightarrow{g_n} & Y_n \\ pr\downarrow & & \Downarrow \psi & & \downarrow qs \\ A & & \xrightarrow[f]{} & & B \end{array} \tag{1.6}$$

  with same target there exists a unique cell χ giving a factorisation

$$\begin{array}{ccccc} Y_0 & \xrightarrow{g_1} & \cdots & \xrightarrow{g_n} & Y_n \\ r\downarrow & & \Downarrow \exists!\chi & & \downarrow s \\ X_0 & & \xrightarrow[e]{} & & X_1 \\ p\downarrow & & \Downarrow \xi & & \downarrow q \\ A & & \xrightarrow[f]{} & & B. \end{array} \tag{1.7}$$

- A cell

$$\begin{array}{ccccc} X_0 & \xrightarrow{e_1} & \cdots & \xrightarrow{e_n} & X_n \\ \mathrm{id}_{X_0}\| & & \Downarrow \xi & & \|\mathrm{id}_{X_n} \\ X_0 & & \xrightarrow[f]{} & & X_n \end{array} \tag{1.8}$$

is **opcartesian** if for any cell

$$\begin{array}{ccccccccccccccccc} Y_0 & \xrightarrow{g_1} & \cdots & \xrightarrow{g_{m-1}} & Y_{m-1} & \xrightarrow{g_m} & X_0 & \xrightarrow{e_1} & \cdots & \xrightarrow{e_n} & X_n & \xrightarrow{h_1} & Z_1 & \xrightarrow{h_2} & \cdots & \xrightarrow{h_p} & Z_p \\ \downarrow & & & & & & & & & & & & & & & & \downarrow \\ C & & & & & & & & \xrightarrow[g]{\qquad\qquad} & & & & & & & & D \end{array} \tag{1.9}$$

containing its domain there exists a unique cell χ giving a factorisation

$$\begin{array}{ccccccccccccccccc} Y_0 & \xrightarrow{g_1} & \cdots & \xrightarrow{g_{m-1}} & Y_{m-1} & \xrightarrow{g_m} & X_0 & \xrightarrow{e_1} & \cdots & \xrightarrow{e_n} & X_n & \xrightarrow{h_1} & Z_1 & \xrightarrow{h_2} & \cdots & \xrightarrow{h_p} & Z_p \\ \| & & & & \| & & \| & & \Downarrow \xi & & \| & & \| & & & & \| \\ Y_0 & \xrightarrow{g_1} & \cdots & \xrightarrow{g_{m-1}} & Y_{m-1} & \xrightarrow{g_m} & X_0 & & \xrightarrow{f} & & X_n & \xrightarrow{h_1} & Z_1 & \xrightarrow{h_2} & \cdots & \xrightarrow{h_p} & Z_p \\ \downarrow & & & & & & & & \Downarrow \exists ! \chi & & & & & & & & \downarrow \\ C & & & & & & & & \xrightarrow[g]{\qquad\qquad} & & & & & & & & D. \end{array} \tag{1.10}$$

Notation 1.1.1.2.4 (Virtual proarrow equipment)**.** Let $\mathfrak{K}$ be a virtual double category. If an object K has a horizontal endomorphism which is the target of a nullary opcartesian cell, we say that K **admits a unit**. We will usually write this horizontal unit also as $K \colon K \nrightarrow K$.

If a solid diagram

$$\begin{array}{ccc} X_0 & \dashrightarrow & X_1 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow[f]{} & B \end{array} \tag{1.11}$$

admits a completion to a cartesian cell, we write the dashed horizontal arrow $f(q, p)$ and call it a **restriction** of f along (q, p).

We say that $\mathfrak{K}$ is a **virtual proarrow equipment**, or that it (virtually) equips its vertical 2-category with proarrows, if it admits all restrictions and all units.

In a virtual proarrow equipment, we will often refer to the horizontal arrows as proarrows.

Theorem 1.1.1.2.5 ([RV21, Theorem 8.2.6])**.** *The homotopy 2-category of an ∞-cosmos is equipped with proarrows by its virtual double category of profunctors.*

We record some formal manipulations with proarrows.

Example 1.1.1.2.6 (Companions and conjoints). Let $f \colon A \to B$ be a vertical arrow in a virtual proarrow equipment. Its **companion** is the proarrow $f_* = B(\mathrm{id}_B, f)$, the restriction of $B(\mathrm{id}_B, \mathrm{id}_B)$ along id_B and f. Its **conjoint** is $f^* = B(f, \mathrm{id}_B)$.

By definition, the companion and conjoint of the vertical arrow f come equipped with canonical cartesian cells

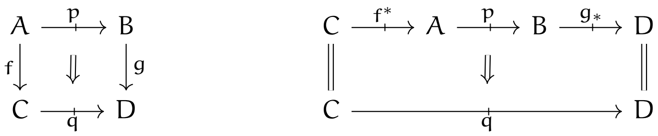
$$\left(\begin{array}{ccc} A & \xrightarrow{f_*} & B \\ {\scriptstyle f}\downarrow & \Downarrow & \| \\ B & \xrightarrow[B]{} & B \end{array} , \quad \begin{array}{ccc} A & \xrightarrow{A} & A \\ \| & \Downarrow & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f_*]{} & B \end{array} \right) \quad \text{and} \quad \left(\begin{array}{ccc} A & \xrightarrow{A} & A \\ {\scriptstyle f}\downarrow & \Downarrow \eta & \| \\ B & \xrightarrow[f^*]{} & A \end{array} , \quad \begin{array}{ccc} B & \xrightarrow{f^*} & A \\ \| & \Downarrow \varepsilon & \downarrow{\scriptstyle f} \\ B & \xrightarrow[B]{} & B \end{array} \right) \tag{1.12}$$

satisfying adjunction-like relations.

Definition 1.1.1.2.7 (Composite proarrows)**.** When there exists an opcartesian cell as in equation (1.8), we say that ξ exhibits the horizontal arrow f as a horizontal **composite** of the sequence $X_0 \nrightarrow \cdots \nrightarrow X_n$, and write f as $e_1 \circledcirc \cdots \circledcirc e_n$.

Remark 1.1.1.2.8. There is an inclusion 2-functor from double categories into virtual double categories. The virtual double categories in its essential image can be characterised as those which admit composites of all paths of horizontal morphisms.

Lemma 1.1.1.2.9 ([CS10, Theorem 7.20])**.** *In a virtual proarrow equipment, consider vertical arrows* $f\colon A \to C$, $g\colon B \to D$ *and horizontal arrows* $p\colon A \nrightarrow B$, $q\colon C \nrightarrow D$, *fitting in the border of the square below-left.*



(1.13)

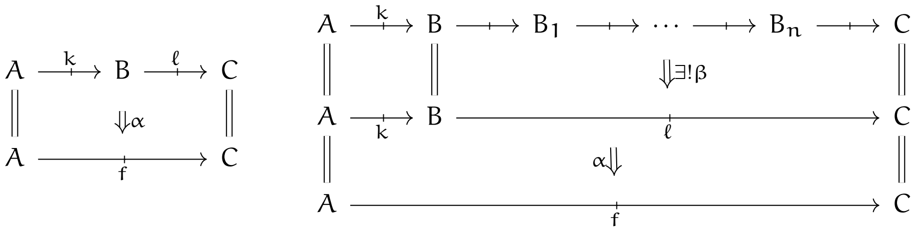
There is a one-to-one correspondence between cells as above-left and cells as above-right.

Corollary 1.1.1.2.10 (Yoneda reduction, [CS10, Theorem 7.16])**.** *In a virtual proarrow equipment, let* $p\colon B \nrightarrow C$ *be a horizontal arrow, and let* $f\colon A \to B$ *and* $g\colon D \to C$ *be vertical arrows. The composite* $f_* \circledcirc p \circledcirc g^*$ *exists and is isomorphic to* $p(g, f)$.

1.1.2 Formal category theory in virtual equipments

1.1.2.1 Weighted (co)limits

Definition 1.1.2.1.1 (Lax extensions and lifts)**.** Let $\mathbb{D}$ be a virtual double category and $k\colon A \nrightarrow B$ and $f\colon A \nrightarrow C$ be horizontal arrows. A **lax extension** of f along k is a horizontal arrow $\ell\colon B \nrightarrow C$ with a cell as below-left



(1.14)

universal among such, *i.e.* such that for any cell α' with boundary as above-right there is a unique cell β above making the diagram commute (*i.e.* such that α' is equal to the composite cell).

Lax lifts are defined as lax extensions in the (horizontally) dual virtual double category.

When they exist, we denote $k \triangleright f$ the lax extension of f along k and $f \triangleleft k$ the lax lift of f along k so as to emphasise their algebraic properties, although later we will write such lax extensions as $k \triangleright f =: \mathrm{Lex}_k f$.

Lemma 1.1.2.1.2 (Formal Yoneda lemma)**.** *Suppose* $\mathbb{D}$ *is a virtual proarrow equipment. Then* $g_* \triangleright H \triangleleft f^*$ *exists and is isomorphic to* $H(g, f)$.

Proposition 1.1.2.1.3 ([RV21, Proposition 9.1.6])**.** *Let* $K\colon A \nrightarrow B$, $H\colon B \nrightarrow C$ *and* $F\colon A \nrightarrow D$ *be proarrows such that the composite* $K \circ H$ *and the lax extension* $K \triangleright F$ *exist. Then* $H \triangleright (K \triangleright F)$ *exists if and only if* $(H \circ K) \triangleright F$, *in which case they are isomorphic.*

Definition 1.1.2.1.4 (Weighted (co)limits)**.** Let $\mathcal{d}\colon \mathbb{I} \to \mathfrak{C}$ be a vertical arrow, and let B be an object. A **limit** of $\mathcal{d}$ weighted by a proarrow $\mathscr{W}\colon \mathbb{I} \nrightarrow \mathfrak{B}$ is a vertical arrow $\{\mathscr{W},\mathcal{d}\}\colon \mathfrak{B} \to \mathfrak{C}$ equipped with a cell as below-left

$$\begin{array}{ccccc} \mathbb{I} & \xrightarrow{\mathscr{W}} & \mathfrak{B} & \xrightarrow{\{\mathscr{W},\mathcal{d}\}_*} & \mathfrak{C} \\ \| & & \Downarrow & & \| \\ \mathbb{I} & & \xrightarrow[\mathcal{d}_*]{} & & \mathfrak{C} \end{array} \qquad \begin{array}{ccccc} \mathfrak{C} & \xrightarrow{(\mathscr{P}\star\mathcal{d})^*} & \mathfrak{B} & \xrightarrow{\mathscr{P}} & \mathbb{I} \\ \| & & \Downarrow & & \| \\ \mathfrak{C} & & \xrightarrow[\mathcal{d}^*]{} & & \mathbb{I} \end{array} \tag{1.15}$$

exhibiting $\{\mathscr{W},\mathcal{d}\}_* = \mathfrak{C}(\mathrm{id}_{\mathfrak{C}},\{\mathscr{W},\mathcal{d}\})$ as a lax extension of $\mathcal{d}_*$ along $\mathscr{W}$.

A **colimit** of $\mathcal{d}$ weighted by $\mathscr{P}\colon \mathfrak{B} \nrightarrow \mathbb{I}$ is a vertical arrow $\mathscr{P}\star\mathcal{d}\colon \mathfrak{B} \to \mathfrak{C}$ equipped with a cell as above-right (in equation (1.15)) exhibiting $(\mathscr{P}\star\mathcal{d})^* = \mathfrak{C}(\mathscr{P}\star\mathcal{d},\mathrm{id}_{\mathfrak{C}})$ as a lax lift of $\mathcal{d}^*$ along $\mathscr{P}$.

Example 1.1.2.1.5 (Representable weights). Let $\mathcal{j}\colon \mathfrak{B} \to \mathbb{I}$ and $\mathcal{k}\colon \mathbb{I} \to \mathfrak{B}$ be vertical arrows. By comparing the universal properties and using corollary 1.1.1.2.10, one verifies that:

- If $\mathscr{W} = \mathcal{j}^*$ is the conjoint of $\mathcal{j}$, then $\{\mathcal{j}^*,\mathcal{d}\} = \mathcal{d}\circ\mathcal{j}$.
- If $\mathscr{W} = \mathcal{k}_*$ is the companion of $\mathcal{k}$, then $\{\mathcal{k}_*,\mathcal{d}\}$ defines a lax extension of $\mathcal{d}$ along $\mathcal{k}$ (in the vertical 2-category), which is pointwise when that makes sense (*e.g.* when working in the virtual equipment of profunctors in an ∞-cosmos, by [RV21, Theorem 9.3.3]).
- If $\mathscr{P} = \mathcal{j}_*$ is the companion of $\mathcal{j}$, then $\mathcal{j}_*\star\mathcal{d} = \mathcal{d}\circ\mathcal{j}$.
- When $\mathscr{P} = \mathcal{k}^*$ is the conjoint of $\mathcal{k}$, then $\mathcal{k}^*\star\mathcal{d}$ defines a pointwise oplax extension of $\mathcal{d}$ along $\mathcal{k}$.

Lemma 1.1.2.1.6 ([RV21, Proposition 9.5.4])**.** *Let* $f\colon A \to E$ *and* $g\colon C \to E$ *be vertical arrows and let* $W\colon A \nrightarrow B$ *and* $V\colon B \nrightarrow C$ *be proarrows. Then* $\{W \circledcirc V, f\} \simeq \{V,\{W,f\}\}$ *and similarly* $W\circledcirc V\star g \simeq W\star V\star g$ *whenever the composite weights and the (co)limits exist.*

Proof. By the proposition 1.1.2.1.3. □

Corollary 1.1.2.1.7. *For any generalised element* $b\colon X \to B$,

$$\{\mathscr{W},\mathcal{d}\}\circ b = \{\mathscr{W}(b,\mathrm{id}_{\mathbb{I}}),\mathcal{d}\}. \tag{1.16}$$

In particular, if the virtual double category has a (vertically) terminal object $*$, let us call **pointlike** the limits (resp. colimits) weighted by a proarrow $\mathbb{I} \to *$ (resp. $* \to \mathbb{I}$). Then the above corollary 1.1.2.1.7 suggests that general weighted limits can be computed pointwise and understood as families of pointlike weighted limits whose weights vary functorially on the elements of $\mathfrak{B}$.

1.1.2.2 Yoneda arguments

Definition 1.1.2.2.1 (Yoneda embedding)**.** A **Yoneda embedding** for an object C in a virtual equipment is a vertical arrow $\text{よ}^C\colon C \to \widehat{C}$

- which is dense, in that the leftmost cell in equation (1.12) exhibits $\mathrm{id}_{\widehat{C}}$ as the pointwise oplax extension of よ^C along $(\text{よ}^C)_*$ in the sense of [Roa19, Definition 4.6]; and

- such that for any horizontal $f \colon B \nrightarrow C$, there exists a cartesian cell

$$\begin{array}{ccc} B & \overset{P}{\nrightarrow} & C \\ {\scriptstyle P^{\dagger}}\downarrow & & \downarrow{\scriptstyle よ^{C}} \\ \widehat{C} & = & \widehat{C}. \end{array} \tag{1.17}$$

Remark 1.1.2.2.2. Our definition is somewhat horizontally dual to the one used by [Roa19], wherein instead of $P^{\dagger}$ it is required the existence of a $P^{\lambda} \colon C \to \widehat{B}$.

Remark 1.1.2.2.3. Yoneda embeddings can be defined in the more general context of augmented virtual double categories, which can have objects without units, but cells with nullary target instead: this is to accommodate the example of non-necessarily small categories, whose unit profunctors would have to be large.

In this case, the notion of Yoneda embedding can be refined to that of *good* Yoneda embeddings, in which the vertical arrows $よ^{C}$ must be required to have companions. It is shown in [Roa19, Proposition 5.6] that a collection of good Yoneda embeddings for each unital object of an augmented virtual double category is equivalent data to that of a good Yoneda structure (in the sense of Street-Walters and Weber) on its vertical 2-category.

Definition 1.1.2.2.4 (Accessible ∞-cosmos, [BL21] (see also [BLV20, §9.4]))**.** An ∞-cosmos $\mathfrak{K}$ is **accessible** if

- it is accessible as an $\mathfrak{sSet}$-enriched category, meaning that its underlying category is accessible and that the functors given by cotensoring with any simplicial set are all accessible,
- the embedding $\mathfrak{sSet}$-functors of the full subcategories of $\mathfrak{K}^{2}$ spanned by the isofibrations and by the equivalences are accessible.

We say furthermore that $\mathfrak{K}$ is **presentable** if it is accessible and admits all small colimits.

Conjecture 1.1.2.2.5. The homotopy 2-category of any presentable ∞-cosmos extends to a virtual proarrow equipment with (good) Yoneda embeddings.

Remark 1.1.2.2.6. One may expect that the virtual equipment defined in subsection 1.1.1.2 should be enough to obtain a good Yoneda structure, making the presentability assumption in the above coonjecture unnecessary. However, the profunctors as defined here do not coincide with the notion of profunctors that are normally exppected, except in the case of an ∞-cosmos of $(\infty, 1)$-categories. Indeed, consider for example an ∞-cosmos of (∞, n)-category. The usual definition of profunctors means that they should be interpreted as presheaves valued in the (∞, n)-category of $(\infty, n-1)$-categories, but the definition given here makes them given by presheaves valued in (∞, n)-categories whose underlying $(\infty, 1)$-category is groupoidal.

As observed in [Str80] and explained in [LR20, §4.3], the correct notion of profunctors is actually given by codiscrete cofibrations rather than discrete fibrations; these are composed using colimits rather than limits and hence necessitate the additional assumption of accessibility.

Construction 1.1.2.2.7. Let $C \colon * \to \mathfrak{C}$ be an element of $\mathfrak{C}$, giving the conjoint proarrow $C^{*} \colon \mathfrak{C} \nrightarrow *$. The Yoneda structure induces $C^{*,\dagger} \colon \mathfrak{C} \to \widehat{*}$, which we view as the arrow corepresented by C, and denote $\mathrm{Map}(C, \mathrm{id}_{C})$.

Proposition 1.1.2.2.8. *In a presentable ∞-cosmos, the Yoneda functors preserves (and detects) limits.*

Proof. We need to check that for any object C, seen equivalently as $C\colon * \to \mathfrak{C}$, the arrow $\mathrm{Map}(C, \mathrm{id}_{\mathfrak{C}})$ commutes with limits. Given such a limit $\mathscr{L}\colon \mathfrak{A} \to \mathfrak{C}$ of $\mathscr{D}\colon \mathbb{I} \to \mathfrak{C}$ weighted by the proarrow $\mathscr{W}\colon \mathbb{I} \nrightarrow \mathfrak{A}$, we must thus check that there is a cell α filling the diagram

$$\begin{array}{ccccc} \mathbb{I} & \overset{\mathscr{W}}{\longrightarrow} & \mathfrak{A} & \overset{\mathrm{Map}(C,\mathscr{L})_*}{\longrightarrow} & \widehat{*} \\ \Vert & & \Downarrow \alpha & & \Vert \\ \mathbb{I} & & \underset{\mathrm{Map}(C,\mathscr{D})_*}{\longrightarrow} & & \widehat{*} \end{array} \tag{1.18}$$

to a lax extension diagram. But the functor $\mathrm{Map}(C, \mathscr{D})$ is nothing but the functor $\text{よ}^{-1}(\mathscr{D}_* \otimes C^*)\colon \mathbb{I} \simeq \mathbb{I} \times *^{\mathrm{op}} \to \mathcal{V}$ corresponding to the profunctor $\mathscr{D}_* \otimes C^*\colon \mathbb{I} \nrightarrow *$ by the Yoneda structure よ induced by presentability. As C^* and $\mathscr{D}_*$ are composable ([RV21, Proposition 8.4.7]), we can obtain the desired cell from that expressing $\mathscr{L}$ as a limit and from the composition cell:

$$\begin{array}{ccccc} \mathbb{I} & \overset{\mathscr{W}}{\longrightarrow} & \mathfrak{A} & \overset{\text{よ}^{-1}(\mathscr{L}_* C^*)_*}{\longrightarrow} & \widehat{*} \\ \Vert & & \Downarrow\alpha & & \Vert \\ \mathbb{I} & & \underset{\text{よ}^{-1}(\mathscr{D}_* C^*)_*}{\longrightarrow} & & \widehat{*} \end{array} = \begin{array}{ccccccc} \mathbb{I} & \overset{\mathscr{W}}{\longrightarrow} & \mathfrak{A} & \overset{\mathscr{L}_*}{\longrightarrow} & \mathfrak{C} & \overset{\text{よ}^{-1}(C^*)_*}{\longrightarrow} & \mathcal{V} \\ \Vert & & \Downarrow & & \Vert & \Downarrow \mathrm{id} & \Vert \\ \mathbb{I} & & \underset{\mathscr{D}_*}{\longrightarrow} & & \mathfrak{C} & \underset{\text{よ}^{-1}(C^*)_*}{\longrightarrow} & \widehat{*} \\ \Vert & & & \Downarrow \mathrm{compos} & & & \Vert \\ \mathbb{I} & & & \underset{\text{よ}^{-1}(\mathscr{D}_* C^*)_*}{\longrightarrow} & & & \widehat{*} \end{array}. \tag{1.19}$$

Its universal property follows from those of the aforementioned universal cells.

The proof straightforwardly dualises to show that $\mathrm{Map}_{\mathfrak{C}}(\mathrm{id}_{\mathfrak{C}}, C)$ sends colimits (*i.e.* limits in the opposite $\mathcal{V}$-category $\mathfrak{C}^{\mathrm{op}}$) to limits in $\mathcal{V}$. □

Example 1.1.2.2.9 (Enriched hom functors). Let $\mathcal{V}^{\otimes}$ be a presentable monoidal $(\infty, 1)$-category. By presentability, [Lur09, Proposition A.3.7.6] and [Lur17, Remark 4.1.8.9] imply that it can be presented as the ∞-categorical localisation of a combinatorial simplicial monoidal model category $\mathcal{V}^{\mathrm{mod}}$ (note that, as per [Lur17, Remark 4.5.4.9], if the monoidal structure of $\mathcal{V}^{\otimes}$ extends to a *symmetric* monoidal one, it is not a given that $\mathcal{V}^{\otimes}$ can be written as the localisation of a symmetric monoidal model category). Then by the rectification results of [Hau15, Theorem 5.8] the $(\infty, 1)$-category of $\mathcal{V}^{\otimes}$-enriched ∞-categories is itself the localisation of the simplicial model category of $\mathcal{V}^{\mathrm{mod}}$-enriched categories[1]. This allows us to use [RV21, Proposition E.1.1] to conclude that there is indeed an ∞-cosmos of $\mathcal{V}^{\otimes}$-enriched ∞-categories. Alternately, we could use the virtual double ∞-category of $\mathcal{V}^{\otimes}$-enriched ∞-categories constructed in [Hau16] and recalled in subsection 1.1.3.2.

Note that when the $\mathcal{V}^{\otimes}$-enriched ∞-category structure comes from a closed monoidal structure, preservation of limits is more simply a consequence of the fact that the internal hom ∞-functor is a right-adjoint.

1.1.2.3 Homotopy-coherent adjunctions and monads

Construction 1.1.2.3.1 (The generic adjunction and (co)monad)**.** The **generic adjunction** $\mathfrak{Adj}$ is the 2-category determined by the universal property that it contains an adjunction, for any

[1] To use the results of [Hau15], $\mathcal{V}^{\mathrm{mod}}$ must be further chosen to be a left proper tractable biclosed monoidal model category satisfying the monoid axiom

2-category $\mathfrak{K}$, evaluation at this adjunction induces an equivalence between the 2-category $2\text{-}\mathfrak{Cat}(\mathfrak{Adj}, \mathfrak{K})$ and that of adjunctions in $\mathfrak{K}$.

This 2-category must hence contain an adjunction and be freely generated by it in an appropriate sense: it contains

- two objects, which shall be denoted $+$ and $-$ (the carriers of the adjunction),
- 1-morphisms generated by an arrow $\ell\colon + \to -$ and an arrow $r\colon - \to +$ (the putative left- and right-adjoints),
- 2-morphisms generated (under composition and whiskering) by a unit $\eta\colon \mathrm{id}_+ \Rightarrow r \circ \ell$ and counit $\varepsilon\colon \ell \circ r \Rightarrow \mathrm{id}_-$,
- 3-morphisms $\mathrm{id}_\ell \xrightarrow{\simeq} (\varepsilon\ell) \circ (\ell\eta)$ and $(r\varepsilon) \circ (\eta r) \xrightarrow{\simeq} \mathrm{id}_r$, necessarily identities since $\mathfrak{Adj}$ is 2-truncated, imposing the triangular identities.

We will give a completely explicit description of this 2-category when we focus on monads, as $\mathfrak{Adj}$ contains a certain amount of redundancy that will be better organised once the generic monad is introduced.

Recall that, through local application of the fully faithful nerve functor $\mathfrak{Cat} \to \mathfrak{sSet}$, every 2-category can be seen as an $\mathfrak{sSet}$-enriched category.

Definition 1.1.2.3.2 (Homotopy coherent adjunction)**.** A **homotopy coherent adjunction** in an ∞-cosmos $\mathfrak{K}$ is an $\mathfrak{sSet}$-enriched functor $\mathfrak{Adj} \to \mathfrak{K}$.

Theorem 1.1.2.3.3 ([RV16, Theorem 4.3.11])**.** *Let $\mathfrak{K}$ be an ∞-cosmos. Every adjunction A in the homotopy 2-category* $\mathrm{Ho}\,\mathfrak{K}$ *extends to a homotopy-coherent adjunction in $\mathfrak{K}$,* i.e. *there is a homotopy coherent adjunction* $\widetilde{\mathsf{A}}\colon \mathfrak{Adj} \to \mathfrak{K}$ *such that* $\mathrm{Ho}\,\widetilde{\mathsf{A}} \simeq \mathsf{A}$.

Every andjunction in a 2-category induces a monad on one of its carriers and a comonad on the other. As such, the generic adjunction contains a generic monad and a generic comonad as well.

Remark 1.1.2.3.4. Let $\mathfrak{Mnd}$ be the full sub-2-category of $\mathfrak{Adj}$ on the object $+$; as a 2-category with a single object it is the classifying 2-category $\mathcal{B}\,\Delta_a^{\boxplus}$ of a monoidal category $\Delta_a^{\boxplus} := \mathfrak{Mnd}(+,+) = \mathfrak{Adj}(+,+)$. According to construction 1.1.2.3.1, this monoidal category should have objects freely generated (under monoidal product) by the endomorphism $r \circ \ell$ of $+$, so that we may denote $[n] = (r \circ \ell)^{n+1}$ these objects and $[n] \boxplus [m] = [n+m-1]$ their product, and morphisms generated by the appropriate whiskerings of the unit and counit, subject to the identifications coming from the triangular identity.

From this description, one sees that Δ_a identifies with the **augmented simplex category**, the full subcategory of $\iota_1\mathfrak{Cat}$ spanned by those categories $[n] = \mathbb{n} + \mathbb{1}$ (for $n \geqslant -1$) that are freely generated by a linear graph, or equivalently (a skeleton of) the category of finite (but possibly empty) totally ordered sets.

Definition 1.1.2.3.5 (Homotopy coherent monad)**.** A **homotopy coherent monad** in an ∞-cosmos $\mathfrak{K}$ is an $\mathfrak{sSet}$-enriched functor $\mathfrak{Mnd} \to \mathfrak{K}$.

The inclusion $\mathfrak{Mnd} \hookrightarrow \mathfrak{Adj}$ induces a restriction $\mathfrak{sSet}$-functor $\mathfrak{K}^{\mathfrak{Adj}} \to \mathfrak{K}^{\mathfrak{Mnd}}$, associating with every homotopy coherent adjunction its underlying homotopy coherent monad.

Definition 1.1.2.3.6. Let $\mathfrak{K}$ be an ∞-cosmos. The **Eilenberg–Moore** adjunction of a homotopy coherent monad $\mathsf{T}\colon \mathfrak{Mnd} \to \mathfrak{K}$ is its lax extension along $\mathfrak{Mnd} \to \mathfrak{Adj}$.

From example 1.1.2.1.5, the Eilenberg–Moore adjunction of T is its limit weighted by the companion of $\mathfrak{Mnd} \to \mathfrak{Adj}$.

1.1.3 Enriched ∞-categories

1.1.3.1 Categorical algebras in a monoidal $(\infty, 1)$-category

For any set S, a definition check shows that, writing $*_S$ for the codiscrete category cogenerated by S (with set of objects S, and a unique arrow between any pair of elements of S), a lax 2-functor $*_S \to \mathcal{B}\mathcal{V}$ (for $\mathcal{V}$ a monoidal category) is the same thing as a $\mathcal{V}$-enriched category with set of objects S (more generally lax 2-functors $*_S \to \mathfrak{K}$ give categories enriched in a 2-category $\mathfrak{K}$). However, for any S the category $*_S$ is equivalent to the terminal category $*$, while not every enriched category is equivalent to one with a single object, showing that lax 2-functors are not invariant under biequivalence of 2-categories and that this definition of enriched categories does not recover the right notion of morphisms between them.

The proper context for performing the procedure described above is to consider the 2-category $*_S$ instead as a vertically discrete double category; as for every $s \in S$ there is now a vertical arrow id_s while there is no vertical arrow $s \to s'$ whenever $s \neq s'$ this double category is no longer equivalent to a $*_T$ for any other set $T \neq S$. Then lax double functors $*_S \to \mathfrak{K}$ give a notion of category enriched in a double category $\mathfrak{K}$. Although we have not yet developed the general formalism of lax morphisms, for double categories (as we will justify in subsection 2.1.1.3) there is a reasonable notion of lax double functors: the functors of underlying virtual double categories.

We start by recalling the relevant algebraic structures.

Definition 1.1.3.1.1 (Virtual double ∞-categories)**.** The category of operators of a **virtual double ∞-category** is an $(\infty, 1)$-functor $\mathcal{O}^\otimes \xrightarrow{p} \Delta^{op}$ such that

1. for every object $O \in \mathcal{O}^\otimes$, every inert arrow $\varphi \colon [m] = pO \to [n]$ in Δ^{op} admits a p-cocartesian lift $\varphi_! \colon O \to \varphi_! O$;
2. for every $[n] \in \Delta^{op}$, the functor $\mathcal{O}^\otimes_{[n]} \to \varprojlim_{[n]\to[i]\in\Delta^{op\,el}_{[n]/}} \mathcal{O}^\otimes_{[i]} \simeq \mathcal{O}^\otimes_{[1]} \times_{\mathcal{O}^\otimes_{[0]}} \cdots \times_{\mathcal{O}^\otimes_{[0]}} \mathcal{O}^\otimes_{[1]}$ induced by the cocartesian lifts of inert maps is an equivalence;
3. for every $O \in \mathcal{O}^\otimes$ and every choice of p-cocartesian lifts $O \to O_i$ of the inert morphisms $[n] \to [0]$ mapping 0 to $i \in [n]$ and every choice of lifts $Y \to \rho_{i,!}O$ of the inert morphisms $\rho_i \colon [n] \to [1]$ including $\{i-1, i\}$, for every $P \in \mathcal{O}^\otimes$ the square
$$\begin{array}{ccc} \mathcal{O}^\otimes(P,O) & \longrightarrow & \mathcal{O}^\otimes(P,\rho_{1,!}O) \underset{\mathcal{O}^\otimes(P,O_1)}{\times} \cdots \underset{\mathcal{O}^\otimes(P,O_{n-1})}{\times} \mathcal{O}^\otimes(P,\rho_{n,!}O) \\ \downarrow & & \downarrow \\ \Delta^{op}([m],[n]) & \longrightarrow & \Delta^{op}([m],[1]) \underset{\Delta^{op}([m],[0])}{\times} \cdots \underset{\Delta^{op}([m],[0])}{\times} \Delta^{op}([m],[1]) \end{array} \tag{1.20}$$
is cartesian.

A **double ∞-category** is a "representable" virtual double ∞-category, that is such that the $(\infty, 1)$-functor $\to \Delta$ is a cartesian fibration. A **non-symmetric (or planar) ∞-operad** is a virtual double ∞-category whose $(\infty, 1)$-category of objects is trivial. A **monoidal ∞-category** is a "representable" non-symmetric ∞-operad (or a double ∞-category with trivial space of objects), that is a virtual double ∞-category that is simultaneously a double ∞-category and a planar ∞-operad.

Remark 1.1.3.1.2. We will see later that this is a particular example of definition 2.1.1.3.1.

Example 1.1.3.1.3. From the description of virtual double categories in [GH15], one sees that every virtual double ∞-category $\mathfrak{O}$ has a homotopy virtual double category, which is a double category when $\mathfrak{O}$ is a double ∞-category.

In addition, [GHK21, Proposition A.4.4] shows that it is sensible to define a (virtual) double ∞-category to be an equipment when its homotopy (virtual) double category is one, as in [GHK21, Definition A.4.3]

Using the theory of categorical patterns introduced in [Lur17, Appendix B], [GH15, Theorem 3.2.5, Definition 3.2.9] obtain an $(\infty,1)$-category, and in fact ([*loc.cit.*, Remark 3.2.12]) an $(\infty,2)$-category, of virtual double ∞-categories.

Example 1.1.3.1.4. Let S be an ∞-groupoid; [GH15] construct a double ∞-category $*_S$ generalising to the ∞-categorical setting the construction we described above. There is an ∞-functor $\mathfrak{VirDbl}_\infty \to (\infty,1)-\mathfrak{Cat}$ mapping a virtual double ∞-category to its vertical $(\infty,1)$-category. By [GH15, Remark 4.1.5] it admits a right-adjoint which we denote $*_{(-)}$; it is given by viewing an $(\infty,1)$-category as an ∞-functor $1 \to (\infty,1)-\mathfrak{Cat}$ and taking a lax extension along $\{1\} \hookrightarrow \Delta^{op}$. More concretely, for a given $(\infty,1)$-category $\mathfrak{C}$, $*_{\mathfrak{C},n}$

Lemma 1.1.3.1.5 (The algebras fibration)**.** *For any virtual double ∞-category $\mathfrak{O}$, there is a cartesian fibration $\mathfrak{Alg}(\mathfrak{O}) \to \mathfrak{VirDbl}_\infty$ whose fibre at $\mathfrak{P}$ is the ∞-category of $\mathfrak{P}$-algebras in $\mathfrak{O}$.*

Definition 1.1.3.1.6 (Categorical algebras)**.** The ∞-category of categorical algebras in a virtual double ∞-category $\mathfrak{O}$ is defined as the fibre product

$$\begin{array}{ccc} \mathfrak{Alg}_{cat}(\mathfrak{O}) & \longrightarrow & \mathfrak{Alg}(\mathfrak{O}) \\ \downarrow & \lrcorner & \downarrow \\ \infty-\mathfrak{Grpd} & \xrightarrow{*_{(-)}} & \mathfrak{VirDbl}_\infty \end{array} . \tag{1.21}$$

This definition will be generalised in subsection 2.1.1.2 to discuss enriched version of more general algebraic structures.

1.1.3.2 A double ∞-category of enriched ∞-categories

Construction 1.1.3.2.1. In [Hau16] is constructed, for $\mathfrak{V}^\otimes$, a double ∞-category whose objects are $\mathfrak{V}^\otimes$-enriched ∞-categories, vertical morphisms the $\mathfrak{V}^\otimes$-functors, horizontal morphisms the $\mathfrak{V}^\otimes$-profunctors, but the sequences of horizontal morphisms only those whose composite exists.

Since we can view $(\infty,2)$-categories as ∞-categories enriched in $(\infty,1)-\mathfrak{Cat}^\times$, this gives an alternative way from ∞-cosmoi of treating $(\infty,2)$-categories with the methods of formal category theory.

A key ingredient in the construction is the following class of maps.

Definition 1.1.3.2.2 (Cellular maps)**.** A map of totally ordered sets $f\colon S \to S'$ is **cellular** if for all $s \in S$, $f(\operatorname{succ} s) \leqslant \operatorname{succ}(f(s))$.

1.2 Coherence for lax morphisms of $(\infty,2)$-algebras

1.2.1 Lax codescent objects from lax 2-monads

1.2.1.1 Soft 2-adjunctions

Example 1.2.1.1.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two 2-categories. A soft 2-adjunction, also called local adjunction, between $\mathfrak{A}$ and $\mathfrak{B}$ consists of a pair of 2-functors $\mathscr{F}\colon \mathfrak{A} \rightleftarrows \mathfrak{B}\colon \mathscr{U}$ and a family of adjunctions

$$\mathfrak{B}(\mathscr{F}A,B) \underset{\tau_{A,B}}{\overset{\sigma_{A,B}}{\rightleftarrows}} \mathfrak{A}(A,\mathscr{U}B) \text{ lax-natural in } A \in \mathfrak{A} \text{ and } B \in \mathfrak{B}.$$

More precisely, as explained in [MS89, §2.1], τ should be strict in its first variable and lax in its second while σ should be colax in its first variable (as $\mathfrak{B}(\mathscr{F}, \mathrm{id}_{\mathfrak{B}})$ is contravariant in its first variable) and strict in its second.

Lax natural transformations are not the 2-cells in any 3-category of 2-categories: given a sequence

$$\mathfrak{A} \underset{\mathscr{F}'}{\overset{\mathscr{F}}{\rightrightarrows}} \!\Downarrow\!\varphi\; \mathfrak{B} \underset{\mathscr{G}'}{\overset{\mathscr{G}}{\rightrightarrows}} \!\Downarrow\!\gamma\; \mathfrak{C} \tag{1.22}$$

of "horizontally composable" lax natural transformations, one may interpret φ as a generalised 1-arrow of $\mathfrak{B}$ (between the generalised objects $\mathscr{F}$ and $\mathscr{F}'$) so that reasoning by components produces a 3-arrow

$$\begin{array}{ccc} \mathscr{G}\mathscr{F} & \xRightarrow{\gamma\mathscr{F}} & \mathscr{G}'\mathscr{F} \\ {\scriptstyle \mathscr{G}\varphi}\Downarrow & \swarrow\!\!\!\swarrow \gamma_\varphi & \Downarrow{\scriptstyle \mathscr{G}'\varphi} \\ \mathscr{G}\mathscr{F}' & \xRightarrow[\gamma\mathscr{F}']{} & \mathscr{G}'\mathscr{F}' \end{array} \tag{1.23}$$

between the (vertical) composites of whiskered transformations.

In other words, the interchange law relating the two ways of horizontally composing γ with φ only holds laxly, up to a non-invertible 3-arrow. The relevant structure containing such 2-arrows, explicitly (in the language of [Str96]) a sesquicategory with lax interchange law, is a lax Gray-category, a category enriched in the monoidal category $2\text{-}\mathfrak{Cat}^{\otimes^\ell}$ which we now recall.

Construction 1.2.1.1.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two 2-categories. Recall that $\mathfrak{B}^{\mathfrak{A}}$ is the 2-category of 2-functors (with strict natural trasformations as 1-cells and modifications as 2-cells), which provides an internal hom for the cartesian product of 2-categories. We let $\mathfrak{Fun}_p(\mathfrak{A},\mathfrak{B})$ and $\mathfrak{Fun}_\ell(\mathfrak{A},\mathfrak{B})$ denote the 2-categories both of whose objects are, again, the (strict) 2-functors $\mathfrak{A} \to \mathfrak{B}$, whose 1-cells are respectively the pseudo-natural and the lax-natural transformations of 2-functors, and whose 2-cells are the obvious notion of modification (described in [Gra74, I.2.4. MQN.]).

All these "hom 2-categories" define closed category structures on $2\text{-}\mathfrak{Cat}$. A **closed category** structure on a category $\mathfrak{C}$ consists in a functor $[\cdot,\cdot]\colon \mathfrak{C}^{\mathrm{op}} \times \mathfrak{C} \to \mathfrak{C}$ and an object $I \in \mathfrak{C}$ together with general (more precisely: extranatural) transformations $I \dot{\Rightarrow} [X,X]$ and $[Y,Z] \dot{\Rightarrow} [[X,Y],[X,Z]]$ satisfying appropriate unitality and associativity constraints.

The main source of closed category structures come from the internal homs of monoidal closed category structures, and for example the closed structure $\mathfrak{B}^{\mathfrak{A}}$ on $2\text{-}\mathfrak{Cat}$ is the cartesian closed one. In fact (as shown in [Man12b, Theorem 5.1]), it is more generally the case that every closed category corresponds to a unital closed multicategory (a.k.a. coloured operad).

Theorem 1.2.1.1.3 ([Gra74]). *The closed category structures $\mathfrak{Fun}_p(,)$ and $\mathfrak{Fun}_\ell(,)$ of construction 1.2.1.1.2 are monoidal, with left-adjoints called respectively the pseudo-Gray and lax Gray tensor products.*

While the pseudo-Gray product will not see any use in this thesis, it shall be convenient to introduce the notation $\otimes^\ell$ for the lax Gray tensor product, determined by the adjunction $\otimes^\ell \mathfrak{B} \dashv \mathfrak{Fun}_\ell(\mathfrak{B},)$ which extends to an enriched adjunction $\mathfrak{Fun}_\ell(\mathfrak{A} \otimes^\ell \mathfrak{B}, \mathfrak{C}) = \mathfrak{Fun}_\ell(\mathfrak{A}, \mathfrak{Fun}_\ell(\mathfrak{B},\mathfrak{C}))$.

Warning 1.2.1.1.4. The lax Gray tensor product is not symmetric: the right-adjoint to $\mathfrak{B} \mapsto \mathfrak{A} \otimes^\ell \mathfrak{B}$ is given by the 2-category of 2-functors with *colax* transformations between them.

Definition 1.2.1.1.5. A **lax Gray-category** is a $2\text{--}\mathfrak{Cat}^{\otimes^{\ell}}$-enriched category.

Example 1.2.1.1.6. By definition, the closed monoidal structure on $2\text{--}\mathfrak{Cat}$ gives a lax Gray-enrichment; the corresponding lax Gray-category (of 2-categories, strict 2-functors, lax natural transformations and modifications) is denoted $2\text{--}\mathfrak{Cat}_{\ell}$.

Example 1.2.1.1.7. By the usual yoga of looping-and-delooping, a one-object lax Gray-category is equivalent datum to a monoid in the enriching category $2\text{--}\mathfrak{Cat}^{\otimes^{\ell}}$: a 2-category with a lax Gray-monoidal structure, also called a lax Gray-monoid. As always, the lax Gray-category $\mathcal{B}\,\mathfrak{V}^{\odot}$ corresponding to $\mathfrak{V}^{\odot}$ has $\mathfrak{V}$ as the endomorphism category of its unique object, with composition given by the monoidal structure.

The data presenting a Gray-monoidal structure on a 2-category $\mathfrak{V}$ is made explicit in [JY20, Explanation 12.2.35] (though see also their Note 12.5.28 for original references) for the pseudo-Gray product; the version relevant to our purposes is readily obtained by not requiring their $\Sigma_{f,g}$ to be invertible. While we do not reproduce the entire description, its highlights are:

- strict 2-functoriality of the product $\odot$ in each variable, in the form of diagrams

$$V \odot W \underset{V\odot f'}{\overset{V\odot f}{\rightrightarrows}} V \odot W' \;\; (V\odot\varphi) \qquad \text{and} \qquad W \odot V \underset{f'\odot V}{\overset{f\odot V}{\rightrightarrows}} W' \odot V \;\; (\varphi\odot V) \tag{1.24}$$

  assigned functorially to any 1-cells and 2-cell $\varphi\colon f \Rightarrow f'\colon W \to W'$,

- the lax interchange law, a 2-cell

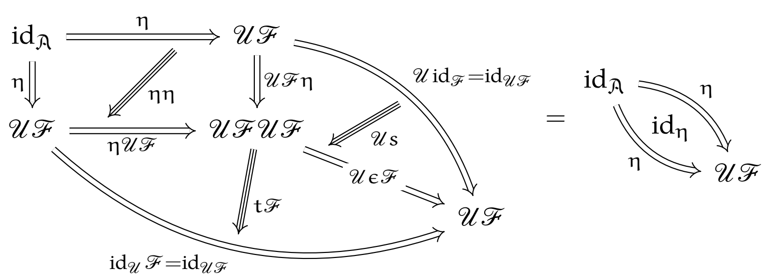
$$\begin{array}{ccc} V \odot W & \xrightarrow{V\odot g} & V \odot W' \\ {\scriptstyle f\odot W}\downarrow & \swarrow\!\!\!\!\swarrow \Sigma_{f,g} & \downarrow{\scriptstyle f\odot W'} \\ V' \odot W & \xrightarrow[V'\odot g]{} & V' \odot W' \end{array} \tag{1.25}$$

  for any pair of 1-cells $f\colon V \to V'$ and $g\colon W \to W'$,

- strict unitality and associativity.

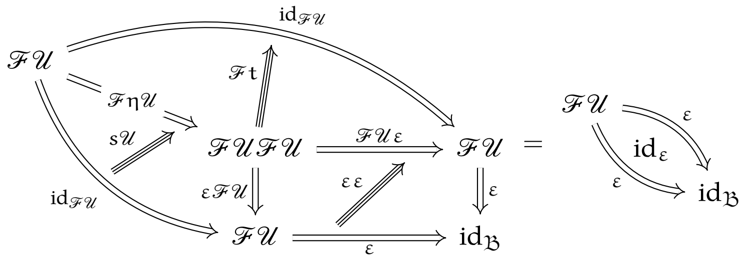
We now have the elements of language necessary to study the general notion of soft adjunctions.

Definition 1.2.1.1.8. A **soft 2-adjunction** in a lax Gray-category $\mathfrak{K}$ is a pair of 1-cells $\mathscr{F}\colon \mathfrak{A} \rightleftarrows \mathfrak{B}\colon \mathscr{U}$ endowed with 2-cells $\eta\colon \mathrm{id}_{\mathfrak{A}} \Rightarrow \mathscr{U}\mathscr{F}$ and $\epsilon\colon \mathscr{F}\mathscr{U} \Rightarrow \mathrm{id}_{\mathfrak{B}}$, along with 3-cells $s\colon \mathrm{id}_{\mathscr{F}} \Rrightarrow (\epsilon\mathscr{F}) \circ (\mathscr{F}\eta)$ and $t\colon (\mathscr{U}\epsilon) \circ (\eta\mathscr{U}) \Rrightarrow \mathrm{id}_{\mathscr{U}}$ (called the **triangulators**) satisfying the swallowtail coherence conditions:



(1.26)

for the lax monad part and



(1.27)

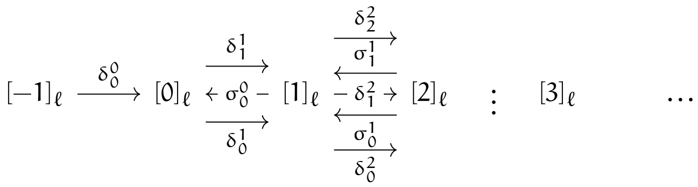
for the oplax comonad part.

When $\mathfrak{K} = 2\text{—}\mathfrak{Cat}_\ell$, the equivalence of the formulations given in example 1.2.1.1.1 and definition 1.2.1.1.8, *i.e.* the fact that example 1.2.1.1.1 is indeed an example (and in fact every example) of a soft adjunction, is the main result of [MS89].

Remark 1.2.1.1.9 (Terminology). The name "soft 2-adjunctions" is taken from [MS89]. They were introduced in [Gra74] as strict weak quasi-adjunctions or the seemingly more general transcendental quasi-adjunctions (the equivalence between the two definitions being established as [Gra74, Theorems I.7.5 iv) and I.7.7 iii)]).

Such 2-adjunctions are also occasionally referred to as local adjunctions (due to example 1.2.1.1.1) or, for example in [Bun74], as lax 2-adjunctions (despite not fitting in the framework of lax algebras over a 2-monad and, as always with adjunctions, exhibiting both lax and oplax aspects).

Construction 1.2.1.1.10. The **lax augmented simplicial indexing 2-category** $\Delta_{a,\ell}$ is the 2-category[2] with underlying 1-category freely generated by the graph



(1.28)

and 2-cells freely generated by

- $\gamma^n_{i,j} : \delta^{n+1}_j \circ \delta^n_i \Rightarrow \delta^{n+1}_i \circ \delta^n_{j-1}$ for $0 \leqslant i < j \leqslant n+1$
- $\alpha^{n+1}_{i,j} : \sigma^n_j \circ \sigma^{n+1}_i \Rightarrow \sigma^n_i \circ \sigma^{n+1}_{j+1}$ for $0 \leqslant i \leqslant j < n$
- $\rho^n_j : \sigma^n_j \circ \delta^{n+1}_j \Rightarrow \mathrm{id}_{[n]_\ell}$ for $0 \leqslant j < n$
- $\lambda^n_j : \mathrm{id}_{[n]_\ell} \Rightarrow \sigma^n_j \circ \delta^{n+1}_{j+1}$ for $0 \leqslant j < n$
- $\upsilon : \sigma^n_j \circ \delta^{n+1}_i \Rightarrow \delta^n_i \circ \sigma^{n-1}_{j-1}$ for $0 \leqslant i < j < n$
- $\delta^n_j \circ \sigma^{n-1}_i \Rightarrow \sigma^n_i \circ \delta^{n+1}_{j+1}$ for $0 \leqslant i$ and $i+1 < j \leqslant n$,

and subject to the identifications

[2] More accurately, we are giving here a presentation from a computad.

- $((\rho^n_j)\delta^n_j) \circ ((\sigma^n_j)(\delta^n_j \delta^n_j)) \circ ((\lambda^n_j)\delta^n_j) = \mathrm{id}_{\delta^n_j}$
- $(\sigma^n_j(\rho^{n+1}_{j+1})) \circ ((\sigma^n_j \sigma^n_j)\delta^{n+1}_j) \circ (\sigma^n_j(\lambda^{n+1}_{j-1})) = \mathrm{id}_{\sigma^n_j}$.

The **lax simplicial indexing 2-category** is the full sub-2-category Δ_ℓ on all objects but $[-1]_\ell$.

The **lax split simplicial indexing 2-category** $\Delta_{s,\ell}$ is the locally full sub-2-category of Δ_ℓ containing everything bar the 1-cells δ^n_n.

Remark 1.2.1.1.11. This 2-category (with its lax Gray-monoidal structure established thereafter) has been independently studied in [MS21] where it(s suspension) is directly defined as the free lax Gray-category containing a lax Gray-monad.

Lemma 1.2.1.1.12. *For any* $n > 0$*, the* 2*-cell*

$$\begin{array}{ccc} [n+1]_\ell & \xleftarrow{\delta^{n+1}_{n+1}} & [n]_\ell \\ \delta^{n+1}_0 \uparrow & \overset{\rho}{\nearrow} & \uparrow \delta^n_0 \\ [n]_\ell & \xleftarrow[\delta^n_n]{} & [n-1]_\ell \end{array} \tag{1.29}$$

defines a universal cocomma cone, exhibiting $[n+1]_\ell = \delta^n_n \uparrow \delta^n_0$.

Proposition 1.2.1.1.13. *1. The ordinal sum* $[n]_\ell \boxplus [m]_\ell = [n+m+1]_\ell$ *is* 2*-functorial on* $\Delta_{a,\ell}$ *and extends to a lax Gray-monoidal structure* $(\Delta_{a,\ell}, \boxplus, [-1]_\ell)$.

2. Restriction of $\boxplus$ *gives left and right lax Gray-module structures* $\Delta_{a,\ell} \otimes^\ell \Delta_\ell \to \Delta_\ell$, $\Delta_\ell \otimes^\ell \Delta_{a,\ell} \to \Delta_\ell$, $\Delta_{a,\ell} \otimes^\ell \Delta_{s,\ell} \to \Delta_{s,\ell}$ *and* $\Delta_{s,\ell} \otimes^\ell \Delta_{a,\ell} \to \Delta_{s,\ell}$.

Proof. The functoriality properties of the sum are implied by the universal property of lemma 1.2.1.1.12. □

Corollary 1.2.1.1.14. *Let* $\mathfrak{lAdj}$ *denote the lax Gray-category with two objects, denoted respectively* $+$ *and* $-$*, and hom-categories* $\mathfrak{lAdj}(+,+) = \Delta_{a,\ell} = \mathfrak{lAdj}(-,-)^{op}$ *and. Then* $\mathfrak{lAdj}$ *satisfies the universal property of the "walking soft adjunction", meaning that for any lax Gray-category* $\mathfrak{K}$*, the lax Gray-category of soft adjunctions in* $\mathfrak{K}$ *is equivalent to* $\mathfrak{K}^{\mathfrak{lAdj}}$.

Proof. Let $\mathsf{A}\colon \mathfrak{lAdj} \to \mathfrak{K}$ be a lax Gray-functor, mapping the two objects to $\mathsf{K} := \mathsf{A}(+)$ and $\mathsf{A} := \mathsf{A}(-)$. Let us also call $\mathscr{U}\colon \mathsf{A} \to \mathsf{K}$ the image of $[0]_\ell \in \mathfrak{lAdj}(-,+)$ and $\mathscr{F}\colon \mathsf{K} \to \mathsf{A}$ the image of $[0]_\ell \in \mathfrak{lAdj}(+,-)$. We explain how to recover from A a lax adjunction structure on $\mathscr{F}\colon \mathsf{K} \rightleftarrows \mathsf{A}\colon \mathscr{U}$.

We will content ourselves with exhibiting the lax monad structure on the composite $\mathscr{T} := \mathscr{U} \circ \mathscr{F} = \mathsf{A}([0]_\ell)$, as the rest of the proof is simply a straightforward transposition, and a direct computation to check the equivalence. Even this part will only deviate from the original arguments of [Aud74] for the interpretation of the 3-cells.

First, functoriality makes it clear that the image of $[n]_\ell \in \mathfrak{lAdj}(+,+)$ is $\mathscr{T}^{n+1}$ (this includes the fact that $[-1]$ is mapped to id_K). Then, as in [Aud74], the image of δ^n_i is $\mathscr{T}^i \eta \mathscr{T}^{n-i}$ while $\mathsf{A}(\sigma^n_i) = \mathscr{T}^i \mu \mathscr{T}^{n-i}\colon \mathscr{T}^{n+2} \Rightarrow \mathscr{T}^{n+1}$.

The 3-cells λ^n_i and ρ^n_i become respectively the triangulators $\mathscr{T}^i \mathscr{U} s \mathscr{T}^{n-i}\colon \mathrm{id}_\mathscr{T} \Rightarrow \mu \circ \mathscr{T}\eta$ and $\mathscr{T}^i t \mathscr{F} \mathscr{T}^{n-i}\colon \mu \circ \eta \mathscr{T} \Rightarrow \mathrm{id}_\mathscr{T}$.

The other 3-cells come from equation (1.23) interpreted with the appropriate interchanges of 2-cells.

$$\begin{array}{ccc}
\mathscr{T}^{n}=\mathscr{T}^{i}\mathscr{T}^{n-i} & \xRightarrow{(\mathscr{T}^{i}\eta)\mathscr{T}^{n-i}} & \mathscr{T}^{n+1}=\mathscr{T}^{i+1}\mathscr{T}^{n-i} \\
{\scriptstyle \mathscr{T}^{i+k-1}\eta\mathscr{T}^{n+1-(i+k)} \atop \scriptstyle =\mathscr{T}^{i}(\mathscr{T}^{k-1}\eta\mathscr{T}^{n+1-(i+k)})}\Big\Downarrow & \swarrow & \Big\Downarrow{\scriptstyle \mathscr{T}^{i+k}\eta\mathscr{T}^{n+1-(i+k)} \atop \scriptstyle =\mathscr{T}^{i+1}(\mathscr{T}^{k-1}\eta\mathscr{T}^{n+1-(i+k)})} \\
\mathscr{T}^{n+1}=\mathscr{T}^{i}\mathscr{T}^{n+1-i} & \xRightarrow[(\mathscr{T}^{i}\eta)\mathscr{T}^{n+1-i}]{} & \mathscr{T}^{n+2}=\mathscr{T}^{i+1}\mathscr{T}^{n+1-i}
\end{array} \tag{1.30}$$

corresponds to $\delta^{n+1}_{i+k}\circ\delta^{n}_{i}\Rightarrow\delta^{n+1}_{i}\delta^{n}_{i+k-1}$.

$$\begin{array}{ccc}
\mathscr{T}^{n+3}=\mathscr{T}^{i+2}\mathscr{T}^{n+1-i} & \xRightarrow{(\mathscr{T}^{i}\mu)\mathscr{T}^{n+1-i}} & \mathscr{T}^{n+2}=\mathscr{T}^{i+1}\mathscr{T}^{n+1-i} \\
{\scriptstyle \mathscr{T}^{i+2}(\mathscr{T}^{k-1}\mu\mathscr{T}^{n-(i+k)})}\Big\Downarrow & \swarrow & \Big\Downarrow{\scriptstyle \mathscr{T}^{i+1}(\mathscr{T}^{k}\mu\mathscr{T}^{n-(i+k)})} \\
\mathscr{T}^{n+2}=\mathscr{T}^{i+2}\mathscr{T}^{n-i} & \xRightarrow[(\mathscr{T}^{i}\mu)\mathscr{T}^{n-i}]{} & \mathscr{T}^{n+1}=\mathscr{T}^{i+1}\mathscr{T}^{n-i}
\end{array} \tag{1.31}$$

corresponds to $\sigma^{n}_{i+k}\circ\sigma^{n+1}_{i}\Rightarrow\sigma^{n}_{i}\circ\sigma^{n+1}_{i+k+1}$.

$$\begin{array}{ccc}
\mathscr{T}^{n+1}=\mathscr{T}^{i}\mathscr{T}^{n+1-i} & \xRightarrow{(\mathscr{T}^{i}\eta)\mathscr{T}^{n+1-i}} & \mathscr{T}^{n+2}=\mathscr{T}^{i+1}\mathscr{T}^{n+1-i} \\
{\scriptstyle \mathscr{T}^{i+k-1}\mu\mathscr{T}^{n-(i+k)} \atop \scriptstyle =\mathscr{T}^{i}(\mathscr{T}^{k-1}\mu\mathscr{T}^{n-(i+k)})}\Big\Downarrow & \swarrow & \Big\Downarrow{\scriptstyle \mathscr{T}^{i+k}\mu\mathscr{T}^{n-(i+k)} \atop \scriptstyle =\mathscr{T}^{i+1}(\mathscr{T}^{k-1}\mu\mathscr{T}^{n-(i+k)})} \\
\mathscr{T}^{n}=\mathscr{T}^{i}\mathscr{T}^{n-i} & \xRightarrow[(\mathscr{T}^{i}\eta)\mathscr{T}^{n-i}]{} & \mathscr{T}^{n+1}=\mathscr{T}^{i+1}\mathscr{T}^{n-i}
\end{array} \tag{1.32}$$

corresponds to $\sigma^{n}_{i+k}\circ\delta^{n+1}_{i}\Rightarrow\delta^{n}_{i}\circ\sigma^{n-1}_{i+k-1}$.

$$\begin{array}{ccc}
\mathscr{T}^{n+1}=\mathscr{T}^{i+1}\mathscr{T}^{n-i} & \xRightarrow{(\mathscr{T}^{i}\mu)\mathscr{T}^{n-i}} & \mathscr{T}^{n}=\mathscr{T}^{i}\mathscr{T}^{n-i} \\
{\scriptstyle \mathscr{T}^{i+k+1}\eta\mathscr{T}^{n-(i+k)} \atop \scriptstyle =\mathscr{T}^{i+1}(\mathscr{T}^{k}\eta\mathscr{T}^{n-(i+k)})}\Big\Downarrow & \swarrow & \Big\Downarrow{\scriptstyle \mathscr{T}^{i+k}\eta\mathscr{T}^{n-(i+k)} \atop \scriptstyle =\mathscr{T}^{i}(\mathscr{T}^{k}\eta\mathscr{T}^{n-(i+k)})} \\
\mathscr{T}^{n+2}=\mathscr{T}^{i+1}\mathscr{T}^{n+1-i} & \xRightarrow[(\mathscr{T}^{i}\mu)\mathscr{T}^{n+1-i}]{} & \mathscr{T}^{n+1}=\mathscr{T}^{i}\mathscr{T}^{n+1-i}
\end{array} \tag{1.33}$$

corresponds to $\delta^{n}_{i+k}\circ\sigma^{n-1}_{i}\Rightarrow\sigma^{n}_{i}\circ\delta^{n+1}_{i+k+1}$. □

Example 1.2.1.1.15. There is a 2-localisation functor $\Delta_{a,\ell}\to\Delta_{a}$, which maps all the 2-cells to identities (it succeeds in being a localisation because each 2-cell to invert is the only one between its source and its target: the 2-category $\Delta_{a,\ell}$ is locally posetal). Thus, by precomposing by the induced lax Gray-functor, any 2-monad can be seen as a lax 2-monad.

1.2.1.2 The weight for lax codescent objects

As motivation, we start by recalling the notion of strong codescent objects in $(\infty,2)$-categories (which is an example of the generalised (co)kernels introduced in unpublished work of Betti–Schumacher–Street).

Construction 1.2.1.2.1. Let $\mathfrak{D} \subset (\infty,1)\text{--}\mathfrak{Cat}^2$ be the full sub-$(\infty,2)$-category spanned by the functors $n \to \mathbb{m}$ for $n > 0$. Note that an arrow $(n \to \mathbb{m}) \to (m \to \mathbb{m})$ in $\mathfrak{D}$ is entirely determined by its underlying functor $\mathbb{n} \to \mathbb{m}$ so that $\mathfrak{D}$ is equivalent to $\mathbb{\Delta}$, with the usual equivalence making correspond $[n-1]$ and $(n \to \mathbb{n})$. Consider the restricted evaluation $(\infty,2)$-functor $\mathbb{2} \times \mathfrak{D} \subset \mathbb{2} \times (\infty,1)\text{--}\mathfrak{Cat}^2 \to (\infty,1)\text{--}\mathfrak{Cat}$.

It determines $(\infty,2)$-profunctors $\mathscr{E}v \colon \mathbb{2} \to\!\!\!\!\!+ \mathfrak{D}^{op}$ and $\mathscr{E}v^{op} \colon \mathfrak{D}^{op} \to\!\!\!\!\!+ \mathbb{2}$.

Let $\mathfrak{C}$ be an $(\infty,2)$-category.

- Let $\mathscr{X} \colon \Delta^{op} \simeq \mathfrak{D}^{op} \to \mathfrak{C}$ be a which we call **strong coherence datum**, or equivalently a simplicial object. Its **codescent object** (or quotient) is the weighted colimit $\mathscr{E}v \star \mathscr{X} \colon \mathbb{2} \to \mathfrak{C}$, seen as an arrow in $\mathfrak{C}$.
- Let f be an arrow of $\mathfrak{C}$, corresponding to an $(\infty,2)$-functor $\ulcorner f \urcorner \colon \mathbb{2} \to \mathfrak{C}$. Its **simplicial kernel** (or higher kernel) is the weighted limit $\{\mathscr{E}v^{op}, \ulcorner f \urcorner\} \colon \mathfrak{D}^{op} \simeq \Delta^{op} \to \mathfrak{C}$.

Remark 1.2.1.2.2. For $\mathscr{X} \colon \mathfrak{D}^{op} \to \mathfrak{C}$, the domain of (the arrow corresponding to) its codescent object is $\mathscr{X}(1 \xrightarrow{=} \mathbb{1})$ (or $\mathscr{X}([0])$ in terms of simplicial objects). This comes from an application of the Yoneda reduction of corollary 1.1.1.2.10 to the fact that $\ulcorner \mathrm{id}_1 \urcorner \colon \mathbb{2} \to (\infty,1)\text{--}\mathfrak{Cat}$ is the corepresentable $\mathbb{2}(0,-)$ (the domain of an arrow being the value at $0 \in \mathbb{2}$ of its characteristic functor).

In conclusion, we recover the codescent object of a simplicial diagram $\mathscr{X} \colon \Delta^{op} \to \mathfrak{C}$ as an arrow $\mathscr{X}([0]) \to \mathrm{codesc}(\mathscr{X})$, where $\mathrm{codesc}(\mathscr{X})$ is the (pointlike) weighted colimit of $\mathscr{X}$ with weight $\mathscr{E}v_1 \colon * \to\!\!\!\!\!+ \mathfrak{D}^{op}$, corresponding to the $(\infty,2)$-functor $\mathfrak{D} \to (\infty,1)\text{--}\mathfrak{Cat}$ mapping $(n \to \mathbb{m})$ to $\mathbb{m} \in (\infty,1)\text{--}\mathfrak{Cat}$, that is the inclusion $\mathbb{\Delta} \hookrightarrow (\infty,1)\text{--}\mathfrak{Cat}$.

Example 1.2.1.2.3 ([Bou10, Example 2.21]). Suppose $X_\bullet \colon {\Delta_{\leqslant 2}}^{op} \to \mathfrak{Set}$ (recall that ${\Delta_{\leqslant 2}}^{op} \hookrightarrow \Delta^{op}$ is a cofinal 2-functor) is a Segal object, defining a small category. Through the inclusion $\mathfrak{Set} \hookrightarrow \mathfrak{Cat}$, $X_\bullet$ can be seen as 2-truncated coherence datum in $\mathfrak{Cat}$, a double category. Its codescent object is the category encoded by $X_\bullet$.

Remark 1.2.1.2.4. Note that there is a further (fully faithful) inclusion of $(\infty,2)$-categories $(\infty,1)\text{--}\mathfrak{Cat} \hookrightarrow (\infty,1)\text{--}\mathfrak{Cat}^{\Delta^{op}}$. By composing with the conclusion of remark 1.2.1.2.2, we obtain that the (pointlike) weight computing codescent objects corresponds to the Yoneda embedding $\mathbb{\Delta} \hookrightarrow (\infty,1)\text{--}\mathfrak{Cat}^{\Delta^{op}}$.

More precisely, this $(\infty,2)$-functor is equivalently given by $\Delta^{op} \to (\infty,1)\text{--}\mathfrak{Cat}^{\mathbb{\Delta}}$, a strong coherence datum in the $(\infty,2)$-category of pointlike weights for colimits of coherence data, and by the arguments giving the previous example, the desired weight is the codescent object of this diagram.

Definition 1.2.1.2.5 (Lax double category)**.** Let $\mathfrak{K}$ be an $(\infty,2)$-category. An $(\infty,2)$-functor $\mathscr{X} \colon {\Delta_\ell}^{op} \to \mathfrak{K}$ preserving commas is called a **lax category object**. In the case where $\mathfrak{K} = (\infty,1)\text{--}\mathfrak{Cat}$, the diagram $\mathscr{X}$ is said to define a **lax double $(\infty,1)$-category**.

In view of lemma 1.2.1.1.12, a lax category object $\mathscr{X}$ in $\mathfrak{K}$ is essentially given by a diagram

$$X_0 \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; X_1 \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; X_1 \downarrow_{X_0} X_1 \qquad X_1 \downarrow_{X_0} X_1 \downarrow_{X_0} X_1 \qquad \cdots \tag{1.34}$$

equipped with the 2-cells filling in for the simplicial identities.

Warning 1.2.1.2.6. Our use of the term "lax double category" is completely unrelated to the one made in [DPP06, §2, p.483], where it played the role of a synonym for virtual double category.

Remark 1.2.1.2.7. A lax double category thus consists of the following data:

- a category C_0 of objects and vertical arrows,
- a category C_1 of horizontal arrows and cells,
- functors $e = s_0$, $s = d_0$ and $t = d_1$ interpreted as horizontal unit, and source and target of horizontal maps (and squares),
- a composition functor $s \downarrow t \to C_1$, associating with any pair of horizontal arrows $f\colon C \nrightarrow C'$, $g\colon D \nrightarrow D'$, joined by a vertical arrow $t\colon C' \to D$, a horizontal arrow $g \circ_t f\colon E \nrightarrow E'$, and a similar operations for linked pairs of squares.

Example 1.2.1.2.8. Since representable functors preserve weighted limits, every object $[n]_\ell$ of Δ_ℓ provides a lax double category よ$_{\Delta_\ell, [n]_\ell}$, and the Yoneda embedding よ$_{\Delta_\ell}$ factors through the 2-category of lax double categories.

To define the weight for lax codescent objects of lax coherence data, we will mimick the conclusion of remark 1.2.1.2.4. Note that the shape Δ_ℓ is a 2-category, so its Yoneda embedding also lands in 2-categories where codescent objects can be defined through explicit diagrams in [Lac02, §2].

Definition 1.2.1.2.9. The weight defining **lax codescent objects** of lax coherence data is the lax codescent object of the lax coherence datum in $\mathfrak{Cat}^{\Delta_\ell}$ provided by the Yoneda embedding.

1.2.2 Lax morphisms classifiers

1.2.2.1 The universal property of the 2-category $\mathfrak{Lax}\text{—}T - \mathfrak{Alg}_l$ of lax algebras over a 2-monad

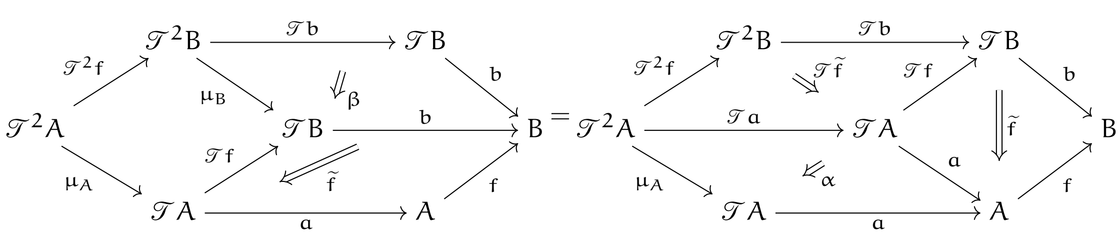
We recall the main definitions and properties from [Lac02].

Unless specified otherwise, 2-monads on 2-categories will be implicitly assumed strict, that is $\mathfrak{Cat}$-monads, monads inside the 2-category $\mathfrak{Cat}^{\times}\text{—}\mathfrak{Cat}$.

Definition 1.2.2.1.1. A **lax** T**-algebra** is a quartet (A, a, α, α_0) consisting of an object $A \in \mathfrak{K}$, a 1-cell $a\colon \mathscr{T}A \to A$, and 2-cells $\alpha\colon a \circ \mathscr{T}a \Rightarrow a \circ \mu_A$ and $\alpha_0\colon \mathrm{id}_A \Rightarrow a \circ \eta_A$, required to satisfy coherence conditions.

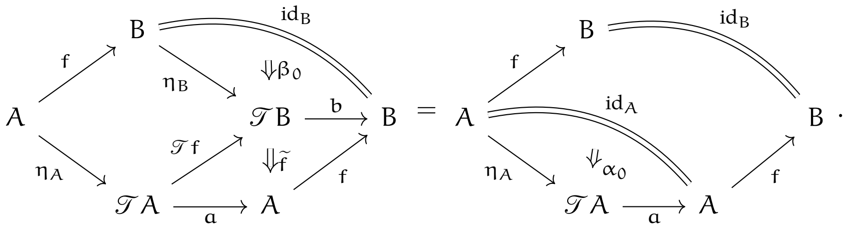
The lax T-algebra (A, a, α, α_0) is said to be **strong** if α and α_0 are invertible.

Definition 1.2.2.1.2. A **lax morphism** of lax T-algebras $(A, a, \alpha, \alpha_0) \to (B, b, \beta, \beta_0)$ is the data of a 1-cell $f\colon A \to B$ and a 2-cell $\widetilde{f}\colon b \circ \mathscr{T}f \Rightarrow f \circ a$ with the identities



(1.35)

and



(1.36)

A lax morphism $(f,\widetilde{f})$ is said to be **strong** if $\widetilde{f}$ is invertible.

Definition 1.2.2.1.3. A transformation $(f,\widetilde{f}) \Rightarrow (g,\widetilde{g})$ between two lax morphisms is a 2-cell $\rho\colon f \Rightarrow g$ satisfying $\widetilde{g} \circ \mathscr{T}\rho = \rho \circ \widetilde{f}\colon b \circ \mathscr{T} f \Rightarrow g \circ a$.

Notation 1.2.2.1.4 (2-category of lax algebras)**.** We let $\mathfrak{Lax}{-}\mathsf{T}-\mathfrak{Alg}_l$ denote the 2-category whose objects are the lax T-algebras, arrows the lax morphisms, and 2-arrows the transformations, with the obvious compositions and units.

We also let $\mathsf{T}{-}\mathfrak{Alg}_l$ denote the full sub-2-category of $\mathfrak{Lax}{-}\mathsf{T}-\mathfrak{Alg}_l$ spanned by the strong T-algebras, and $\mathsf{T}{-}\mathfrak{Alg}_s$ the wide and locally full sub-2-category of $\mathsf{T}{-}\mathfrak{Alg}_l$ on the strong morphisms.

Theorem 1.2.2.1.5. *Let* T *be a 2-monad in* $2{-}\mathfrak{Cat}_\ell$*, seen as a lax 2-monad* $\mathcal{B}\,\Delta_{a,\ell} \to 2{-}\mathfrak{Cat}_\ell$ *that factors through* $\mathcal{B}\,\Delta_a$*. The weighted limit* $\left\{\mathfrak{LAdj}(\mathrm{id}_{\mathfrak{LAdj}},-)|_{\mathcal{B}\,\Delta_{a,\ell}}, \mathsf{T}\right\}$ *is the 2-category* $\mathfrak{Lax}{-}\mathsf{T}-\mathfrak{Alg}_l$.

Proof. As T takes its values in the enriching category $2{-}\mathfrak{Cat}$, the weighted limit is the object (the 2-category) of transformations $\mathfrak{LAdj}(\mathrm{id}_{\mathfrak{LAdj}},-)|_{\mathcal{B}\,\Delta_{a,\ell}} \Rightarrow \mathsf{T}$. Note that the category of lax Gray-categories is considered with its *cartesian* monoidal structure, so that the transformations of Gray-functors are strict and not lax, as explained for example in [Gur13, Proposition 12.2] for the pseudo-Gray case.

As $\mathcal{B}\,\Delta_{a,\ell}$ has a single object, a natural transformation between Gray-functors with source $\mathcal{B}\,\Delta_{a,\ell}$ will consist of a single morphism of 2-categories between the images of $+ \in \mathcal{B}\,\Delta_{a,\ell}$, with additional naturality data. Here the basic datum for a transformation $\mathcal{B}\,\mathscr{a}\colon \mathfrak{LAdj}(\mathrm{id}_{\mathfrak{LAdj}},-)|_{\mathcal{B}\,\Delta_{a,\ell}} \Rightarrow \mathsf{T}$ is thus that of a 2-functor $\mathscr{a}\colon \mathfrak{LAdj}(+,-) = \Delta_{s,\ell} \to \mathsf{T}(+)$, interpreted as a lax split simplicial object in $\mathsf{T}(+)$. This corresponds to a collection of objects $A_n := \mathscr{a}([n]_\ell) \in \mathsf{T}(+)$, with morphisms and 2-cells as in construction 1.2.1.1.10 and which we will interpret in time.

Next, strict naturality of $\mathcal{B}\,\mathscr{a}$ translates to the strict commutativity of

$$\begin{array}{ccc} \Delta_{s,\ell} & \xrightarrow{\mathscr{a}} & \mathsf{T}(+) \\ {\scriptstyle \boxplus[i]_\ell}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{T}^i} \\ \Delta_{s,\ell} & \xrightarrow[\mathscr{a}]{} & \mathsf{T}(+) \end{array} \tag{1.37}$$

for every map $\boxplus[i]_\ell$ of $\mathcal{B}\,\Delta_{a,\ell}$. In other words, for every $[n]_\ell \in \Delta_{s,\ell}$ (and for $i \geqslant -1$) one must have the equalities $A_{n+i+1} = \mathscr{T}^{i+1} A_n$ of objects in $\mathsf{T}(+)$, so that the collection of objects $(A_n)_{n\in\mathbb{N}}$ is completely determined by A_0 and the requirement that $A_n = \mathscr{T}^i A_0$.

The same logic applies to the higher cells, which can be separated and then checked on by one to correspond to the ones forming the definition of $\mathfrak{Lax}{-}\mathsf{T}-\mathfrak{Alg}_l$ in definition 1.2.2.1.1 and definition 1.2.2.1.2. □

Definition 1.2.2.1.6 (Lax algebras and lax morphisms for $(\infty,2)$-monads)**.** Let $\mathfrak{K}$ be an $(\infty,2)$-category and let $\mathsf{T}\colon \mathfrak{lMnd} \to (\infty,2)\text{—}\mathfrak{Cat}$ be an $(\infty,2)$-monad on $\mathfrak{K}$. The **Eilenberg–Moore soft adjunction** of T is the oplax extension of T along $\mathfrak{lMnd} \hookrightarrow \mathfrak{lAdj}$.

By example 1.1.2.1.5, the evaluation of the Eilenberg–Moore adjunction of T at the object $+ \in \mathfrak{lAdj}$ recovers $\mathsf{T}(+) =: \mathfrak{K}$. Its evaluation at $- \in \mathfrak{lAdj}$ is denoted $\mathfrak{Lax}\text{—}\mathsf{T}-\mathfrak{Alg}_l$ and called the $(\infty,2)$**-category of lax** T**-algebras**.

Remark 1.2.2.1.7. Unfolding the definition, we find an explicit description of lax algebras akin to the one given in [Str04, §2], that is recovering the one given in definition 1.2.2.1.1.

1.2.2.2 Construction of the right-adjoint

Conjecture 1.2.2.2.1. The $(\infty,2)$-category $\mathsf{T}\text{—}\mathfrak{Alg}_l$ is the very lax descent object of the lax co-coherence datum

$$C \underset{\mathrm{id}_C}{\overset{\mathscr{T}}{\substack{\longrightarrow\\ \leftarrow\, \mathrm{id}_C\, -\\ \longrightarrow}}} C \underset{\mathrm{id}_C}{\overset{\mathscr{T}}{\substack{\longrightarrow\\ -\, \mathrm{id}_C \,\rightarrow\\ \longrightarrow}}} C \qquad \cdots \tag{1.38}$$

Remark 1.2.2.2.2. Evidence for this conjecture can be found by looking at the lower-dimensional case: the category of algebras over a 1-monad (in a 2-category $\mathfrak{K}$) is the limit of the corresponding 2-functor $\mathfrak{Mnd}\,\mathcal{B}\,\Delta_a$ weighted by the inclusion into $\mathfrak{Adj}$ as recalled in subsection 1.1.2.3, but can also be constructed as a sequence of three successive limits which is known from [Lac02] to construct lax descent objects.

Corollary 1.2.2.2.3. *The hom-$(\infty,1)$-categories in $\mathfrak{Lax}\text{—}\mathsf{T}-\mathfrak{Alg}_l$ are given by the lax descent objects of the lax co-coherence datum*

$$\mathfrak{K}(A,B) \underset{\mathfrak{K}(a,B)}{\overset{\mathfrak{K}(\mathscr{T}A,b)\circ\mathscr{T}}{\substack{\longrightarrow\\ \leftarrow\, \mathfrak{K}(\eta_A,B)\, -\\ \longrightarrow}}} \mathfrak{K}(\mathscr{T}A,B) \underset{\mathfrak{K}(\mathscr{T}a,B)}{\overset{\mathfrak{K}(\mathscr{T}^2A,b)\circ\mathscr{T}}{\substack{\longrightarrow\\ -\, \mathfrak{K}(\mu_A,B) \rightarrow\\ \longrightarrow}}} \mathfrak{K}(\mathscr{T}^2A,B) \qquad \cdots. \tag{1.39}$$

Theorem 1.2.2.2.4. *Let T be an $(\infty,2)$-monad. The $(\infty,2)$-functor $\mathsf{T}\text{—}\mathfrak{Alg}_s \hookrightarrow \mathfrak{Lax}\text{—}\mathsf{T}-\mathfrak{Alg}_l$ admits a left-adjoint.*

Proof. We need to construct, for any $A \in \mathfrak{Lax}\text{—}\mathsf{T}-\mathfrak{Alg}_l$, a lax morphisms classifier $A' \in \mathsf{T}\text{—}\mathfrak{Alg}_s$ equipped with a "universal" lax morphism $A \to A'$ such that for any strong T-algebra B, $\mathfrak{Lax}\text{—}\mathsf{T}-\mathfrak{Alg}_l(A,B) \simeq \mathsf{T}\text{—}\mathfrak{Alg}_s(A',B)$.

Since B is a strong algebra, we can use the adjunction $\mathfrak{K} \rightleftarrows \mathsf{T}\text{—}\mathfrak{Alg}_s$ in equation (1.39) to rewrite it as a diagram of hom $(\infty,1)$-categories in $\mathsf{T}\text{—}\mathfrak{Alg}_s$ as in [Bou10, Remark 6.9].

This diagram is exactly the image of the bar resolution of A under $\mathsf{T}\text{—}\mathfrak{Alg}_s(-,B)$. Since by example 1.1.2.2.9 the hom ∞-functors detect (weighted) limits, its lax descent object must be given by the lax codescent object of the bar construction. □

Corollary 1.2.2.2.5 (Lax Morita equivalence)**.** *Let T and T' be two $(\infty,2)$-monads such that $\mathsf{T}\text{—}\mathfrak{Alg}_s \simeq \mathsf{T}'\text{—}\mathfrak{Alg}_s$. Then $\mathsf{T}\text{—}\mathfrak{Alg}_l \simeq \mathsf{T}'\text{—}\mathfrak{Alg}_l$.*

Proof. We use the fact that lax morphisms can be recast as strong morphisms, and transfer the lax morphisms classifiers along the equivalence. □

CHAPTER

2

BRANE ACTIONS FOR PARTLY UNITAL COLOURED ∞-OPERADS

This chapter is centered around the notion of *brane action* for ∞-operads, which can be understood from the geometric example of the operad of little 2-disks. Let $\mathscr{E}_2$ denote the ∞-operad of little disks, obtained for example from the (monochromatic) topological operad whose space of n-ary operations $\mathscr{E}_2(n)$ is the configuration space of n disjoint (labeled) disks in the unit 2-disk, and whose composition is given by insertion of disks and renumbering. This operad is *unital*, in that its space of nullary operations $\mathscr{E}_2(0)$ is contractible, and *reduced*, in that its space $\mathscr{E}_2(1)$ of unary (or linear) operations is contractible. The spaces of higher-arity operations can be understood fibrewise: since $\mathscr{E}_2(0) \simeq *$ there is for any $n \geqslant 0$ a canonical map $\mathscr{E}_2(n+1) \to \mathscr{E}_2(n)$, forgetting the last little disk. Its fibre at an operation $\sigma \in \mathscr{E}_2(n)$, known as the space of *extensions* of σ, is homotopy equivalent to a wedge $\bigvee^n S^1$ of n circles.

Note that, since the 2-disk is contractible, the space $\mathrm{Ext}(\sigma)$ is homotopy equivalent to the complement of the configuration of disks classified by σ itself. For any such $\sigma \in \mathscr{E}_2(n)$, the space $\mathrm{Ext}(\sigma)$ can be used to define a natural cobordism from $\coprod_n S^1$ to S_1: the first n copies of the circle are included as the boundaries of the little disks, while the last copy of S^1 is included as the boundary of the unit disk. Forgetting the manifold structure on the cobordism and using the equivalence with $\mathrm{Ext}(\sigma)$, we obtain a mere cospan from $\coprod_n S^1$ to S_1. In fact, the cospans thus obtained for each operation σ of $\mathscr{E}_2$ assemble together to give a structure of $\mathscr{E}_2$-algebra on S^1 in cospans of spaces.

This is an example of the more general phenomenon of **brane actions** for reduced ∞-operads discovered in [Toë13]. Let $\mathscr{O}$ be a reduced unital ∞-operad, so that (by unitality) we again have canonical maps $\mathscr{O}(n+1) \to \mathscr{O}(n)$ forgetting the last input by plugging in the unique nullary operation and can similarly define spaces of extensions as $* \times_{\mathscr{O}(n)} \mathscr{O}(n+1)$. Letting id denote the unique (by reducedness) unary operation, for every operation $\sigma \in \mathcal{O}(n)$ there is a cospan

$$\coprod^{n} \mathrm{Ext}(\mathrm{id}) \xrightarrow{\sigma \circ_i -} \mathrm{Ext}(\sigma) \xleftarrow{- \circ_1 \sigma} \mathrm{Ext}(\mathrm{id}) \tag{2.1}$$

whose structure maps are given by partial compositions with σ. These cospans once again assemble into something this is almost, but not quite, an $\mathcal{O}$-algebra structure on $\mathrm{Ext}(\mathrm{id})$. Recall that cospans are naturally the morphisms not of an $(\infty,1)$-category but an $(\infty,2)$-category (whose non-invertible 2-arrows are the arrows between the roofs of cospans). This means that it is possible to talk about lax structures in it, and indeed the spans of equation (2.1) define a *lax* $\mathcal{O}$-algebra structure.

The construction of the brane action in [Toë13] was performed using model-categorical arguments with the model of Segal dendroidal spaces for ∞-operads. An alternate, ∞-categorical, construction was then given in [MR18], using this time the model of $(\infty,1)$-categories of operators of [Lur17]. In this chapter, our main goal will be to adapt their construction to the case of non-reduced, and in fact non-monochromatic (that is coloured), ∞-operads, albeit with a weak version of the unitality condition (which we call hapaxunitality, see definition 2.2.1.2.8) that will reduce the construction to one close to the monochromatic case.

However, this model of $(\infty,1)$-categories of operators is ill-suited for two purposes we are interested in:

- it lacks the flexibility to easily describe other operadic structures, such as cyclic and modular operads;

- we need to access several algebraic properties of ∞-operads, in particular the possibility to enrich them, at least in $(\infty,1)-\mathfrak{Cat}$ to talk of $(\infty,2)$-operads, or to define them internally to another ∞-category such as $(\infty,1)-\mathfrak{Cat}$ to talk of "categorical ∞-operads" (generalising slightly $(\infty,2)$-operads) and an $(\infty,1)$-topos to describe sheaves of ∞-operads. The languages of $(\infty,2)$-operads or of categorical ∞-operads are also necessary to talk clearly about lax morphisms such as obtained for a brane action.

We find that the most appropriate language to address these concerns is that of Segal dendroidal objects as was used in [Toë13]. Indeed, a systematic use of Segal conditions to describe generalised algebraic and operadic structures was developed in [CH21], in a way which is both flexible and algebraically transparent. We will thus start this chapter by recalling in subsection 2.1.1 the essentials of this language of *algebraic patterns* and explaining how the formalism of section 1.2 can be used to discuss lax morphisms between their Segal $(\infty,1)$-categories.

We should note however that we have not yet been able to exploit the full flexibility of the theory of algebraic patterns, and eventually specialise to the case of the pattern for operads, in a way which ends up essentially equivalent (by [Bar18, §10]) to $(\infty,1)$-categories of operators. We still believe that this more general story is worth telling, for the reasons which follow.

- First, although we end up specialising to operads, we expect that the ideas developed here can help reveal a path to constructing the fully general versions.

- Second, even when specialising to Ω, we use the language of algebraic patterns rather than the specific combinatorics of this category, and we believe that this makes the proofs and constructions more transparent.

Due to the assumption of hapaxunitality, the existence of a single distinguished colour which is unital, the extensions and the brane actions behave very closely to the monochromatic case, and despite working with a different model we will essentially follow the construction and proof of [MR18]. It consists of passing to the monoidal envelope of the ∞-operad $\mathscr{O}$ to reformulate the sought (lax) morphism of $(\infty,2)$-operads as a (lax) monoidal $(\infty,2)$-functor $\mathscr{E}nv(\mathscr{O}) \to \mathfrak{Cospan}(\infty\text{-}\mathfrak{Grpd})^{\amalg}$, which can then be recast as a discrete cartesian fibration. For this, it is necessary to have a notion of "monoidal envelopes" of Segal objects.

We model ∞-operads as Segal objects in $\infty\text{-}\mathfrak{Grpd}$ for an algebraic pattern based on the dendroidal category Ω of trees, while monoidal $(\infty,1)$-categories are Segal objects in $(\infty,1)\text{-}\mathfrak{Cat}$ for an algebraic pattern based on Segal's category $\mathbb{F}$ (opposite to that of pointed sets); this suggests that in order to construct the monoidal envelope one must use a relationship between Ω and $\mathbb{F}$. This relationship is the one established in [Bar18, §10] and corresponds to the equivalence between the dendroidal model and the model of $(\infty,1)$-categories of operators for ∞-operads. More precisely, for a class of algebraic patterns $\mathfrak{O}$, we introduce in subsection 2.1.2.1 a "plus construction" $\Delta_{\mathfrak{O}}$ whose Segal objects are the $\mathfrak{O}$-operads; for example, in the case $\mathfrak{O} = \mathbb{F}$, this construction recovers the dendroidal category. This is in fact one main source of our inability to use the full generality of the framework of algebraic patters: we can only construct "monoidal envelopes" for patterns which are known to be the plus construction of some other pattern. Then this construction does allow us in subsection 2.1.2.2 to construct monoidal envelopes of ∞-operads.

This leaves us equipped with most of the technology necessary to go through the construction of the brane actions. The only missing piece is the construction of the $(\infty,2)$-category of spans where the action is to take place, and whose construction in terms of twisted arrow categories is recalled in subsection 2.2.1.1. Then, using the monoidal envelopes developed previously, we give a definition of the ∞-categories of extensions of multimorphisms, adapting that of [Lur17]. We put this technology to work in subsection 2.2.2 to prove our main result, theorem 2.2.2.0.1, the existence of brane actions for coloured ∞-operads with a single distinguished unital colour, of which we also recall a graded variant in subsection 2.2.2.1.

Finally, for applications to Gromov–Witten theory, we shall need to work with an operad consisting of moduli stacks, that is an operad in the $(\infty,1)$-topos of derived stacks. For this, we need to have the brane action not only in the terminal $(\infty,1)$-topos $\infty\text{-}\mathfrak{Grpd}$ as was constructed in subsection 2.2.2, but in an arbitrary $(\infty,1)$-topos, which is the object of section 2.3. In subsection 2.3.1, we recall the main notions of $(\infty,1)$-topos theory which are useful for our operadic purposes, and discuss the construction of operads in this context. Since general $(\infty,1)$-topoi share a great closeness to that of ∞-groupoids, the brane action does not need to be constructed from scratch but can be deduced, in the same way as in [MR18], by gluing the ones obtained in subsection 2.2.2. This is achieved in subsection 2.3.2.

2.1 Segal conditions for enriched (generalised) ∞-operads

2.1.1 The language of algebraic patterns

2.1.1.1 Algebraic patterns and their Segal algebras

Definition 2.1.1.1.1 (Algebraic pattern)**.** An **algebraic pattern** is an $(\infty,1)$-category endowed with a unique factorisation system and a selected class of objects called **elementary**. The morphisms in the left class of the factorisation system are called **inert**, and those in the right class **active**.

Notation 2.1.1.1.2. If $\mathfrak{O}$ is the underlying $(\infty,1)$-category of the algebraic pattern considered, one writes $\mathcal{O}^{\mathrm{inrt}}$ and $\mathcal{O}^{\mathrm{act}}$ for the wide and locally full sub-$(\infty,1)$-categories whose morphisms are respectively the inert and the active morphisms.

We also denote $\mathcal{O}^{\mathrm{el}}$ the full sub-$(\infty,1)$-category of $\mathcal{O}^{\mathrm{inrt}}$ on the selected elementary objects. For any object $O \in \mathcal{O}$, we further write $\mathcal{O}^{\mathrm{el}}_{O/} := \mathcal{O}^{\mathrm{el}} \times_{\mathcal{O}^{\mathrm{inrt}}} \mathcal{O}^{\mathrm{inrt}}_{O/}$, the $(\infty,1)$-category of inert morphisms from O to an elementary object.

Example 2.1.1.1.3. Segal's category $\mathbb{\Gamma}$ is a skeleton of the opposite of the category $\mathfrak{FinSet}_*$ of pointed finite sets. It consists of the pointed sets of the form $\langle n\rangle$, the set $[\![0,n]\!]$ pointed at 0, for some natural integer n) and pointed morphisms between them. It admits an active-inert factorisation system where a morphism of pointed finite sets $f\colon (S,s) \to (T,t)$ is

inert if for every $x \in T\setminus\{t\}$, the preimage $f^{-1}(x)$ consists of exactly one element, and

active if the only element of S mapped to t is s.

The induced inert-active factorisation system on $\mathbb{\Gamma}^{\mathrm{op}}$ gives rise to two algebraic pattern structures, from two choices of elementary objects. The algebraic pattern $\mathbb{\Gamma}^{\mathrm{op}\flat}$ has as elementaries the two-element sets (isomorphic to $\langle 1\rangle$), while the pattern $\mathbb{\Gamma}^{\mathrm{op}\natural}$ has as elementaries the singletons (isomorphic to $\langle 0\rangle$) and the two-element sets.

Example 2.1.1.1.4. The (non-augmented) simplicial indexing category $\mathbb{\Delta}$ of remark 1.1.2.3.4, identified with the category of ordered non-empty finite sets and order-preserving maps between them, admits a factorisation system in which a map is

active if it preserves the top and bottom elements,

inert if it corresponds to the inclusion of a linear subset.

The induced inert-active factorisation system on $\mathbb{\Delta}^{\mathrm{op}}$ gives rise to two algebraic pattern structures: the algebraic pattern $\mathbb{\Delta}^{\mathrm{op}\flat}$ has as elementaries the two-element sets (isomorphic to $[1]$), while the pattern $\mathbb{\Gamma}^{\mathrm{op}\natural}$ has as elementaries the singletons (isomorphic to $[0]$) and the two-element sets.

Example 2.1.1.1.5. The dendroidal category Ω of Moerdijk–Weiss can be described as a category of (non-planar) rooted trees, with morphisms the morphisms of free (coloured, symmetric) operads generated by these trees. We shall give an alternate construction of (a sufficient subcategory of) it in subsection 2.1.2.1.

Certain particularly interesting trees can be distinguished:

the free-living edge is the tree, denoted η, consisting of one edge but no vertex (so the operad it freely generates has one colour, and only its identity unary morphism);

the corollas are the trees, denoted $\star_n$ with $n \in \mathbb{N}$, with a single vertex and $n+1$ edges (one of which is the root) attached to it. Note that the corolla $\star_n$ with n leaves has as automorphism group the symmetric group $\mathfrak{S}_n$.

The category Ω admits a factorisation system in which a morphism is

active if it is boundary-preserving,

inert if it corresponds to a subtree inclusion (which, in particular, is valence-preserving on the vertices).

The induced inert-active factorisation system on Ω^{op} gives rise to two algebraic pattern structures: the algebraic pattern $\Omega^{\mathrm{op}\flat}$ has as elementaries the corollas $\star_n$, while the pattern $\Omega^{\mathrm{op}\natural}$ has as elementaries the corollas and the free-living edge η.

Example 2.1.1.1.6. The graphical category Υ of [HRY20] has as objects the modular graphs. Once again there are two main families of trees to distinguish:

The free-living edge is the tree, also denoted η here, with no vertex and one edge. Its automorphism group is now $\mathbb{Z}/(2)$.

The corollas are the graphs, again denoted $\star_n$, consisting of one vertex and $n+1$ edges attached to it. Its automorphism group is the symmetric group $\mathfrak{S}_{n+1}$.

We will not give a direct description of the morphisms in Υ, but all can be obtained from the two halves of a factorisation system. By [HRY20, Theorem 2.15], Υ admits a factorisation system similar to that of Ω, in which a morphism is

active if it is boundary-preserving,

inert if it corresponds to a subgraph inclusion.

The induced inert-active factorisation system on Υ^{op} gives rise to two algebraic pattern structures: the algebraic pattern $\Upsilon^{op\flat}$ has as elementaries the corollas $\star_n$, while the pattern $\Upsilon^{op\natural}$ has as elementaries the corollas and the free-living edge η.

We may summarise these examples in the following table:

Category	Inerts	Actives	$\natural$ elementaries	$\flat$ elementaries
$\mathbb{\Gamma}^{op}$	partial bijections	(non-partial) functions	$\langle 1\rangle$, $\langle 0\rangle$	$\langle 1\rangle$
$\mathbb{\Delta}^{op}$	linear inclusions	endpoint-preserving	$[1]$, $[0]$	$[1]$
Ω^{op}	subtree inclusions	boundary-preserving	$\star_n$, η	$\star_n$
Υ^{op}	subgraph inclusions	boundary-preserving	$\star_n$, η	$\star_n$

We now record a definition of the $(\infty,1)$-category of algebraic patterns which will be useful when describing their limits in lemma 2.1.1.2.5.

Construction 2.1.1.1.7 (The ∞-category of algebraic patterns)**.** Recall from example 1.1.1.1.7 that $\mathfrak{cocorr}$ denotes the category $\cdot \to \cdot \leftarrow \cdot$, the generic cospan. The $(\infty,1)$-category of factorisation systems is the full sub-$(\infty,1)$-category of $(\infty,1)\text{–}\mathfrak{Cat}^{\mathfrak{cocorr}}$ on those cospans corresponding to the inclusions of the left and right classes of a factorisation systems.

The $(\infty,1)$**-category of algebraic patterns** $\mathfrak{AlgPatrn}$ can then be defined as the full sub-$(\infty,1)$-category of $(\infty,1)\text{–}\mathfrak{Cat}^{\mathfrak{cocorr}} \times_{(\infty,1)\text{–}\mathfrak{Cat}} (\infty,1)\text{–}\mathfrak{Cat}^2$ on those diagrams $\mathfrak{O}^{el} \hookrightarrow \mathfrak{O}^{inrt} \hookrightarrow \mathfrak{O} \hookleftarrow \mathfrak{O}^{act}$ which define an algebraic pattern.

Let $\mathfrak{O}$ and $\mathfrak{P}$ be algebraic patterns. We thus see that a **morphism of algebraic patterns** from $\mathfrak{O}$ to $\mathfrak{P}$ is an $(\infty,1)$-functor $\mathfrak{O} \to \mathfrak{P}$ which preserves active and inert morphisms and elementary objects.

Example 2.1.1.1.8. There is a functor $\mathbb{\Delta} \to \Omega$ which, taking the standard skeleton of $\mathbb{\Delta}$, sends $[n]$ to the linear tree with n nodes and $n+1$ edges. One may remark that it is fully faitful, and can also be identified with the canonical projection functor $\Omega_{/\eta} \to \Omega$. By translating the definition of inert and active morphisms for the given factorisation system on $\mathbb{\Delta}^{op}$, one immediately sees that this functor induces morphisms of algebraic patterns $\mathbb{\Delta}^{op\natural} \to \Omega^{op\natural}$ and $\mathbb{\Delta}^{op\flat} \to \Omega^{op\flat}$.

Example 2.1.1.1.9. There is a functor $\Omega \to \Upsilon$, forgetting the rooting of a tree. Once again it clearly induces a morphism of algebraic for each of the mutually compatible pattern structures exhibited on these categories.

Definition 2.1.1.1.10 (Segal objects)**.** Let $\mathfrak{O}$ be an algebraic pattern. An $(\infty,1)$-category $\mathfrak{C}$ is said to be **$\mathfrak{O}$-complete** if it admits limits of diagrams with shape $\mathfrak{O}^{\mathrm{el}}_{O/}$ for any $O \in \mathfrak{O}$.

Let $\mathfrak{C}$ be an $\mathfrak{O}$-complete $(\infty,1)$-category. A **Segal $\mathfrak{O}$-object** in $\mathfrak{C}$ is an $(\infty,1)$-functor $\mathscr{X}\colon \mathfrak{O} \to \mathfrak{C}$ such that $\mathscr{X}|_{\mathfrak{O}^{\mathrm{inrt}}}$ is a lax extension of $\mathscr{X}|_{\mathfrak{O}^{\mathrm{el}}}$ (along the inclusion). Explicitly, this means that for any $O \in \mathfrak{O}$ the canonical map

$$\mathscr{X}(O) \to \varprojlim_{E \in \mathfrak{O}^{\mathrm{el}}_{O/}} \mathscr{X}(E) \tag{2.2}$$

is invertible.

The $(\infty,1)$-category $\mathfrak{Seg}_{\mathfrak{O}}(\mathfrak{C})$ of Segal $\mathfrak{O}$-objects in $\mathfrak{C}$ is the full sub-$(\infty,1)$-category of $\mathfrak{C}^{\mathfrak{O}}$ spanned by the Segal objects.

Example 2.1.1.1.11.
- A Segal $\mathbb{\Gamma}^{\mathrm{op}\flat}$-object is a commutative algebra object (or $\mathscr{E}_\infty$-algebra object).
- A Segal $\mathbb{\Delta}^{\mathrm{op}\flat}$-object is an associative algebra object (or $\mathscr{A}_\infty$-algebra object, or $\mathscr{E}_1$-algebra), while a Segal $\mathbb{\Delta}^{\mathrm{op}\natural}$-object is an internal category.
- A Segal $\mathbb{\Omega}^{\mathrm{op}\flat}$-object is an internal monochromatic operad, while a Segal $\mathbb{\Omega}^{\mathrm{op}\natural}$-object is an internal (coloured) operad.
- A Segal $\mathbb{\Upsilon}^{\mathrm{op}\flat}$-object is an internal monochromatic modular operad, while a Segal $\mathbb{\Upsilon}^{\mathrm{op}\natural}$-object is an internal (coloured) modular operad.

Remark 2.1.1.1.12. Let $\mathfrak{O}$ be an algebraic pattern such that the inclusion $\mathfrak{O}^{\mathrm{el}} \hookrightarrow \mathfrak{O}^{\mathrm{inrt}}$ is codense. Then for any $O \in \mathfrak{O}$, the corepresentable $よ_{\mathfrak{O}^{\mathrm{op}}}(O)\colon \mathfrak{O} \to \infty\text{–}\mathfrak{Grpd}$ is a Segal $\mathfrak{O}$-∞-groupoid; this is an immediate consequence of example 1.1.2.2.9. It should be viewed as the Segal object generated by O.

Example 2.1.1.1.13. Over $\mathbb{\Delta}^{\mathrm{op}}$, the Segal object $よ[n]$ corresponds to the linear category $\mathbb{n}+\mathbb{1}$ with n successive arrows.

Example 2.1.1.1.14. Over $\mathbb{\Omega}^{\mathrm{op}}$ (respectively $\mathbb{\Upsilon}^{\mathrm{op}}$), the Segal object $よ\star_n$ generated by the corolla with $n+1$ flags is also denoted $\star_n$ and called the corolla. It corresponds to the operad (resp. modular operad) with $n+1$ colours $O_1,\dots,O_{n+1}$ and, for each permutation $\sigma \in \mathfrak{S}_n$ (resp. $\in \mathfrak{S}_{n+1}$) a single operation of signature $(O_{\sigma(1)},\dots,O_{\sigma(n)};O_{n+1})$ (resp. $(O_{\sigma(1)},\dots,O_{\sigma(n+1)})$).

Definition 2.1.1.1.15 (Segal morphism of algebraic patterns)**.** A morphism of algebraic patterns $\mathscr{F}\colon \mathfrak{O} \to \mathfrak{P}$ is said to be a **Segal morphism** if "it preserves Segal conditions", that is if for any $\mathfrak{P}$-complete $(\infty,1)$-category $\mathfrak{C}$, the induced $(\infty,1)$-functor $\mathscr{F}^*|_{\mathfrak{Seg}_{\mathfrak{P}}}\colon \mathfrak{Seg}_{\mathfrak{P}}(\mathfrak{C}) \subset \mathfrak{C}^{\mathfrak{P}} \to \mathfrak{C}^{\mathfrak{O}}$ factors through $\mathfrak{Seg}_{\mathfrak{O}}(\mathfrak{C}) \hookrightarrow \mathfrak{C}^{\mathfrak{O}}$.

Remark 2.1.1.1.16. By [CH21, Lemma 4.5], it is enough to check Segality of a morphism with $\mathfrak{C} = \infty\text{–}\mathfrak{Grpd}$, that is to check preservation of Segal ∞-groupoids. Then the condition for $\mathscr{F}$ to be a Segal morphism can be written in a formula as: for every $O \in \mathfrak{O}$, for every Segal $\mathscr{P}$-∞-groupoid $\mathscr{X}$, the morphism of ∞-groupoids

$$\varprojlim_{\mathfrak{P}^{\mathrm{el}}_{\mathscr{F}(O)/}} \mathscr{X} \to \varprojlim_{\mathfrak{O}^{\mathrm{el}}_{O/}} \mathscr{X} \circ \mathscr{F}^{\mathrm{el}} \tag{2.3}$$

induced by the $(\infty,1)$-functor $\mathfrak{O}^{\mathrm{el}}_{O/} \to \mathfrak{P}^{\mathrm{el}}_{\mathscr{F}(O)/}$ is an equivalence.

Theorem 2.1.1.1.17 ([CH21, Proposition 8.1]). *Let $\mathfrak{C}$ be a locally presentable $(\infty,1)$-category. Let $\mathscr{F}: \mathfrak{O} \to \mathfrak{P}$ be an essentially surjective Segal morphism of algebraic patterns. The $(\infty,1)$-functor $\mathscr{F}^*$ admits a left-adjoint, and the adjunction is monadic.*

Example 2.1.1.1.18. In the case where $\mathscr{F}$ is the inclusion of the wide sub-$(\infty,1)$-category $\mathfrak{O}^{\mathrm{inrt}} \hookrightarrow \mathfrak{O}$. We find that $\mathfrak{Seg}_{\mathfrak{O}}(\mathfrak{C})$ is monadic over $\mathfrak{C}^{\mathfrak{O}^{\mathrm{el}}}$.

2.1.1.2 Enriched operadic structures

In this section we recall the constructions[1] of the forthcoming paper [CH22].

If $\mathfrak{V}^{\otimes}$ is a symmetric monoidal $(\infty,1)$-category and $\mathscr{O}$ is, say, a $\mathfrak{V}^{\otimes}$-enriched ∞-operad, one would expect $\mathscr{O}$ to have an ∞-groupoid $\mathscr{O}(\eta) \in \infty{-}\mathfrak{Grpd}$ of colours but objects $\mathscr{O}(\star_n) \in \mathfrak{V}$ of operations. In that capacity, the would-be Segal decompositions do not make sense to write as the values of $\mathscr{O}$ on elementaries can live in different $(\infty,1)$-categories. The idea of [CH22] is to separate the two kinds of elementary objects η and $\star_n$ and to recast the general Segal condition as a family of "monochromatic" Segal conditions, which are finite-product conditions and thus make sense in $\mathfrak{V}^{\otimes}$, indexed by $\mathscr{O}(\eta)$.

Definition 2.1.1.2.1 (Cartesian pattern). A **cartesian pattern** is an algebraic pattern $\mathfrak{O}$ endowed with a morphism of patterns $|{-}|\colon \mathfrak{O} \to \mathbb{\Gamma}^{\mathrm{op}^\flat}$ such that for any $O \in \mathfrak{O}$, the morphism $\mathfrak{O}^{\mathrm{el}}_{O/} \to \mathbb{\Gamma}^{\mathrm{op}^\flat,\mathrm{el}}_{|O|/}$ is an equivalence.

Remark 2.1.1.2.2. Viewing $\mathbb{\Gamma}^{\mathrm{op}}$ as (the standard skeleton of) the category of pointed finite sets, one verifies that any $\langle n\rangle \in \mathbb{\Gamma}^{\mathrm{op}}$ admits exactly n inert morphisms to the unique elementary $\langle 1\rangle$, the pointed morphisms $\rho_i\colon \langle n\rangle \to \langle 1\rangle$ mapping i to 1 and every other element of $\langle n\rangle$ to the base-point. The condition of being a cartesian pattern then means that, for any $O \in \mathfrak{O}$, the $(\infty,1)$-category $\mathfrak{O}^{\mathrm{el}}_{O/}$ must be equivalent to the discrete set of the (essentially unique) lifts $\rho_{i,!}$ of the ρ_i. In particular, the Segal condition for a precosheaf $\mathscr{X}$ on $\mathfrak{O}$ is constrained to being the finite product condition

$$\mathscr{X}(O) \simeq \prod_{i=1}^{|O|} \mathscr{X}(\rho_{i,!}O). \tag{2.4}$$

Example 2.1.1.2.3. There is a functor $\mathbb{\Delta}^{\mathrm{op}} \to \mathbb{\Gamma}^{\mathrm{op}}$ mapping $[n]$ to $\langle n\rangle$ and sending an arrow of $\mathbb{\Delta}^{\mathrm{op}}$ corresponding to $\phi\colon [n] \to [m]$ in $\mathbb{\Delta}$ to $|\phi|\colon \langle m\rangle \to \langle n\rangle$ given by

$$|\phi|(i) = \begin{cases} j & \text{if } \phi(j-1) < i \leqslant \phi(j) \\ * & \text{otherwise.} \end{cases} \tag{2.5}$$

It can be checked directly that this is a structure of enrichable pattern on $\mathbb{\Delta}^{\mathrm{op}^\flat}$.

Example 2.1.1.2.4. A functor $\mathbb{\Omega}^{\mathrm{op}} \to \mathbb{\Gamma}^{\mathrm{op}} \simeq \mathfrak{FinSet}_*$ is defined in [Enriched operads, Definition 4.1.16] in the following way. A tree T with set of vertices $V(T)$ is mapped to the freely pointed set $V(T)_+$. A morphism $T' \leftarrow T$ in $\mathbb{\Omega}^{\mathrm{op}}$ is mapped to the pointed morphism $V(T')_+ \to V(T)_+$ which sends a vertex $v \in V(T')$ to the unique vertex of T whose image subtree contains v, or to the basepoint if there is no such vertex. By [loc. cit., Lemma 4.1.18], this functor preserves the inert-active factorisation system, aand thus defines a morphism of algebraic patterns.

proposition 2.1.2.1.20

[1] which were presented in the seminar talk available at `https://www.msri.org/seminars/25057`

Lemma 2.1.1.2.5 ([CH21, Corollary 5.5]). *The $(\infty,1)$-category of algebraic patterns as defined in construction 2.1.1.1.7 admits all limits and filtered colimits, and they are created by the forgetful $(\infty,1)$-functor to $(\infty,1)-\mathfrak{Cat}^{\mathfrak{cocorr}} \times_{(\infty,1)-\mathfrak{Cat}} (\infty,1)-\mathfrak{Cat}^{2}$ and preserved by the forgetful $(\infty,1)$-functor to $(\infty,1)-\mathfrak{Cat}$.*

Definition 2.1.1.2.6 (Enrichable pattern). An **enrichable pattern** is an algebraic pattern $\mathcal{O}$ equipped with a morphism of patterns $|-|\colon \mathcal{O} \to \Gamma^{\mathrm{op}\natural}$ such that $(\mathcal{O} \times_{\Gamma^{\mathrm{op}\natural}} \Gamma^{\mathrm{op}\flat}, |-| \times_{\Gamma^{\mathrm{op}\natural}} \Gamma^{\mathrm{op}\flat})$ is a cartesian pattern.

Thanks to lemma 2.1.1.2.5, we see that the fibre product $\mathcal{O} \times_{\Gamma^{\mathrm{op}\natural}} \Gamma^{\mathrm{op}\flat}$ appearing in the definition of enrichable patterns has a very simple description: it consists of the category $\mathcal{O}$ equipped with its same factorisation system, and the choice of only those elementary objects living over $\langle 1 \rangle \in \Gamma^{\mathrm{op}}$ (*i.e.* excluding those over $\langle 0 \rangle$).

Example 2.1.1.2.7. All the $\natural$-decorated patterns.

Lemma 2.1.1.2.8. *Let $\mathcal{O}$ be an enrichable pattern, and let $i\colon \mathcal{O}_0 \hookrightarrow \mathcal{O}$ denote the inclusion of the full sub-$(\infty,1)$-category on the objects lying over $\langle 0 \rangle$. The right adjoint $i_* = \mathrm{Lex}_i$ to i^*, given by lax extension along i, is a Segal morphism.*

Construction 2.1.1.2.9. Let $\mathcal{X}$ be a Segal $\mathcal{O}_0$-object in $\infty-\mathfrak{Grpd}$. We let $\mathcal{O}_{\mathcal{X}} \to \mathcal{O}$ denote the discrete cocartesian fibration corresponding to $i_*\mathcal{X}\colon \mathcal{O} \to \infty-\mathfrak{Grpd}$.

The pattern structures on $\mathcal{O}$ and $\mathcal{O}^{\flat} := \mathcal{O} \times_{\Gamma^{\mathrm{op}\natural}} \Gamma^{\mathrm{op}\flat}$ may be lifted along $\mathcal{O}_{\mathcal{X}} \to \mathcal{O}$ to give two algebraic patterns denoted $\mathcal{O}_{\mathcal{X}}$ and $\mathcal{O}^{\flat}_{\mathcal{X}}$.

Proposition 2.1.1.2.10. *The assignment $\mathcal{X} \mapsto \mathcal{O}^{\flat}_{\mathcal{X}}$ is functorial in $\mathcal{X} \in \mathfrak{Seg}_{\mathcal{O}_0}(\infty-\mathfrak{Grpd})$.*

Hence $\mathcal{X} \mapsto \mathfrak{Seg}_{\mathcal{O}^{\flat}_{\mathcal{X}}}(\infty-\mathfrak{Grpd})$ defines an $(\infty,1)$-functor $\mathfrak{Seg}_{\mathcal{O}_0}(\infty-\mathfrak{Grpd}) \to (\infty,1)-\mathfrak{Cat}$. We let $\mathfrak{Seg}_{\mathcal{O}^{\flat}_{\bullet}}(\infty-\mathfrak{Grpd}) \to \mathfrak{Seg}_{\mathcal{O}_0}(\infty-\mathfrak{Grpd})$ denote the associated cartesian fibration.

Theorem 2.1.1.2.11 ([CH22]). *For any enrichable pattern $\mathcal{O}$, there is an equivalence of $(\infty,1)$-categories $\mathfrak{Seg}_{\mathcal{O}}(\infty-\mathfrak{Grpd}) \simeq \mathfrak{Seg}_{\mathcal{O}^{\flat}_{\bullet}}(\infty-\mathfrak{Grpd})$.*

Definition 2.1.1.2.12 ($\mathcal{O}$-monoidal ∞-category). Let $\mathcal{O}$ be an algebraic pattern. An **$\mathcal{O}$-monoidal $(\infty,1)$-category** is a Segal $\mathcal{O}$-object in $(\infty,1)-\mathfrak{Cat}$.

Construction 2.1.1.2.13. Suppose $\mathcal{V}\colon \mathcal{O} \to (\infty,1)-\mathfrak{Cat}$ is an $\mathcal{O}$-monoidal $(\infty,1)$-category. Passing to the Grothendieck construction of the functor produces a cocartesian fibration $\int\mathcal{V} \to \mathcal{O}$. We shall refer to a cocartesian fibration over $\mathcal{O}$ whose associated $(\infty,1)$-functor $\mathcal{O} \to (\infty,1)-\mathfrak{Cat}$ satisfies the Segal conditions as a **Segal fibration** over $\mathcal{O}$.

By [Lur17, Proposition 2.1.2.5], if $\int\mathcal{V} \to \mathcal{O}$ is a Segal $\mathcal{O}$-fibration, the inert-active factorisation system on $\mathcal{O}$ lifts to one on $\int\mathcal{V}$, endowing it with a structure of algebraic pattern. If $\mathfrak{P} \to \mathcal{O}$ is another algebraic pattern over $\mathcal{O}$, we shall define a **$\mathfrak{P}$-algebra** in $\int\mathcal{V}$ to be a morphism of algbraic pattern over $\mathcal{O}$ from $\mathfrak{P}$ to $\int\mathcal{V}$.

Definition 2.1.1.2.14 (Algebroid). Let $\mathcal{O}$ be an enrichable algebraic pattern and let $\mathcal{V}$ be an $\mathcal{O}$-monoidal $(\infty,1)$-category. An $\mathcal{O}$-algebroid in $\mathcal{V}$ is an $\mathcal{O}_{\mathcal{X}}$-algebra in $\mathcal{V}$ for some $\mathcal{X} \in \mathfrak{Seg}_{\mathcal{O}_0}(\infty-\mathfrak{Grpd})$.

2.1.1.3 Lax morphisms and generalised Segal fibrations

Definition 2.1.1.3.1 (Weak Segal fibration). Let $\mathcal{O}$ be an algebraic pattern. A **weak Segal $\mathcal{O}$-fibration** (also called $\mathcal{O}$-operad) is an $(\infty,1)$-functor $p\colon \mathfrak{X} \to \mathcal{O}$ such that:

1. for every object $X \in \mathfrak{X}$, every inert arrow $i\colon pX \to O$ in $\mathfrak{O}$ admits a p-cocartesian lift $i_!\colon X \to i_!X$;

2. for every object $O \in \mathfrak{O}$, the $(\infty,1)$-functor $\mathfrak{X}_O \to \varprojlim_{E\in\mathfrak{O}^{el}_{O/}} \mathfrak{X}_E$ induced by the cocartesian morphisms over inert arrows is invertible;

3. for every $X \in \mathfrak{X}$ and every choice of p-cocartesian lift of the tautological diagram $\iota\colon \mathfrak{O}^{el}_{pX/} \to \mathfrak{O}$ (of inert morphisms from pX) to an $\iota_!\colon (\mathfrak{O}^{el}_{pX/})^{\triangleleft} \to \mathfrak{X}$ taking the cone point to X, for every $Y \in \mathfrak{X}$, the commutative square

$$\begin{array}{ccc} \mathfrak{X}(Y,X) & \longrightarrow & \varprojlim_{E\in\mathfrak{O}^{el}_{pX/}} \mathfrak{X}(Y,\iota_!(E)) \\ \downarrow & & \downarrow \\ \mathfrak{O}(pY,pX) & \longrightarrow & \varprojlim_{E\in\mathfrak{O}^{el}_{pX/}} \mathfrak{O}(pY,\iota(E)=E) \end{array} \tag{2.6}$$

is cartesian.

Example 2.1.1.3.2.
- A weak Segal $\Gamma^{op\flat}$-fibration is the $(\infty,1)$-category of operators of an ∞-operad in the sense of [Lur17, Definition 2.1.1.10], while a weak Segal $\Gamma^{op\natural}$-fibration is a generalised ∞-operad in the sense of [Lur17, Definition 2.3.2.1].

- A weak Segal $\Delta^{op\flat}$-fibration is the $(\infty,1)$-category of operators of a non-symmetric ∞-operad in the sense of [GH15, Definition 2.2.6, Definition 3.1.3] while a weak Segal $\Delta^{op\natural}$-fibration is virtual double ∞-category, or generalised ∞-operad in [GH15, Definition 2.4.1, Definition 3.1.13], recovering definition 1.1.3.1.1.

Definition 2.1.1.3.3 (Morphisms of weak Segal fibrations)**.** By [CH21], the source $\mathfrak{X}$ of a weak Segal $\mathfrak{O}$-fibration $p\colon \mathfrak{X} \to \mathfrak{O}$ inherits an algebraic pattern structure in which active morphisms are those lying over an active morphism in $\mathfrak{O}$, inert morphisms are the p-cocartesian morphisms lying over inert arrows of $\mathcal{O}$, and elementaries are the objects lying over elementary objects.

The $(\infty,2)$-category of weak Segal $\mathfrak{O}$-fibrations is the locally full sub-$(\infty,2)$-category of $(\infty,1)\text{−}\mathfrak{Cat}_{/\mathfrak{O}}$ spanned by the weak Segal $\mathfrak{O}$-fibrations and Segal morphisms thereof.

It can be checked directly that any Segal fibration as in construction 2.1.1.2.13 is in particular a weak Segal fibration. In fact Segal $\mathfrak{O}$-fibrations are exactly those $(\infty,1)$-functors to $\mathfrak{O}$ which are both weak Segal fibrations and cocartesian fibrations. This provides a (non-full) inclusion $(\infty,2)$-functor from the $(\infty,2)$-category of Segal fibrations into that of weak Segal fibrations.

Lemma 2.1.1.3.4. *The inclusion $(\infty,2)$-functor is monadic.*

Proof. We first note that it is indeed a right-adjoint; this follows from the construction of the envelope in subsection 2.1.2.2 and in [Lur17]. It is in addition conservative, as any (*a priori* lax) equivalence is necessarily strong. □

Proposition 2.1.1.3.5. *Lax morphisms of Segal $\mathfrak{O}$-∞-categories are exactly the morphisms of the underlying Segal $\mathfrak{O}$-fibrations.*

Proof. This is equivalent to saying that the lax morphisms classifier of a Segal $\mathfrak{O}$-∞-category is its image under the comonad induced by the adjunction. □

Example 2.1.1.3.6. In the case of the pattern $\mathbb{F}^{\mathrm{op}^\flat}$, we obtain that lax morphisms of symmetric monoidal $(\infty, 1)$-categories are the same as "∞-operads maps" between the $(\infty, 1)$-category of operators of their underlying ∞-operads, as in [Lur17, Before Definition 2.1.3.7]

Example 2.1.1.3.7. In the case of the pattern $\Delta^{\mathrm{op}^\natural}$, we find that lax morphisms of double ∞-categories are morphisms of their underlying virtual double ∞-categories, generalising the identification made in [DPP06, Theorem 2.6].

2.1.2 Monoidal envelopes of Segal objects

Any monoidal category $\mathcal{V}^\otimes$ defines an operad whose colours are the objects of $\mathcal{V}$ and whose multimorphisms $C_1, \cdots, C_n \to D$ are given by the morphisms $C_1 \otimes \cdots \otimes C_n \to D$. The operads which arise in this way are said to be representable; we will call them *representably monoidal*. Indeed, the tautological mutlimorphism $C_1, \cdots, C_n \to C_1 \otimes \cdots \otimes C_n$ (corresponding to $\mathrm{id}_{C_1 \otimes \cdots \otimes C_n}$) carries the universal property that every multimorphism with source $(C_1, \cdots, C_n)$ must factor through it, by a unique unary morphism of source $C_1 \otimes \cdots \otimes C_n$. This universal property can be seen as a cocartesianity condition.

Recall indeed that, if $p \colon \mathfrak{E} \to \mathfrak{B}$ is a cocartesian fibration (or opfibration), there is a factorisation system on $\mathfrak{E}$ whose left class consists of p-cocartesian morphisms and whose right class consists of purely p-vertical morphisms. One can also define a notion of opfibration of multicategories, as done for example in [Her04] for generalised multicategories, and more generally of cocartesian (multi)morphisms therein, so that a multicategory is representably monoidal if and only if its morphism to the terminal operad is an opfibration. In general, an opfibration of multicagories can be thought of as a morphism whose fibres are monoidal categories: since the selection of a colour comes from the operad generated by η, which only has unary morphisms, the vertical arrows are always unary one, while the cocartesian arrows are those exhibiting tensor products.

Thus one would like to define a notion of cartesian operations and fibrations for Segal objects over algebraic patterns, and in addition construct the free fibration generated by an arbitrary morphism. But to do this requires a notion of direction for the operations of Segal objects: in categories, there are two directions for cartesianity, from the source or from the target (giving rise to cartesian and cocartesian morphisms), while for operads the notion of (co)cartesian morphisms which has so far been explored looks from the target (as in definition 1.1.1.2.3), but we expect that any choice of input to separate would give a notion of cartesianity. Not all choices of direction are as good as others: for operads, the choice of the output to keep apart leads to functoriality for composition, while the other choices do not (due to the absence of a duality operation as for categories). As a consequence, one may need to restrict the study to "good" orientations; this seems likely to be impossible to find for modular operads.

In this work, we have elected to eschew the problem by replacing a perhaps more canonical definition of directability by a more practical one. The presence of a notion of direction for operations means that we can think of any composite of them as having an order of progression. This is what is captured by the objects of the simplex category Δ, chains of morphisms going from a beginning to an end. Hence we will base our notion of direction on this category, declaring an algebraic pattern to be *well-directed* if it can be written as the output of a certain construction involving Δ. The appropriate construction to consider turns out to be a variant of the plus construction suggested by Baez–Dolan and studied by, among others, [Bar18] and [Ber21].

In subsection 2.1.2.1, we define this plus construction for appropriately complete algebraic patterns. Then, in subsection 2.1.2.2, we will use it to construct the "representably monoidal" envelope of a Segal object. While the construction makes sense for general well-directed patterns, we are only able to exhibit its good monoidal properties by restricting to the pattern $\Omega^{\mathrm{op}^\flat}$ for operads.

2.1.2.1 The plus construction

Definition 2.1.2.1.1 (Semi-inert morphism)**.** An arrow $f\colon O \to O'$ of $\mathbb{O}$ is **semi-inert** if it is weakly left-orthogonal to the class of active morphisms, that is: for any active morphism $P \rightsquigarrow P'$ and any commutative square

$$\begin{array}{ccc} O & \longrightarrow & P \\ {\scriptstyle f}\downarrow & \nearrow & \wr\downarrow \\ O' & \longrightarrow & P', \end{array} \tag{2.7}$$

there exists a dashed lift to a commutative diagram.

We record a pair of examples which are known from their explicit descriptions to be of independent interest.

Proposition 2.1.2.1.2. *In Γ^{op}, seen as the category of finite pointed sets, semi-inert morphisms are exactly the semi-inert maps in the sense of [Lur17].*

Proof. We need to show that the semi-inert maps in $\Gamma^{op} \simeq \mathfrak{FinSet}_*$ are weakly left-orthogonal to the active maps.

Let $f\colon \langle m\rangle \to \langle n\rangle$ be a map of finite pointed sets, whose weak left-orthogonality to active maps we wish to test. Assume that there is some $i \in \langle n\rangle \setminus \{0\}$ such that the cardinality of $f^{-1}(\{i\})$ is strictly greater than 1, and pick two distinct elements $a, b \in f^{-1}(\{i\})$. Consider the unique active map $\langle 2\rangle \to \langle 1\rangle$ as well as maps $\langle m\rangle \to \langle 2\rangle$ mapping a to 1 and b to 2, and $\langle n\rangle \to \langle 1\rangle$ sending i to 1, fitting in the commuting square

$$\begin{array}{ccc} \langle m\rangle & \longrightarrow & \langle 2\rangle \\ {\scriptstyle f}\downarrow & \overset{?}{\nearrow} & \wr\downarrow{\scriptstyle !} \\ \langle n\rangle & \xrightarrow[i\mapsto 1]{} & \langle 1\rangle. \end{array} \tag{2.8}$$

It is impossible to have any dashed filler $\langle n\rangle \to \langle 2\rangle$ rendering the upper triangle commuting. Hence f is not weakly left-orthogonal to all active maps. ☐

Proposition 2.1.2.1.3. *In Δ^{op}, the semi-inert morphisms are those corresponding to the cellular morphisms of Δ as defined in definition 1.1.3.2.2*

Proof. We need to show that the cellular maps in Δ are weakly right-orthogonal to the boundary preserving maps.

Let $f\colon [m] \to [n]$ whose weak right orthogonality to all boundary-preserving morphisms we wish to check. Suppose that there is some $i \in [m]$ such that $f(i+1) > f(i)+1$. Consider $[0] \to [1]$ selecting 0 (which is an active morphism) as well as the maps $[0] \to [m]$ selecting i and $[1] \to [n]$ mapping $0 < 1$ to $f(i) < f(i)+1$, all forming a nicely commuting solid square

$$\begin{array}{ccc} [0] & \xrightarrow{\{i\}} & [m] \\ {\scriptstyle \{1\}}\wr\downarrow & \overset{?}{\nearrow} & \downarrow{\scriptstyle f} \\ [1] & \xrightarrow[\{f(i),f(i)+1\}]{} & [n]. \end{array} \tag{2.9}$$

Any dashed filler $[1] \to [m]$ must, so that the upper-left triangle commute, send $0 \in [1]$ to $i \in [m]$, which means that the image of 1 will be either i or greater than i. In either case, its further image by f will be (respectively) either strictly smaller or strictly bigger than the only value $f(i)+1$ which would make the lower-right triangle commute, so the filler cannot exist and f is not weakly right-orthogonal to boundary-preserving maps. ☐

Definition 2.1.2.1.4. An algebraic pattern $\mathcal{O}$ is **of operator type** if its active sub-$(\infty,1)$-category $\mathcal{O}^{act}$ admits pullbacks.

Example 2.1.2.1.5. By [Ber21, Lemma 1.16], algebraic patterns coming from moment categories (as defined in [Ber21]) are of operator type. This is the case of $\mathbb{F}^{op}$ and Δ^{op}.

Remark 2.1.2.1.6. For the plus construction, it is likely sufficient to work with a weaker condition, such as requiring only the existence of a choice of fibres rather than requiring them to be fibre products, as is done in the operadic categories of [BM21].

The following construction is a variant of one due to [Bar18] in the setting of operator categories, inspired by the plus construction or "slice operads" of [BD98], and which was also studied in [Ber21] in the setting of hypermoment categories and [BM21] for operadic categories, and used in [CHH18].

Construction 2.1.2.1.7. Let $\mathcal{O}$ be an algebraic pattern of operator type. Consider the restriction $よ_{(\infty,1)-\mathfrak{Cat},\mathcal{O}^{act}}|_{\Delta}$ to $\Delta \subset (\infty,1)-\mathfrak{Cat}$ of the $(\infty,1)$-functor represented by $\mathcal{O}^{act}$ and let $\Delta^{pre}_{\mathcal{O}} \to \Delta$ be its Grothendieck construction. Thus an

object of $\Delta^{pre}_{\mathcal{O}}$ consists of a pair $([n], O_\bullet)$ where $[n] \in \Delta$ and $O_\bullet \colon [n] \to \mathcal{O}^{act}$ is a linear diagram in $\mathcal{O}^{act}$, that is a sequence $O_0 \rightsquigarrow O_1 \rightsquigarrow \cdots \rightsquigarrow O_n$ of active morphisms in $\mathcal{O}$, while a

morphism $([n], O_\bullet) \to ([n'], O'_\bullet)$ consists of a pair $(\phi, f_\bullet)$ where $\phi \colon [n] \to [n']$ is a map in Δ and $f_\bullet \colon O_\bullet \Rightarrow O'_{\phi(\bullet)} = O'_\bullet \circ \phi$ is a natural transformation of $[n]$-shaped diagrams in $\mathcal{O}^{act}$.

We define $\Delta_{\mathcal{O}}$ to be the wide and locally full (*i.e.*, recall from the conventions, containing all objects and all higher morphisms between a selection of 1-morphisms) sub-$(\infty,1)$-category of $\Delta^{pre}_{\mathcal{O}}$ on those morphisms $(\phi, f_\bullet)$ such that

- $f_\bullet$ is component-wise semi-inert (in addition to active), that is each $f_i \colon O_i \rightsquigarrow O'_{\phi(i)}$ (for $i \in [n]$) is semi-inert, and
- $f_\bullet$ is a cartesian natural transformation, that is for any morphism $i < j$ in $[n]$ the naturality square

$$\begin{array}{ccc} O_i & \overset{f_i}{\rightsquigarrow} & O'_{\phi(i)} \\ {\scriptstyle O_{i<j}} \wr\downarrow & & \wr\downarrow {\scriptstyle O'_{\phi(i<j)}} \\ O_j & \underset{f_j}{\rightsquigarrow} & O'_{\phi(j)} \end{array} \tag{2.10}$$

  is cartesian (in $\mathcal{O}^{act}$, where such pullbacks exist thanks to the operator type hypothesis on $\mathcal{O}$).

Remark 2.1.2.1.8. Given fixed $\phi \colon [m] \to [n]$ in Δ and $([n], P_\bullet)$ over $[n]$, morphisms $(\phi, f_\bullet) \colon ([m], O_\bullet) \to ([n], P_\bullet)$ in $\Delta_{\mathcal{O}}$ lifting ϕ are essentially determined by their underlying arrow $f_m \colon O_m \to P_{\phi(m)}$. Indeed, for each $i \in [m]$ the object O_i and arrow f_i are required by the pullback condition in equation (2.10) to be the base-change of O_m and f_m along $i \leqslant m$.

Definition 2.1.2.1.9. Let $\mathcal{O}$ be any algebraic pattern. The $(\infty,1)$-category $\Delta_{\mathcal{O}}$ is called the $(\infty,1)$-category of **$\mathcal{O}$-forests**.

Let $|\cdot|\colon \mathcal{O} \to \Gamma^{\mathrm{op}\natural}$ be a structure of enrichable pattern on $\mathcal{O}$. The full sub-$(\infty,1)$-category $\Delta^1_{\mathcal{O}}$ of $\Delta_{\mathcal{O}}$ on the $([n], O_\bullet)$ such that $O_n \in \mathcal{O}^{\flat,\mathrm{el}}$ is called the $(\infty,1)$-category of **$\mathcal{O}$-trees**.

Example 2.1.2.1.10. For the terminal algebraic pattern $*$, the $(\infty,1)$-categories of $*$-forests and of $*$-trees both recover the simplex category Δ.

Example 2.1.2.1.11. For $\Gamma^{\mathrm{op}\flat}$, we obtain a category of forests, *i.e.* disjoint unions of trees, with level structures, whose tree-like subcategory is identified in [Ber21] with a full subcategory of Ω. Indeed, recalling that the active subcategory of Γ^{op} is equivalent to the category of finite sets, the object O_n is to be thought of as the set of roots of the forest, and each O_i is the set of leaves at level $n-i$. The morphisms in $(\Gamma^{\mathrm{op}})^{\mathrm{act}}$ give partitions of the leaves at levels ℓ corresponding to the node (recognised by its unique output leaf at level $\ell+1$) to which they lead.

Example 2.1.2.1.12. For $\Delta^{\mathrm{op}\flat}$, we similarly obtain a category of planar trees (or rather, forests thereof) with level structures.

Lemma 2.1.2.1.13. *The ∞-functor $d_{\mathcal{O}}\colon \Delta_{\mathcal{O}} \subset \Delta^{\mathrm{pre}}_{\mathcal{O}} \to \Delta$ is a cartesian fibration.*

Proof. We first observe as in[CHH18] that, due to the pullback condition in equation (2.10), a morphism $(\phi, f_\bullet)\colon ([m], O_\bullet) \to ([n], P_\bullet)$ in $\Delta_{\mathcal{O}}$ is $d_{\mathcal{O}}$-cartesian if and only if $f_\bullet$ is a natural equivalence.

Let $\phi\colon [m] \to [n]$ be a morphism in Δ, and let $([n], O_\bullet)$ be a lift of $[n]$ in $\Delta_{\mathcal{O}}$. We define a cartesian lift of $([n], O_\bullet)$ along ϕ to be $([m], O_{\phi(\bullet)})$ with $(\phi, \mathrm{id}_{O_\bullet \circ \phi})$. ☐

Corollary 2.1.2.1.14. *Say that an arrow $(\phi, f_\bullet)$ in $\Delta_{\mathcal{O}}$ is inert if ϕ is inert in Δ and active if ϕ is active in Δ and $f_\bullet$ is an equivalence. Then $({\Delta_{\mathcal{O}}}^{\mathrm{op\,inrt}}, {\Delta_{\mathcal{O}}}^{\mathrm{op\,act}})$ defines a factorisation system on ${\Delta_{\mathcal{O}}}^{\mathrm{op}}$.*

Proof. By [Lur17, Proposition 2.1.2.5], since $d_{\mathcal{O}}\colon \Delta_{\mathcal{O}} \subset \Delta^{\mathrm{pre}}_{\mathcal{O}} \to \Delta$ is a cartesian fibration, the factorisation system on its base can be lifted as required. ☐

Definition 2.1.2.1.15. Let $\mathcal{O}$ be an algebraic pattern of operator type. Its **plus construction** ${\Delta_{\mathcal{O}}}^{\mathrm{op}\natural}$ is the $(\infty,1)$-category ${\Delta_{\mathcal{O}}}^{\mathrm{op}}$ equipped with the inert-active factorisation of corollary 2.1.2.1.14 and as elementary objects those $([n], O_\bullet)$ with $[n] \in \Delta^{\mathrm{op}\natural,\mathrm{el}}$ (*i.e.* either $[0]$ or $[1]$) and $O_n \in \mathcal{O}^{\mathrm{el}}$.

If $\mathcal{O}$ has a structure of enrichable pattern, its **tree-like plus construction** is the algebraic pattern induced on ${\Delta^1_{\mathcal{O}}}^{\mathrm{op}}$.

We shall say that an algebraic pattern is **well-directed** if it can be written as the plus construction of some algebraic pattern of operator type.

Example 2.1.2.1.16. The main result of [CHH18] shows that ${\Delta_{\Gamma^{\mathrm{op}\flat}}}^{\mathrm{op}\natural}$ is Morita-equivalent to $\Omega^{\mathrm{op}\natural}$ in the sense that their $(\infty,1)$-categories of Segal ∞-groupoids are equivalent. We will henceforth conflate operad objects with Segal ${\Delta_{\Gamma^{\mathrm{op}\flat}}}^{\mathrm{op}\natural}$-objects.

Remark 2.1.2.1.17. The Segal condition for a Segal ${\Delta_{\mathcal{O}}}^{\mathrm{op}\natural}$-object $\mathcal{F}$ can be explicitly written as the following set of conditions:

level decomposition For any $([n], O_\bullet)$, the canonical arrow

$$\mathcal{F}([n], O_\bullet) \to \mathcal{F}([1], (O_0 \rightsquigarrow O_1)) \underset{\mathcal{F}([0],(O_1))}{\times} \cdots \underset{\mathcal{F}([0],(O_{n-1}))}{\times} \mathcal{F}([1], (O_{n-1} \rightsquigarrow O_n)) \quad (2.11)$$

is an equivalence.

forest decomposition For any height-0 forest of the form $([0], O_0)$, the canonical map

$$\mathscr{F}([0],(O_0)) \to \prod_{i=1}^{|O_0|} \mathscr{F}([0],(E_i^{O_0})) \tag{2.12}$$

is an equivalence (where $E_i^O = \rho_{i,!}O$ in the notation of remark 2.1.1.2.2).

$\mathbb{O}$-Segal condition For any $([1],(O_0 \rightsquigarrow O_1))$, the canonical arrow

$$\mathscr{F}([1],(O_0 \rightsquigarrow O_1)) \to \prod_{i=1}^{|O_1|} \mathscr{F}([1], O_{0,i} \to E_i^{O_1}) \tag{2.13}$$

is an equivalence, where $O_{0,i}$ is the fibre product $O_0 \times_{O_1} E_i^{O_1}$.

Lemma 2.1.2.1.18 ([CHH18, Lemma 2.11])**.** *Forests and trees have the same Segal objects: if $\mathbb{O}$ has an enrichable structure, then the $(\infty, 1)$-functor $\mathfrak{Seg}_{\Delta_{\mathbb{O}}^{\mathrm{op}\natural}}(\infty\text{–}\mathfrak{Grpd}) \to \mathfrak{Seg}_{\Delta_{\mathbb{O}}^{1}{}^{\mathrm{op}\natural}}(\infty\text{–}\mathfrak{Grpd})$ of restriction along the inclusion $\Delta^1_{\mathbb{O}} \hookrightarrow \Delta_{\mathbb{O}}$ is an equivalence of $(\infty, 1)$-categories.*

Proof. For $([n], O_\bullet)$ be an $\mathbb{O}$-forest, and denote the decomposition of $|O_n|$ into its fibres. We then write any forest as a union of trees, and use the forest decomposition condition of remark 2.1.2.1.17. □

Construction 2.1.2.1.19. We define a **corolla** of an $\mathbb{O}$-forest $F \in \Delta_{\mathbb{O}}$ to be an equivalence class of inert morphisms $F \to E$ where E is elementary.

We can now define a functor $|{-}|^{\mathrm{op}}\colon \Delta_{\mathbb{O}} \to \Gamma$ by counting the numbers of corollas in an $\mathbb{O}$-forest.

Proposition 2.1.2.1.20. *The functor $|{-}|\colon \Delta_{\mathbb{O}}{}^{\mathrm{op}} \to \Gamma^{\mathrm{op}}$ gives a structure of enrichable pattern on $\Delta_{\mathbb{O}}{}^{\mathrm{op}\natural}$.*

Proof. By direct verification; this follows essentially by definition of corollas. □

2.1.2.2 The monad for monoidal envelopes

Let $\mathscr{O}$ be an operad. Its monoidal envelope is constructed as a monoidal category $\mathscr{E}nv(\mathscr{O})$ which has as set of objects the free monoid generated by the colours of $\mathscr{O}$, whose elements are denoted as $C_1 \otimes \cdots \otimes C_n$ or simply $C_1 \cdots C_n$. If $C_1 \otimes \cdots \otimes C_n$ is such a string of colours of $\mathscr{O}$ and D is one colour, a morphism $C_1 \otimes \cdots \otimes C_n \to D$ is given by a multimorphism $C_1, \dots, C_n \to D$ in $\mathscr{O}$. If $C_1 \otimes \cdots \otimes C_n$ and $D_1 \otimes \cdots \otimes D_m$ are two such strings of colours of $\mathscr{O}$, to defines a morphism $C_1 \otimes \cdots \otimes C_n \to D_1 \otimes \cdots \otimes D_m$ one needs to further select of partition of the inputs $(C_1, \dots, C_n)$ into m (possibly empty) parts.

The above is so far just a description of the underlying category $\mathfrak{Env}(\mathscr{O})$ of the envelope of $\mathscr{O}$; to define it as a monoidal category, or representably monoidal operad, one must also define multimorphisms of higher arity in the operadic structure. Let $(C^i_1 \otimes \cdots \otimes C^i_{p_i})_{i \in [\![1,n]\!]}$ be n colours of $\mathscr{E}nv(\mathscr{O})$ and let $D_1 \otimes \cdots \otimes D_m$ be a further colour. By the representability condition, a multimorphism $(C^1_1 \cdots C^1_{p_1}, \dots, C^n_1 \cdots C^n_{p_n}) \to D_1 \otimes \cdots \otimes D_m$ is given by a morphism $C^1_1 \otimes \cdots \otimes C^n_{p_n} \to D_1 \otimes \cdots \otimes D_m$, that is a partition of the entries and a collection of multimorphisms to each D_i.

In the dendroidal model, one simply defines the object $\mathscr{E}nv(\mathscr{O})(\star_n)$ of all n-ary morphisms without specifying their sources and targets. To describe this, it becomes useful to reverse the thinking: taking a family of multimorphisms of $\mathscr{O}$, we can ask how to interpret it as a multimorphism in $\mathscr{E}nv(\mathscr{O})$. If $C_1, \cdots, C_r$ is the union of the domains of the multimorphisms in the family considered, the decomposition in family provides a partition of p indexed by the targets; however, from the point of view of $\mathscr{E}nv(\mathscr{O})$, this partition is completely artificial as it is only used to construct a morphism whose target may consist of several colours. Thus it must be forgotten. Meanwhile, if the family is to be interpreted as a multimorphism of specified arity n in $\mathscr{E}nv(\mathscr{O})$, the set of colours $(C_i)_{i\in[[1,r]]}$ must be endowed with a partition into n parts.

In a formula, we have that

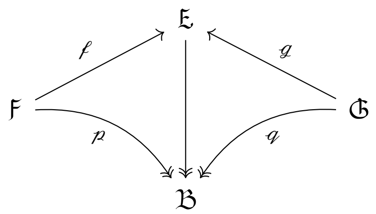
$$\mathscr{E}nv(\mathscr{O})(\star_n) = \coprod_{\substack{r\in\mathbb{N} \\ \lambda \text{ partition of } r \text{ in } n}} \prod_{i=1}^{m} \mathscr{O}(\star_{\lambda(i)}). \tag{2.14}$$

Recall that partitions are the same thing as active morphisms in $\mathbb{F}^{\mathrm{op}}$. We can then interpret the coproduct on partitions as a colimit over morphisms in $(\mathbb{F}^{\mathrm{op}})^{\mathrm{act}}$.

Viewed in this way, this formula is very reminiscent to the one computing oplax extensions, with one difference: the colimit is taken only of the trees whose height is that of a corolla, *i.e.* compatibly with the projection to Δ. To that end, the notion of extension needs to be refined to a "fibrewise" one.

To define the necessary fibrewise extensions, we need to briefly work in the generality of formal ∞-category theory introduced in section 1.1, that is in the framework of ∞-cosmoi, modelling the $(\infty, 2)$-category of ∞-categories. Recall from example 1.1.2.1.5 that oplax extensions along an ∞-functor can be expressed as colimits weighted by the conjoint of this ∞-functor. We will define fibrewise extensions similarly, replacing this conjoint by a relative version.

Construction 2.1.2.2.1 (Relative comma ∞-category)**.** In an ∞-cosmos $\mathfrak{K}$, consider a cocorrespondence in the sliced ∞-cosmos $\mathfrak{K}_{/B}$:



(2.15)

We let $\mathscr{f} \downarrow_{/\mathfrak{B}} \mathscr{g}$ denote the comma object in $\mathfrak{K}_{/B}$, and call it the **relative comma ∞-category** over B.

Lemma 2.1.2.2.2. *The canonical projection $\mathscr{f} \downarrow_{/\mathfrak{B}} \mathscr{g} \to \mathfrak{F} \times \mathfrak{G}$ is a discrete two-sided fibration in $\mathfrak{K}$.*

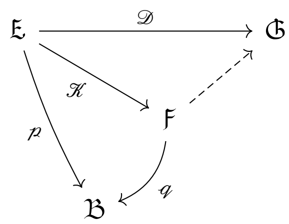
Proof. As in [RV21, Proposition 7.4.6]. □

Remark 2.1.2.2.3. By [RV21, Proposition 1.2.22, (iv)–(vi)], the relative comma ∞-category can be constructed as

$$\mathscr{f} \downarrow_{/\mathfrak{B}} \mathscr{g} \simeq (\mathfrak{F} \times_{\mathfrak{B}} \mathfrak{G}) \underset{(\mathfrak{E}\times_{\mathfrak{B}}\mathfrak{E})}{\times} (\mathfrak{B} \times_{\mathfrak{B}^2} \mathfrak{E}^2) \tag{2.16}$$

(where the map $\mathfrak{B} \to \mathfrak{B}^2$ is the diagonal). That is, informally, an object of $\mathscr{f} \downarrow_{/\mathfrak{B}} \mathscr{g}$ consists of a triple (F, G, α) where F and G are objects of $\mathfrak{F}$ and $\mathfrak{G}$ respectively and $\alpha\colon \mathscr{f}(F) \to \mathscr{g}(G)$ is an arrow in $\mathfrak{E}$, such that all data live above the same object of $\mathfrak{B}$ (*i.e.* there is an object $B \in \mathfrak{B}$ and isomorphisms $\mathscr{p}F \xrightarrow{\simeq} B$, $\mathscr{q}G \xrightarrow{\simeq} B$, and $\alpha \xRightarrow{\simeq} \mathrm{id}_B$).

Definition 2.1.2.2.4 (Fibrewise (op)lax extensions)**.** Let $\mathfrak{E} \xrightarrow{p} \mathfrak{B}$ and $\mathfrak{F} \xrightarrow{q} \mathfrak{B}$ be two ∞-categories defined over a base $\mathfrak{B}$, and let $\mathscr{K}\colon \mathfrak{E} \to \mathfrak{F}$ be an ∞-functor defined over $\mathfrak{B}$. Let $\mathscr{D}\colon \mathfrak{E} \to \mathfrak{G}$ be an $(\infty, 1)$-functor, so that we have the solid diagram



(2.17)

A **fibrewise lax extension** of $\mathscr{D}$ along $\mathscr{K}$ (relative to $\mathfrak{B}$) is a limit

$$\mathrm{Lex}_{\mathscr{K}/\mathfrak{B}}\mathscr{D} := \left\{\mathscr{K}_*^{/\mathfrak{B}}, \mathscr{D}\right\} : \mathfrak{F} \to \mathfrak{G} \tag{2.18}$$

of $\mathscr{D}$ weighted by the ∞-profunctor $\mathscr{K}_*^{/\mathfrak{B}}\colon \mathfrak{E} \nrightarrow \mathfrak{F}$ associated with the comma ∞-category $\mathrm{id}_{\mathfrak{F}} \downarrow_{/\mathfrak{B}} \mathscr{K}$.

A **fibrewise oplax extension** of $\mathscr{D}$ along $\mathscr{K}$ (relative to $\mathfrak{B}$) is a colimit

$$\mathrm{Opex}_{\mathscr{K}/\mathfrak{B}}\mathscr{D} := \mathscr{K}^*_{/\mathfrak{B}} \star \mathscr{D}\colon \mathfrak{F} \to \mathfrak{G} \tag{2.19}$$

of $\mathscr{D}$ weighted by the ∞-profunctor $\mathscr{K}^*_{/\mathfrak{B}}\colon \mathfrak{F} \nrightarrow \mathfrak{E}$ corresponding to the relative comma ∞-category $\mathscr{K} \downarrow_{/\mathfrak{B}} \mathrm{id}_{\mathfrak{F}}$.

Remark 2.1.2.2.5. We have the explicit formulae, deduced from corollary 1.1.2.1.7, computing fibrewise extensions: the fibrewise oplax extension of $\mathscr{D}$ along $\mathscr{K}$, evaluated at F, is

$$\mathrm{Opex}_{\mathscr{K}/\mathfrak{B}}\mathscr{D}(F) = \varinjlim_{\substack{\mathscr{K}(E)\to F\\ E\in\mathfrak{E}_{q(F)}}} \mathscr{D}(E), \tag{2.20}$$

and the fibrewise lax extension of $\mathscr{D}$ along $\mathscr{K}$, evaluated at F, is

$$\mathrm{Lex}_{\mathscr{K}/\mathfrak{B}}\mathscr{D}(F) = \varprojlim_{\substack{F\to\mathscr{K}(E)\\ E\in\mathfrak{E}_{q(F)}}} \mathscr{D}(E). \tag{2.21}$$

Definition 2.1.2.2.6 (Envelope)**.** Let $\mathscr{F}\colon {\Delta_{\mathfrak{O}}}^{\mathrm{op}} \to \infty\text{—}\mathfrak{Grpd}$ be a precosheaf on ${\Delta_{\mathfrak{O}}}^{\mathrm{op}}$. Its **envelope** is the precosheaf

$$\mathscr{E}nv(\mathscr{F}) = \imath^* \,\mathrm{Opex}_{\imath/\Delta^{\mathrm{op}}} \mathscr{F} \tag{2.22}$$

where $\imath^*$ denotes the $(\infty, 1)$-functor of (fibrewise) restriction along $\imath\colon {\Delta_{\mathfrak{O}}}^{\mathrm{op}} \hookrightarrow (\Delta_{\mathfrak{O}}^{\mathrm{pre}})^{\mathrm{op}}$, right-adjoint to fibrewise oplax extension.

Remark 2.1.2.2.7. From the formula for fibrewise extensions (equation (2.20)), we see that the value taken by the envelope of $\mathscr{F} \in \mathfrak{Seg}_{{\Delta_{\mathfrak{O}}}^{\mathrm{op}\natural}}(\mathfrak{C})$ at an $\mathfrak{O}$-tree $T_\bullet$ is computed by

$$\mathscr{E}nv(\mathscr{F})(T_\bullet) = \varinjlim_{B_\bullet \rightsquigarrow T_\bullet} \prod_{i=1}^{|B_n|} \mathscr{F}(B_{\bullet,i}) \tag{2.23}$$

where $(B_{\bullet,i})_i$ denotes the forest decomposition in fibres (as in equation (2.12)).

Example 2.1.2.2.8. If $*$ denotes the terminal $\Delta_{\mathbb{O}}{}^{\mathrm{op}\natural}$-Segal ∞-groupoid, then its envelope $\mathscr{E}nv(*)$ is given by the colimits

$$\mathscr{E}nv(*)(T_\bullet) = \varinjlim_{B_\bullet \rightsquigarrow T_\bullet} * \tag{2.24}$$

(where the last $*$ is the terminal ∞-groupoid).

Remark 2.1.2.2.9. In the case $\mathbb{O} = \Gamma^{\mathrm{op}\flat}$, we may understand the envelope in the following way. Recall that an active morphism in Γ^{op} can be seen as giving a partition of its source indexed by its target (possibly with empty parts). Then the colimit creates several copies of $\mathscr{F}(\star_n)$, each equipped with a new partition specifying how to distribute its inputs.

Lemma 2.1.2.2.10. *For any Segal $\Delta_{\mathbb{O}}{}^{\mathrm{op}\natural}$-object $\mathscr{F}$, its envelope $\mathscr{E}nv(\mathscr{F})$ is a Segal $\Delta_{\mathbb{O}}{}^{\mathrm{op}\natural}$-object.*

Proof. The general form of the Segal conditions would be

$$\mathscr{E}nv(\mathscr{F})(O_n) = \varinjlim_{B_n \to O_n} \mathscr{F}(B_n) = \varinjlim_{B_n \to O_n} \varprojlim_{B_n \to E} \mathscr{F}(E) \tag{2.25}$$

$$\varprojlim_{O_n \to E} \mathscr{E}nv(\mathscr{F})(E) = \varprojlim_{O_n \to E} \varinjlim_{B_{0,1} \to E} \mathscr{F}(B_{0,1}) \tag{2.26}$$

Their equality is the condition of distributivity of limits over colimits as made explicit in [CH21, Definition 7.12], and using [CH21, Corollary 7.17] which shows that $\infty{-}\mathfrak{Grpd}$ is admissible (or even any $(\infty,1)$-topos since $\mathbb{O}$ is enrichable).

We can use the explicit form of the Segal conditions given in remark 2.1.2.1.17, which can be checked explicitly. □

For the cocartesian properties of the envelope, we now need to specialise to the case $\mathbb{O} = \Gamma^{\mathrm{op}\flat}$. Indeed, following [Her04, Theorem 2.4], we will characterise cocartesian fibrations of Segal $\mathfrak{P}$-objects (for $\mathfrak{P} = \Delta_{\mathbb{O}}{}^{\mathrm{op}\natural}$ a well-directed algebraic pattern) in terms of cocartesian fibrations of the underlying $\mathfrak{P}_{\mathfrak{P}_0/}$-objects of their envelopes. This presupposes having already a good understanding of cocartesian fibrations of $\mathfrak{P}_{\mathfrak{P}_0/}$-objects, which is only the case when $\mathfrak{P}_{\mathfrak{P}_0/}$ is $\Delta^{\mathrm{op}\natural}$, in particular when $\mathfrak{P} = \Omega^{\mathrm{op}\natural}$ (and $\mathfrak{P}_0 = \{\eta\}$).

Definition 2.1.2.2.11 (Cocartesian fibration)**.** A morphism $\mathscr{F} \to \mathscr{B}$ of Segal $\Delta_{\Gamma^{\mathrm{op}\flat}}{}^{\mathrm{op}\natural}$-∞-groupoids is a **cocartesian fibration of Segal objects** if $\mathfrak{Env}(\mathscr{F}) \to \mathfrak{Env}(\mathscr{B})$ is a cocartesian fibration (of $(\infty,1)$-categories), where $\mathfrak{Env}(\mathscr{F})$ denotes the underlying $(\infty,1)$-category of the ∞-operad $\mathscr{E}nv(\mathscr{F})$.

A Segal object $\mathscr{F}$ is **representably monoidal** if the unique morphism $\mathscr{F} \to *$ is a cocartesian fibration of Segal objects.

Proposition 2.1.2.2.12. *For any Segal object $\mathscr{F}$, $\mathscr{E}nv(\mathscr{F})$ is a representably monoidal.*

Proof. Let $\phi\colon \langle n\rangle \to \langle m\rangle$ be a morphism in $\mathfrak{Env}(*)$, given by a partition of n in m subsets $n_1, \cdots, n_m$, and let $C \in \mathfrak{Env}(\mathscr{E}nv(\mathscr{F}))$ lying over $\langle n\rangle$, which we write in terms of colours of $\mathscr{F}$ as

$$(C_{1,1} \cdots C_{1,i_1}) \cdots (C_{n,1} \cdots C_{n,i_n}) \tag{2.27}$$

We set

$$\phi_! C = (C_{1,1} \cdots C_{n_1,i_{n_1}}) \cdots (C_{n_{m-1}+1,1} \cdots C_{n_m,i_{nm}}) \tag{2.28}$$

which lies over $\langle m\rangle$.

A morphism $C \to \phi_! C$ in $\mathfrak{Env}(\mathscr{Env}(\mathscr{F}))$ is given by a moprhism $C_{1,1} \cdots C_{n_m, i_{n_m}} \to C_{1,1} \cdots C_{n_m, i_{n_m}}$ in $\mathfrak{Env}(\mathscr{F})$ along with a partition of $\sum_{k=1}^{m} i_{n_k}$ into n parts. We define the lift $C \to \phi_! C$ of ϕ to be given by the identity arrow of $C_{1,1} \cdots C_{n_m, i_{n_m}}$ along with the partition exhibited in equation (2.27). This lift is cocartesian. □

Thus the construction $\mathscr{F} \mapsto \mathscr{Env}(\mathscr{F})$ defines an $(\infty,1)$-functor $\mathscr{Env}\colon \infty\text{−}\mathfrak{Oprd} \to \mathfrak{Mon}(\infty,1)\text{−}\mathfrak{Cat}$.

Remark 2.1.2.2.13 (Monoidal structure on monoidal ∞-categories). Using the idea from the proof of proposition 2.1.2.2.12, we can construct a product on the colours of a representably monoidal ∞-operad $\mathscr{F}$. First note that (regardless of monoidality) the colours of $\mathscr{Env}(\mathscr{F})$ in the image of the unit map $\mathscr{F} \to \mathscr{Env}(\mathscr{F})$ are exactly those whose image under the morphism $\mathscr{Env}(\mathscr{F}) \to \mathscr{Env}(*)$ is 1.

Consider an n-uple of colours of $\mathscr{F}$, given by n morphisms $C_1, \cdots, C_n\colon \text{よ}\eta \to \mathscr{F}$. Those define a morphism $(C_1, \cdots, C_n)\colon \text{よ}\eta \to \mathscr{Env}(\mathscr{F})$, whose image lies over the colour n of $\mathscr{Env}(*)$. Now since $\mathscr{F}$ is representably monoidal, the morphism $n \to 1$ in $\mathscr{Env}(*)$ has a cocartesian lift from $(C_1, \cdots, C_n)$, whose target is then a colour $C_1 \otimes \cdots \otimes C_n$ of $\mathscr{F}$. Clearly, the same construction can be applied to obtain a product of morphisms as well, with appropriate functoriality.

Theorem 2.1.2.2.14. *The $(\infty,1)$-functor $\mathscr{Env}$ is left-adjoint to the inclusion $\mathfrak{Mon}(\infty,1)\text{−}\mathfrak{Cat} \hookrightarrow \infty\text{−}\mathfrak{Oprd}$.*

Proof. By the construction of the envelope, we have a unit morphism $\eta_{\mathscr{F}}\colon \mathscr{F} \to \mathscr{Env}(\mathscr{F})$ for any ∞-operad $\mathscr{F}$. We need to construct a counit $\varepsilon_{\mathscr{V}}\colon \mathscr{Env}(\mathscr{V}) \to \mathscr{V}$ for any representably monoidal ∞-operad $\mathscr{V}$, which is a morphism of monoidal ∞-categories.

This morphism is provided by the construction of remark 2.1.2.2.13: since $\mathscr{V}$ is representably monoidal, it admits a monoidal product, and its envelope simply corresponds to adding a second level a parenthesising to the products.

We will construct $\varepsilon_{\mathscr{V}}$ componentwise, as a natural transformation of ∞-functors ${\Delta_{\Gamma^{\mathrm{op}\flat}}}^{\mathrm{op}} \to \infty\text{−}\mathfrak{Grpd}$. Let $T_\bullet$ be a tree. By the formula equation (2.23), giving a map $\mathscr{Env}(\mathscr{V})(T_\bullet) \to \mathscr{V}(T_\bullet)$ is equivalent to giving, for each morphism $B_\bullet \rightsquigarrow T_\bullet$, a map $\mathscr{F}(B_\bullet) \to \mathscr{F}(T_\bullet)$. But recall that $B_\bullet \rightsquigarrow T_\bullet$ can be interpreted as a $T_\bullet$-partition of $B_\bullet$. Following the previous remark, we can use the product to simply reorganise the parenthesising levels according to the partition, which produces the desired morphism.

Finally, it is directly checked that ε and η satisfy the triangular equalities, so that they do exhibit an adjunction. □

2.2 Brane actions

2.2.1 Preliminary constructions

2.2.1.1 Correspondences and twisted arrows

Definition 2.2.1.1.1 (Twisted arrows category)**.** Let $\mathfrak{C}$ be an $(\infty,1)$-category, and consider the $(\infty,1)$-functor $\mathfrak{C}^{\mathrm{op}} \times \mathfrak{C} \to \infty\text{−}\mathfrak{Grpd}$ adjunct to $\text{よ}_{\mathfrak{C}}\colon \mathfrak{C} \to \infty\text{−}\mathfrak{Grpd}^{\mathfrak{C}^{\mathrm{op}}}$. Its associated discrete cartesian fibration is called the **twisted arrows** $(\infty,1)$**-category of** $\mathfrak{C}$, and denoted $\mathfrak{Tw}(\mathfrak{C}) \to \mathfrak{C} \times \mathfrak{C}^{\mathrm{op}}$.

Construction 2.2.1.1.2 (Twisted arrows quasicategories)**.** Since both the operation of taking the opposite of a category and that of taking the join of two categories are functorial, the rule mapping the category $[n]$ to the join $[n] \star [n]^{\mathrm{op}}$ defines a functor $\vartheta\colon \Delta \to \Delta$. Note that there is an equivalence $\vartheta([n]) := [n] \star [n]^{\mathrm{op}} \simeq [2n+1]$.

For any simplicial set $C_\bullet$, one may consider the simplicial set $(\vartheta^{op})^* C_\bullet$ whose n-simplices are the $\vartheta(n)$-simplices of $C_\bullet$. By [Lur17, Proposition 5.2.1.3], if $C_\bullet$ is a quasicategory, the projection $(\vartheta^{op})^* C_\bullet \to C_\bullet \times {C_\bullet}^{op}$ induced by the inclusions $[n], [n]^{op} \hookrightarrow \vartheta([n])$ is a right fibration of simplicial sets, implying that $(\vartheta^{op})^* C_\bullet$ is a quasicategory. Writing $C_\bullet = N(\mathfrak{C})_\bullet$ as the nerve of an essentially unique $(\infty, 1)$-category $\mathfrak{C}$, the quasicategory $(\vartheta^{op})^* C_\bullet$ can be seen by [Lur17, Proposition 5.2.1.11] as the nerve of $\mathfrak{Tw}(\mathfrak{C})$.

In particular one obtains from this construction an explicit description of the structure of $\mathfrak{Tw}(\mathfrak{C})$. The set of n-simplices of $N(\mathfrak{Tw}(\mathfrak{C}))_\bullet$ is C_{2n+1}. That is, a 0-simplex is an arrow $X \xrightarrow{f} Y$ of $\mathfrak{C}$, a 1-simplex is a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$ of three composable arrows, viewed as a morphism from the composite hgf to g and visualised as the completed square below-left,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle hgf}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xleftarrow[h]{} & T \end{array} \qquad \begin{array}{ccccccccccc} \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \cdots & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot & \cdots & \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot \end{array} \tag{2.29}$$

and more generally an n-simplex will be seen as a sequence of squares as above-right.

Example 2.2.1.1.3. The twisted arrows category of the category $\mathbb{n}$, for $n \in \mathbb{N}$, can be seen as the partially ordered set whose objects are pairs $(i, j) \in \mathbb{n} \times \mathbb{n}$ with $i \leqslant j$, and order given by $(i, j) \leqslant (i', j')$ if $i \leqslant i'$ and $j' \leqslant j$.

For the rest of the section, we let $\mathfrak{C}$ be an $(\infty, 1)$-category admitting pullbacks.

Construction 2.2.1.1.4. Consider the $(\infty, 1)$-functor $\mathcal{Sq}_{\mathfrak{C}} \colon \Delta^{op} \to (\infty,1){-}\mathfrak{Cat}$ defined as the composite of (the opposite of) the inclusion $\Delta \to (\infty,1){-}\mathfrak{Cat}, [n] \mapsto \mathbb{n} + \mathbb{1}$ and the exponential $\mathfrak{C}^{(-)}$, and let $\int \mathcal{Sq}_{\mathfrak{C}} \to \Delta$ be the associated cartesian fibration.

For any $n \in \mathbb{N}$, let $\mathfrak{Tw}(\mathbb{n})^{el}$ denote the full subcategory of $\mathfrak{Tw}(\mathbb{n})$ on those pairs (i, j) such that $j - i \leqslant 1$. We let $\int \overline{\mathfrak{Span}}(\mathfrak{C})$ be the full sub-$(\infty, 1)$-category of $\int \mathcal{Sq}_{\mathfrak{C}}$ on those $(\infty, 1)$-functors $\mathfrak{Tw}(\mathbb{n}) \to \mathfrak{C}$ which are the lax extension of their restriction to $\mathfrak{Tw}(\mathbb{n})^{el}$.

Proposition 2.2.1.1.5 ([Hau18, Corollary 5.12 and Proposition 5.14])**.** *The projection* $\int \overline{\mathfrak{Span}}(\mathfrak{C}) \to \Delta$ *is a cartesian fibration, and the associated* $(\infty, 1)$*-functor* $\overline{\mathfrak{Span}}(\mathfrak{C}) \colon \Delta^{op} \to (\infty,1){-}\mathfrak{Cat}$ *is an internal category, that is a Segal* $\Delta^{op\natural}$*-object (and so a double* ∞*-category).*

Construction 2.2.1.1.6 (Horizontal $(\infty, 2)$-category of a double ∞-category, [Ste20, Remark 3.1.8])**.** For $\mathfrak{X}$ a double ∞-category, its **horizontal** $(\infty, 2)$**-category** $\mathfrak{Hor}(\mathfrak{X}) \hookrightarrow \mathfrak{X}$ is defined as universal among the double ∞-categories $\mathfrak{H} \colon \Delta^{op} \to (\infty,1){-}\mathfrak{Cat}$ equipped with a double ∞-functor $\mathfrak{H} \to \mathfrak{X}$, and such that the $(\infty, 1)$-category of objects $\mathfrak{H}([0])$ is an ∞-groupoid. An explicit construction is given in [Ste20, Notation 3.1.6].

Definition 2.2.1.1.7 (Correspondences)**.** The $(\infty, 2)$**-category of correspondences** or **spans** in an $(\infty, 1)$-category $\mathfrak{C}$ with pullbacks is the horizontal category $\mathfrak{Span}(\mathfrak{C})$ of the Segal Δ^{op}-object $\overline{\mathfrak{Span}}(\mathfrak{C})$.

In our definition of horizontal $(\infty, 2)$-categories, we have used the term "$(\infty, 2)$-category" loosely: they are only double ∞-categories with appropriate discreteness condition in the vertical direction, what one might call Segal pre-2-categories. In order to properly view $\mathfrak{Span}(\mathfrak{C})$ as an ∞-bicategory, it is necessary to ensure that it is local with respect to essentially surjective and fully faithful $(\infty, 2)$-functors, that is that it is *complete*. Recall that a Segal $\Delta^{op\natural}$-∞-groupoid $\mathfrak{X}$ is said to be **univalent complete**, or **Rezk-complete**, if it is local with respect to the morphism $\mathbb{2}[\to^{-1}] \to *$, where we recall that $\mathbb{2}[\to^{-1}]$ is the walking equivalence.

Proposition 2.2.1.1.8 ([Ste20, Corollary 3.1.12]). *The underlying Segal $\Delta^{\mathrm{op}\natural}$-∞-groupoid of the Segal $\Delta^{\mathrm{op}\natural}$-object $\mathfrak{Span}(\mathfrak{C})$ is complete.*

Remark 2.2.1.1.9. Another construction of $\mathfrak{Span}(\infty\text{−}\mathfrak{Grpd})$ as a Segal $\Delta^{\mathrm{op}\natural}$-object is given in [Kos21, Remark 2.18].

Remark 2.2.1.1.10 (Limits and colimits in correspondences). As the $(\infty,2)$-category $\mathfrak{Span}(\mathfrak{C})$ is self-dual, its limits and colimits coincide. In fact, many colimits in $\mathfrak{Span}(\mathfrak{C})$ come from colimits in $\mathfrak{C}$.

By [SH11, Theorem 6.3], in the 1-categorical case, colimits in $\mathfrak{C}$ are preserved by the 2-functor $\mathfrak{C} \to \mathfrak{Span}(\mathfrak{C})$ exactly when they are Van Kampen colimits (*i.e.* preserved by the $(\infty,2)$-functor $\mathfrak{C}_{/(-)}\colon \mathfrak{C}^{\mathrm{op}} \to 2\text{−}\mathfrak{Cat}$). We would expect the same result to hold in the $(\infty,1)$-categorical setting.

By [Lur09, Theorem 6.1.3.9], locality of all morphisms in an $(\infty,1)$-topos translates into saying that the universality of colimits is the Van Kampen property. So when $\mathfrak{C}$ is an $(\infty,1)$-topos, all colimits of $\mathfrak{C}$ give rise to colimits (and thus also limits) in$\mathfrak{Span}(\mathfrak{C})$.

Proposition 2.2.1.1.11 ([Hau18, Proposition 12.1]). *The $(\infty,2)$-category of correspondences in $\mathfrak{C}$ admits a canonical symmetric monoidal structure, given on objects by the cartesian product of $\mathfrak{C}$.*

Idea of proof. The construction of spans can be iterated to produce an $(\infty,3)$-category, still with the same objects as $\mathfrak{C}$ but with hom $(\infty,2)$-categories between objects (C,C') given by $\mathfrak{Span}(\mathfrak{C})_{/C\times C'}$. In particular, $\mathfrak{Span}(\mathfrak{C})$ is recover as the $(\infty,2)$-category of endomorphisms of the terminal object $*$, which endows it canonically with a monoidal structure. The fact that this construction can be iterated to a sequence of (∞,n)-categories for all n is what makes $\mathfrak{Span}(\mathfrak{C})$ an infinite loop space, and thus extends the monoidal structure to a symmetric one. □

Proposition 2.2.1.1.12 ([Hau18, Corollary 12.5]). *All objects of $\mathfrak{Span}(\mathfrak{C})$ are dualisable for the monoidal structure $\mathfrak{Span}(\mathfrak{C})^{\times}$.*

Proof. Using the iterated construction in the previous proof, this corresponds to showing that the morphisms in the higher span ∞-categories have adjoints, which can be checked in the homotopy n-categories. □

Lemma 2.2.1.1.13 ([Lur17, Example 5.2.2.23]). *Let $\mathfrak{V}^{\otimes}$ be a symmetric monoidal $(\infty,1)$-category. Then $\mathfrak{Tw}(\mathfrak{V})$ admits a structure of symmetric monoidal $(\infty,1)$-category $\mathfrak{Tw}(\mathfrak{V})^{\otimes}$.*

Proposition 2.2.1.1.14 ([MR18, Corollary 2.1.3]). *Let $\mathfrak{V}^{\otimes}$ be a symmetric monoidal $(\infty,1)$-category and let $\mathfrak{C}$ be a category. There is a natural equivalence between the ∞-groupoid of symmetric monoidal $(\infty,1)$-functors $\mathfrak{V}^{\otimes} \to \iota_1\mathfrak{Span}(\mathfrak{C})^{\times}$ and that of symmetric monoidal $(\infty,1)$-functors $\mathscr{F}\colon \mathfrak{Tw}(\mathfrak{V})^{\otimes} \to \mathfrak{C}^{\times}$ such that, for every sequence $V \xrightarrow{f} W \xrightarrow{g} Z$ of composable arrows of $\mathfrak{V}$, there is an equivalence*

$$\mathscr{F}(g\circ f) \simeq \mathscr{F}(g) \underset{\mathscr{F}(\mathrm{id}_W)}{\times} \mathscr{F}(f). \tag{2.30}$$

Remark 2.2.1.1.15 (Lax morphisms). General symmetric monoidal ∞-functors $\mathfrak{Tw}(\mathfrak{V})^{\otimes} \to \mathfrak{C}^{\times}$ which do not necessarily satisfy the condition of equation (2.30) correspond to symmetric monoidal *lax* $(\infty,2)$-functors $\mathfrak{V}^{\otimes} \to \mathfrak{Span}(\mathfrak{C})^{\times}$.

2.2.1.2 Extensions

Definition 2.2.1.2.1 (Atomic morphism). An arrow is **atomic** it is not an identity and admits no non-trivial factorisation.

We will be interested in the atomic morphisms among those which are simultaneously semi-inert but not inert and active.

Example 2.2.1.2.2. In Γ, the atomic non-inert semi-inert active morphisms are the monomorphisms $\langle n\rangle \to \langle n+1\rangle$ missing a single element.

Example 2.2.1.2.3. In Δ, the atomic non-inert semi-inert active morphisms are the monomorphisms $[n] \to [n+1]$ skipping a single element.

Remark 2.2.1.2.4. More generally, in an operator category as in [Bar18], the interval inclusions are explicitly defined as the composites of finite sequences of fibre inclusions (as in [Bar18, Definition 2.1]).

Although we will only apply it to operads, the definition of extensions can be given in the context of a well-directed algebraic pattern ${\Delta_{\mathbb{O}}}^{\mathrm{op}\natural}$.

Construction 2.2.1.2.5. Let $\mathscr{F}$ be a ${\Delta_{\mathbb{O}}}^{\mathrm{op}\natural}$-Segal ∞-groupoid, and let $B = ([n], O_\bullet) \in {\Delta_{\mathbb{O}}}^{\mathrm{op}}$ be an $\mathbb{O}$-forest. The expression $i_*\mathscr{F}(B) \simeq \varinjlim_{B\to T} \mathscr{F}(T)$ induces (from $\mathrm{id}_B \colon B \to B$) a canonical arrow $\gamma_B \colon \mathscr{F}(B) \to i_*\mathscr{F}(B)$. Informally, it consists of viewing $\sigma \in \mathscr{F}(B)$ as endowed with the tautological partition of its inputs.

By its construction in [Ngu18, Proposition 3.4.5], the Grothendieck construction is compatible with pullback, so there is an $(\infty,1)$-functor $\int i^*i_!\mathscr{F} \to \int i_!\mathscr{F}$. Its composition with (the Grothendieck construction of the unit $\mathscr{F} \to i^*i_!\mathscr{F}$) is an $(\infty,1)$-functor γ giving (on each fibre) the ∞-functors γ_B constructed above.

Definition 2.2.1.2.6 (Extensions)**.** Let $\mathscr{F}$ be a ${\Delta_{\mathbb{O}}}^{\mathrm{op}\natural}$-Segal ∞-groupoid. Let $B = ([n], O_\bullet) \in {\Delta_{\mathbb{O}}}^{\mathrm{op}}$ be an $\mathbb{O}$-forest, let $\ell \in [n]$, and let $\sigma \in \mathscr{F}(B)$. The $(\infty,1)$-category of **extensions of** σ **at level** ℓ is the full sub-$(\infty,1)$-category $\mathrm{Ext}(\sigma; S)$ of $\gamma_B(\sigma) \downarrow \gamma$ spanned by the cartesian morphisms $e \colon \gamma_B(\sigma) \to \gamma_T(\tau)$ whose projection $(\phi, f_\bullet)$ satisfies

- ϕ is an equivalence in Δ (so that e is actually a morphism in the fibre over $[n]$)
- if $i > \ell$, then f_i is an equivalence
- if $i \leqslant \ell$, then f_i is an atomic semi-inert non-inert (active) morphism
- if $i < \ell$, the square

$$\begin{array}{ccc} O_i & \longrightarrow & P_i \\ \downarrow & & \downarrow \\ O_{i+1} & \longrightarrow & P_{i+1} \end{array} \tag{2.31}$$

  is cartesian.

Remark 2.2.1.2.7. Only the square

$$\begin{array}{ccc} O_\ell & \longrightarrow & P_\ell \\ \downarrow & & \downarrow \\ O_{\ell+1} & \xrightarrow[\simeq]{} & P_{\ell+1} \end{array} \tag{2.32}$$

is not in the image of γ.

Specialising back to the pattern $\Delta_{\Gamma^{\mathrm{op}\flat}}{}^{\mathrm{op}\natural}$ modelling that $\Omega^{\mathrm{op}\natural}$ for operads, we can see that as in [Lur17] the definition of extensions is only well-behaved for *unital* ∞-operads, where we may here say that an ∞-operad $\mathscr{F}$ is **unital** if its morphism $\mathscr{F}(\star_0) \to \mathscr{F}(\eta)$ is an equivalence.

In order to talk sensibly of colours, it is necessary to enforce the condition of univalent completeness (or Rezk completeness) to ensure that the colours of the Segal object correspond to the actual colours of the ∞-operad it presents. Recall that a Segal $\Omega^{\mathrm{op}\natural}$-object is **univalent complete** if its underlying $\Delta^{\mathrm{op}\natural}$-object is univalent complete (as defined ahead of proposition 2.2.1.1.8).

For our applications to Gromov–Witten theory, we will be concerned with operads which are not unital, and only have one colour whose ∞-groupoid of unary endo-operations is contractible.

Definition 2.2.1.2.8. A pointed ∞-operad $(\mathscr{F}, C_0)$ is **hapaxunital**[2] if its distinguished colour $C_0 \in \mathscr{F}(\eta)$ is such that $\mathscr{F}(\star_0) \times_{\mathscr{F}(\eta)} \{C_0\}$ is contractible.

Construction 2.2.1.2.9. If $\mathscr{F}$ is hapaxunital with distinguished colour C_0, we define extensions of its forests to be those extensions where the added colour is C_0.

Remark 2.2.1.2.10. Suppose $\mathscr{F}$ is hapaxunital and call C_0 its distinguished colour. Contractibility of $\mathscr{F}(\star_0) \times_{\mathscr{F}(\eta)} \{C_0\}$ furnishes, for every natural integer n, a map $\mathscr{F}(\star_{n+1}) \to \mathscr{F}(\star_n)$. Then, as in [MR18, Remark 2.1.5], $\mathrm{Ext}(\sigma) \simeq \{\sigma\} \times_{\mathscr{F}(\star_n)} \mathscr{F}(\star_{n+1}) \times_{\mathscr{F}(\eta)} \{C_0\}$.

2.2.2 Constructing the brane action

Theorem 2.2.2.0.1. *Let $\mathscr{O}$ be a hapaxunital ∞-operad. There is a lax morphism of categorical ∞-operads $\mathscr{O} \to \mathfrak{Cocorr}(\infty\text{–}\mathfrak{Grpd})^{\amalg}$, mapping each colour C to the ∞-groupoid $\mathrm{Ext}(\mathrm{id}_C)$.*

To construct the brane action, our strategy, following that of [MR18], will be to

- recast the morphism of ∞-operads $\mathscr{O} \to \mathfrak{Cocorr}(\infty\text{–}\mathfrak{Grpd})^{\amalg}$ as a monoidal ∞-functor $\mathscr{E}nv(\mathscr{O}) \to \mathfrak{Cocorr}(\infty\text{–}\mathfrak{Grpd})^{\amalg}$,
- which is then equivalent to a monoidal ∞-functor $\mathfrak{Tw}(\mathscr{E}nv(\mathscr{O}))^{\otimes} \to \infty\text{–}\mathfrak{Grpd}^{\mathrm{op}\amalg}$.
- We can then use the formalism of weak cartesian structures of [Lur17, §2.4.1] to recast it as an ∞-functor $\mathfrak{Env}(\mathfrak{Tw}(\mathscr{E}nv(\mathscr{O}))) \to \infty\text{–}\mathfrak{Grpd}^{\mathrm{op}}$,
- which finally is given by a discrete cartesian fibration over $\mathfrak{Env}(\mathfrak{Tw}(\mathscr{E}nv(\mathscr{O})))$.

Remark 2.2.2.0.2 (Twisted arrows in a monoidal envelope). Let $\mathscr{O}$ be an ∞-operad. The morphisms in $\mathfrak{Env}(\mathscr{O})$, that is the unary operations in $\mathscr{E}nv(\mathscr{O})$, and which will be the objects of $\mathfrak{Tw}(\mathscr{E}nv(\mathscr{O}))$, are forests of corollas in $\mathfrak{O}$. Passing once more to the monoidal envelope, we see that as in the proof of proposition 2.1.2.2.12 the objects of $\mathfrak{Env}(\mathfrak{Tw}(\mathscr{E}nv(\mathscr{O})))$ are forests of corollas of $\mathscr{O}$ with two levels of parenthesising.

Construction 2.2.2.0.3. Let $\mathfrak{B}\mathscr{O}$ be the locally full sub-$(\infty,1)$-category of $\mathfrak{Env}(\mathfrak{Tw}(\mathscr{E}nv(\mathscr{O})))^2$ with

[2] From the Greek adverb $\alpha\pi\alpha\xi$ meaning "once only"

objects the forests of twisted arrows (reorganised as twisted arrows between concatenations of forests)

$$\begin{array}{ccc} X=(X_1^1\otimes\cdots\otimes X_{n_1}^1)\otimes\cdots\otimes(X_1^m\otimes\cdots\otimes X_{n_m}^m) & \xrightarrow{f} & U=(U_1\otimes\cdots\otimes U_{\sum_{i=1}^m n_i+1}) \\ \sigma=\sigma^1\otimes\cdots\otimes\sigma^m\big\downarrow & & \big\downarrow\delta \\ Y=(Y_1^1\otimes\cdots\otimes Y_{k_1}^1)\otimes\cdots\otimes(Y_1^m\otimes\cdots\otimes Y_{k_m}^m) & \xleftarrow[g]{} & V=(V_1\otimes\cdots\otimes V_{\sum_{i=1}^m k_i}) \end{array} \tag{2.33}$$

such that the projection of $X\to U$ is semi-inert atomic, $V\to Y$ is an equivalence, and the added colour is the distinguished colour.

morphisms from $\sigma\xrightarrow{(f,g)}\delta$ to $\tau\xrightarrow{(a,b)}\epsilon$ (where $\tau\colon S\to T$, $\epsilon\colon A\to B$, and $a\colon S\to A$ and $b\colon B\to T$)

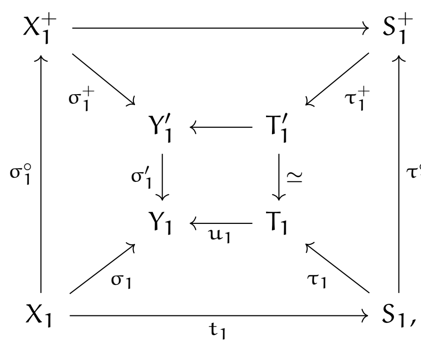
$$\begin{array}{ccc} \delta & \xrightarrow{(r,s)} & \epsilon \\ {\scriptstyle (f,g)}\big\uparrow & & \big\uparrow{\scriptstyle (a,b)} \\ \sigma & \xrightarrow[(t,u)]{} & \tau \end{array} \tag{2.34}$$

such that the induced square is cartesian.

Proof (of theorem 2.2.2.0.1). **Step 1** We first observe that the fibre over $\bigotimes_i \sigma_i$ is $\coprod_i \mathrm{Ext}(\sigma_i)$. This comes down to comparing the definitions.

Step 2 We must next show that $\mathfrak{B}\mathcal{O}\to\mathfrak{Env}(\mathfrak{Tw}(\mathcal{E}nv(\mathcal{O})))$ is a discrete cartesian fibration, that is that every morphism is cartesian.

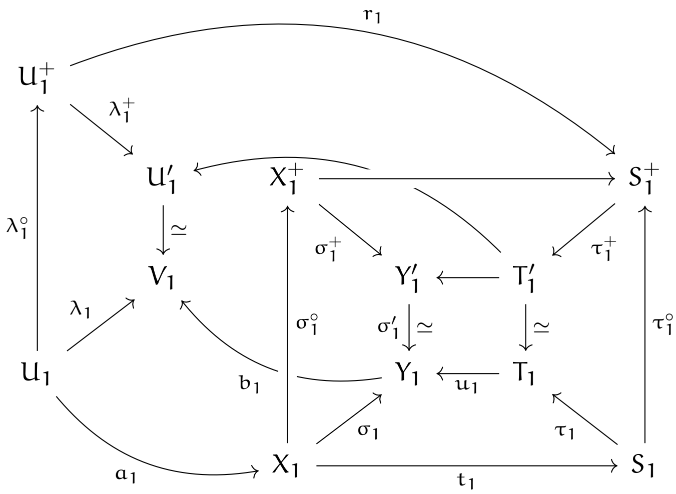
Let us consider a generic morphism of $\mathfrak{B}\mathcal{O}$ as laid out in the diagram



(2.35)

and whose image in $\mathfrak{Env}(\mathfrak{Tw}(\mathcal{E}nv(\mathcal{O})))$ can be read in the lower half of the diagram.

Let $\lambda=(\lambda_1\colon U_1\to V_1)$ be another forest, and consider an extension $\lambda^+=(\lambda_1^\circ\colon U_1\to U_1^+,\lambda_1^+\colon U_1^+\to V_1)$, a twisted arrow $(a_1\colon U_1\to X_1, b_1\colon Y_1\to V_1)\colon\lambda\to\sigma$, and a morphism $r=(r_1\colon U_1^+\to S_1^+)\colon\lambda^+\to\tau^+$. We check that there is a unique filling $\lambda^+\to\sigma^+$.



(2.36)

This is because, due to the conditions defining $\mathfrak{B}\mathcal{O}$, and the commutativity of the diagram, a lift $U_1^+ \to X_1^+$ must be split into a morphism $U_1 \to X_1$, which is fixed to be a_1, and a morphism between the added colours in the extension. But the hypothesis of hapaxunitality implies that there is only (essentially) one such morphism, so the lift is indeed (essentially) unique.

Step 3 Finally, we show that is a weak cartesian structure on $\mathfrak{Tw}(\mathscr{E}nv(\mathcal{O}))$. This is clear because, for a forest $B = \sigma_1 \otimes \cdots \otimes \sigma_n$ we have $\mathrm{Ext}(B) = \coprod_{i=1}^{n} \mathrm{Ext}(\sigma_i)$.

□

Corollary 2.2.2.0.4. *Let X be an ∞-groupoid and let $\mathcal{O}$ be a hapaxunital ∞-operad. There is a lax morphism of categorical ∞-operads $\mathcal{O} \to \mathfrak{Span}(\infty\text{—}\mathfrak{Grpd})^\times$ sending each colour O to $\infty\text{—}\mathfrak{Grpd}(\mathrm{Ext}(\mathrm{id}_O), X)$.*

Proof. The $(\infty, 1)$-functor $\infty\text{—}\mathfrak{Grpd}(-, X)\colon \infty\text{—}\mathfrak{Grpd}^{op} \to \infty\text{—}\mathfrak{Grpd}$ preserves limits, so it defines a morphism of categorical ∞-operads $\mathfrak{Cospan}(\infty\text{—}\mathfrak{Grpd})^{\amalg} \to \mathfrak{Span}(\infty\text{—}\mathfrak{Grpd})^\times$. The desired action $\mathcal{O} \to \mathfrak{Span}(\infty\text{—}\mathfrak{Grpd})^\times$ is obtained by composing the brane action $\mathcal{O} \to \mathfrak{Cospan}(\infty\text{—}\mathfrak{Grpd})^{\amalg}$ with this morphism. □

2.2.2.1 Graded brane actions

In our application to Gromov–Witten theory, the operad in play will carry the further structure of a grading of its components by a certain monoid. Hence, the brane action we will need to use is not the plain one of theorem 2.2.2.0.1, but one incorporating this grading.

Construction 2.2.2.1.1. Let B be a monoid with indecomposable zero. We let $\Gamma^{op}{}_B$ be the category whose

objects are those of Γ^{op},

morphisms from $\langle m \rangle$ to $\langle n \rangle$ are pairs (f, β) of an arrow $f\colon \langle m \rangle \to \langle n \rangle$ in Γ^{op} and a function $\beta\colon \langle n \rangle^\circ \to B$,

composition of $(f,\beta)\colon \langle m\rangle \to \langle n\rangle$ and $(g,\gamma)\colon \langle n\rangle \to \langle p\rangle$ is $(g\circ f, \gamma\circ\beta)$ where $\gamma\circ\beta\colon \langle p\rangle^\circ \to B$ is given by

$$(\gamma\circ\beta)(i) = \begin{cases} \gamma(i) & g^{-1}(i) = \emptyset \\ \gamma(i) + \sum_{j\in g^{-1}(i)} \beta(j) & \text{otherwise.} \end{cases} \tag{2.37}$$

Lemma 2.2.2.1.2 ([MR18, Proposition 2.3.2]). *The functor* ${\Gamma_B}^{op} \to \Gamma^{op}$ *is a weak Segal fibration for the pattern* $\Gamma^{op\flat}$.

Corollary 2.2.2.1.3. *There is an induced algebraic pattern structure on* Γ_B^{op}. □

Remark 2.2.2.1.4. We can describe ${\Delta_{\Gamma_B^{op\flat}}}^{op}$ as in example 2.1.2.1.11: its objects are trees equipped with a grading of every vertex.

We define B-graded ∞-operads to be the Segal ${\Delta_{\Gamma_B^{op\flat}}}^{op\natural}$-objects.

Remark 2.2.2.1.5 (Extensions in graded ∞-operads). The semi-inert morphisms in $\Gamma^{op\flat}$ are those semi-inerts of degree 0.

Proposition 2.2.2.1.6. *Let* $\mathcal{O}$ *be a* B*-graded* ∞*-operad. There is a lax morphism of* B*-graded* $(\infty,2)$*-operads* $\mathcal{O} \to \mathfrak{Cospan}(\infty\text{—}\mathfrak{Grpd})^{\amalg} \times B$.

Proof. Using the same construction as for theorem 2.2.2.0.1. □

2.3 Descent for brane actions

We are mainly interested in applying the formalism of brane actions to an operad coming from algebraic geometry. This is therefore not a standard operad, in ∞-groupoids, but one internal to the $(\infty,1)$-category of derived stacks, which is an $(\infty,1)$-topos. We shall thus have the need for a brane action for ∞-operads in general $(\infty,1)$-topoi rather than just the terminal one $\infty\text{—}\mathfrak{Grpd}$.

2.3.1 Operads in ∞-topoi

2.3.1.1 Some higher topos theory

In this section, it will be crucial to keep track of the relative sizes of categories in order to make meaningful statements. We thus fix a universe $\mathbb{U}$ with regard to which all the $(\infty,1)$-categories we consider are assumed to be locally small, and we will let κ denote a (varying as needed) *small* regular cardinal, that is one in $\mathbb{U}$.

Definition 2.3.1.1.1 ([Lur09, Proposition 5.3.3.3]). An $(\infty,1)$-category $\mathfrak{C}$ is said to be **κ-filtered** if the $(\infty,1)$-functor $\varinjlim_{\mathfrak{C}}\colon \infty\text{—}\mathfrak{Grpd}^{\mathfrak{C}} \to \infty\text{—}\mathfrak{Grpd}$ preserves κ-small limits (that is if $\mathfrak{C}$-indexed colimits commute with κ-small limits).

Remark 2.3.1.1.2. A (co)-limit indexed by a filtered $(\infty,1)$-category is usually called a filtered (co)limit. By [Lur09, Proposition 5.3.1.18], for any filtered $(\infty,1)$-category there is a filtered poset and a cofinal $(\infty,1)$-functor.

Definition 2.3.1.1.3 (Flat presheaf). Let $\mathfrak{C}$ be an $(\infty,1)$-category. A presheaf $\mathscr{F}\colon \mathfrak{C}^{op} \to \infty\text{—}\mathfrak{Grpd}$ is **κ-flat** if its $(\infty,1)$-category of elements $\int\mathscr{F}$ is κ-filtered.

Example 2.3.1.1.4 ([Lur09, Remark 5.3.2.11, Proposition 5.3.2.9]). If $\mathfrak{C}$ admits all κ-small colimits, then $\mathscr{F}$ is κ-flat if and only if it preserves all κ-small limits (that is sends κ-small colimits in $\mathfrak{C}$ to limits in $\infty-\mathfrak{Grpd}$).

Definition 2.3.1.1.5 (Compact object). An object C of an $(\infty,1)$-category $\mathfrak{C}$ with κ-filtered colimits is **κ-compact** if it commutes with κ-filtered colimits (meaning that the covariant hom-functor $よ^{\mathfrak{C},C}$ preserves κ-filtered colimits).

Example 2.3.1.1.6 ([Lur09, Proposition 5.3.4.17]). If $\mathfrak{C}$ is small, the κ-compact objets of the presheaf $(\infty,1)$-category $\infty-\mathfrak{Grpd}^{\mathfrak{C}^{op}}$ are exactly the retracts of κ-small colimits of representable presheaves.

Definition 2.3.1.1.7 (Accessible and presentable $(\infty,1)$-categories). A locally small $(\infty,1)$-category is **κ-accessible** if it admits all κ-filtered colimits and is generated under κ-filtered colimits by a small class of κ-compact objects.

An $(\infty,1)$-category is accessible if it is κ-accessible for some regular cardinal κ.

A **locally presentable** $(\infty,1)$**-category** is an accessible $(\infty,1)$-category which is cocomplete.

We let $(\infty,1)-\mathfrak{Cat}^{\text{pres}} \subset (\infty,1)-\mathfrak{Cat}$ denote the locally full sub-$(\infty,1)$-category of $(\infty,1)-\mathfrak{Cat}$ whose objects are the locally presentable $(\infty,1)$-categories and whose 1-arrows are the colimit-preserving $(\infty,1)$-functors.

Writing $\mathrm{Ind}_\kappa(\mathfrak{C})$, for any small $(\infty,1)$-category $\mathfrak{C}$, for the full sub-$(\infty,1)$-category of the free cocompletion $\infty-\mathfrak{Grpd}^{\mathfrak{C}^{op}}$ on the κ-filtered colimits of representable presheaves, this means that an accessible $(\infty,1)$-category is an $(\infty,1)$-category of the form $\mathrm{Ind}_\kappa(\mathfrak{C})$ for some small $(\infty,1)$-category $\mathfrak{C}$ (and some regular cardinal κ), and a locally presentable $(\infty,1)$-category is of the form $\mathrm{Ind}_\kappa(\mathfrak{C})$ for a small κ-cocomplete $(\infty,1)$-category $\mathfrak{C}$.

Lemma 2.3.1.1.8 ([Lur09, Corollary 5.3.5.4]). *Let $\mathfrak{C}$ be a small $(\infty,1)$-category. An object of $\infty-\mathfrak{Grpd}^{\mathfrak{C}^{op}}$ is in $\mathrm{Ind}_\kappa(\mathfrak{C})$ if and only if it is a κ-flat presheaf.*

In particular, if $\mathfrak{C}$ is κ-cocomplete so that $\mathrm{Ind}_\kappa(\mathfrak{C})$ is locally presentable, a presheaf is in $\mathrm{Ind}_\kappa(\mathfrak{C})$ if and only if it sends κ-small colimits in $\mathfrak{C}$ (the limits in $\mathfrak{C}^{op}$) to limits in $\infty-\mathfrak{Grpd}$.

Indeed, the category of ind-objects can be seen as a limit-preserving filtered cocompletion. That this is indeed an operation of completion is expressed by its idempotence.

Proposition 2.3.1.1.9 ([Lur09, Proposition 5.5.2.2]). *Let $\mathfrak{C}$ be a locally presentable $(\infty,1)$-category. A presheaf on $\mathfrak{C}$ is representable if and only if it is flat (that is if, as an $(\infty,1)$-functor $\mathfrak{C}^{op} \to \infty-\mathfrak{Grpd}$, it preserves all small limits).*

Proposition 2.3.1.1.10 ([Lur09, Corollary 5.5.2.4]). *A locally presentable category is complete.*

Presentability is the first exactness condition characterising $(\infty,1)$-topoi; to get to the full definition there remains to express the notion of descent.

Lemma 2.3.1.1.11 ([Lur09, Lemma 6.1.1.1]). *For any $(\infty,1)$-category $\mathfrak{C}$, let $\mathfrak{Arr}(\mathfrak{C}) = \mathfrak{C}^2 \xrightarrow{\mathcal{cod}_{\mathfrak{C}}=\mathrm{ev}_1} \mathfrak{C}$ denote the codomain fibration (which is a cocartesian fibration by [Lur09, Corollary 2.4.7.11 and Lemma 2.4.7.5]). An arrow of $\mathfrak{Arr}(\mathfrak{C})$, given by a commutative square in $\mathfrak{C}$, is $\mathcal{cod}_{\mathfrak{C}}$-cartesian if and only if corresponds to a cartesian square in $\mathfrak{C}$.*

In particular, if $\mathfrak{C}$ admits all pullbacks, $\mathcal{cod}_{\mathfrak{C}}$ is in fact a bicartesian fibration, whose underlying cartesian fibration classifies the $(\infty,1)$-functor $\mathscr{U}\colon \mathfrak{C}^{op} \to (\infty,1)-\mathfrak{Cat}, C \mapsto \mathfrak{C}_{/C}$.

Corollary 2.3.1.1.12. *The (restriction of $\mathcal{cod}_{\mathfrak{C}}$ to the) wide and locally full sub-$(\infty,1)$-category of $\mathfrak{Arr}(\mathfrak{C})$ whose arrows are the squares of $\mathfrak{C}$ which are cartesian is a discrete cartesian fibration, classifying the $(\infty,1)$-functor $\iota_0\mathscr{U}\colon \mathfrak{C}^{op} \to (\infty,1)-\mathfrak{Cat}, C \mapsto \iota_0\mathfrak{C}_{/C}$.*

Remark 2.3.1.1.13. For every map $f\colon C \to D$ in $\mathfrak{C}$, a $\mathit{cod}_{\mathfrak{C}}$-cocartesian lift $\mathfrak{C}_{/C} \to \mathfrak{C}_{/D}$ is given by the $(\infty,1)$-functor Σ_f mapping $B \xrightarrow{g} C$ to $B \xrightarrow{f\circ g} D$. In the presence of pullbacks in $\mathfrak{C}$, the (cartesian) right adjoint to Σ_f is the $(\infty,1)$-functor $\mathcal{U}(f) = f^*$ mapping $D' \to D$ to $f^*D' = C \times_D D' \to D$.

Definition 2.3.1.1.14 (Universality of colimits)**.** Let $\mathfrak{C}$ be a locally presentable $(\infty,1)$-category. One says that **colimits are universal in** $\mathfrak{C}$ if the $(\infty,1)$-functors f^* preserves small colimits.

Remark 2.3.1.1.15. By the adjoint functor theorem for locally presentable $(\infty,1)$-categories of [Lur09, Corollary 5.5.2.9], the colimit preservation condition in definition 2.3.1.1.14 is equivalent to the existence of right adjoints $\Pi_f\colon \mathfrak{C}_{/D} \to \mathfrak{C}_{/C}$ to the $(\infty,1)$-functors $f^*\colon \mathfrak{C}_{/C} \to \mathfrak{C}_{/D}$. Furthermore, the existence of these dependent product $(\infty,1)$-functors can be seen to be equivalent to local cartesian closedness of $\mathfrak{C}$.

Indeed, when $f = !_D\colon D \to *$ is the unique morphism to the final object (so that $f^*\colon \mathfrak{C}_{/*} \simeq \mathfrak{C} \to \mathfrak{C}_{/D}$ is given by product with D), a direct comparison of the universal properties shows that, for any $D' \xrightarrow{\varpi} D$ and any $E \in \mathfrak{C}$,

$$\begin{aligned}\mathfrak{C}(E, \Pi_{!_D} D') &= \mathfrak{C}_{/D}(E \times D, D') = \mathfrak{C}(E \times D, D') \times_{\mathfrak{C}(E\times D, D)} \{!_E \times D\} \\ &= \mathfrak{C}(E, (D')^D) \times_{\mathfrak{C}(E, D^D)} * \\ &= \mathfrak{C}\big(E, ((D')^D) \times_{D^D} *\big),\end{aligned} \tag{2.38}$$

meaning that Π_f can be constructed from exponentials and pullbacks, and computes the object of sections of f. Conversely, for any object $D' \in \mathfrak{C}$, one recovers the internal hom $(D')^D$ as the space of sections of the constant bundle $(!_D)^*D'$.

In the general case, one may use the equivalence $(\mathfrak{C}_{/C})_{/g} \simeq \mathfrak{C}_{/C'}$ (where $g\colon C' \to C$) to again obtain the dependent product from internal homs in the slice $(\infty,1)$-categories.

Proposition 2.3.1.1.16 ([Lur09, Proposition 6.1.1.4])**.** *An $(\infty,1)$-category admitting finite limits is locally presentable with universal colimits if and only if the $(\infty,1)$-functor $\mathcal{U}$ factors through $(\infty,1){-}\mathfrak{Cat}^{\mathrm{pres}} \hookrightarrow (\infty,1){-}\mathfrak{Cat}$.*

Warning 2.3.1.1.17. Although our notation does not reflect it, the self-indexing ∞-functors $\mathcal{U}$ and $\iota_0\mathcal{U}$ only land in the (very large) $(\infty,1)$-categories of *large* $(\infty,1)$-categories and ∞-groupoids. Hence there is no hope of their being representable.

Definition 2.3.1.1.18 (Relatively compact morphism)**.** A morphism $f\colon X \to Y$ is **relatively κ-compact** if for any morphism $U \to Y$ from a κ-compact object U, the base-change f^*U is also κ-compact.

Proposition 2.3.1.1.19 ([Lur09, Lemma 6.1.3.7, Proposition 6.1.6.7])**.** *Let $\mathfrak{E}$ be a locally presentable $(\infty,1)$-category with universal colimits. Let S be a class of morphisms stable under pullback, and write $\mathcal{U}^S$ for the (full on arrows) sub-$(\infty,1)$-functor of $\mathcal{U}$ assigning to any $E \in \mathfrak{E}$ the full sub-$(\infty,1)$-category $\mathfrak{E}^S_{/E} \subset \mathfrak{E}_{/E}$ on arrows belonging to S. Then $\mathcal{U}^S$ preserves limits if and only if $\iota_0\mathcal{U}^S$ does (*i.e. *is a flat presheaf), and for this it is necessary and sufficient that the class S be stable under amalgamated sums in $\mathfrak{Arr}(\mathfrak{C})$ and that pushouts of cartesian squares between S-morphisms be cartesian (in which case the class S is said to be **local**).*

Therefore (by proposition 2.3.1.1.9), $\iota_0\mathcal{U}^S$ is representable if and only if the class S satisfies the above equivalent conditions and, for any object $E \in \mathfrak{E}$, the $(\infty,1)$-category $\mathfrak{E}^S_{/E}$ is essentially small.

In addition, writing (for any regular cardinal κ) S_κ for the intersection of S with the class of relatively κ-compact morphisms, there is a sufficiently large regular cardinal κ such that $\iota_0\mathcal{U}^{S_\kappa}$ is representable.

A representing object for $\iota_0\mathscr{U}^S$ is called a **classifying object for** S**-morphisms**; by corollary 2.3.1.1.12 it is equivalently given by a terminal object of the $(\infty,1)$-category $\mathfrak{Arr}_S(\mathfrak{C})$, that is an S-morphism $U_* \to U$ such that for any other morphism $E \to B$ in S there is an essentially unique morphism $B \to U$ inducing a pullback square.

Definition 2.3.1.1.20 ($(\infty,1)$-topos)**.** A **Grothendieck–Rezk–Lurie** $(\infty,1)$**-topos** is a locally presentable $(\infty,1)$-category whose colimits are universal and which admits a classifying object for all relatively κ-compact morphisms for sufficiently large regular cardinals κ.

In view of the proposition 2.3.1.1.19, this means that a locally presentable $(\infty,1)$-category with universal colimits is an $(\infty,1)$-topos exactly when the class of *all* its arrows is local.

Example 2.3.1.1.21. The $(\infty,1)$-category $\infty-\mathfrak{Grpd}$ of ∞-groupoids is an $(\infty,1)$-topos; its classifying object for κ-compact morphisms is $\iota_0\infty-\mathfrak{Grpd}_\kappa$, where $\infty-\mathfrak{Grpd}_\kappa$ is the $(\infty,1)$-category of κ-small ∞-groupoids.

More generally, for any small $(\infty,1)$-category $\mathfrak{C}$, the presheaf $(\infty,1)$-category $\infty-\mathfrak{Grpd}^{\mathfrak{C}^{op}}$ is an $(\infty,1)$-topos, with κ-compact morphisms classifier the presheaf $\iota_0\mathscr{U}_\kappa\colon C \mapsto \iota_0\mathfrak{C}_{/C,\kappa}$ (where $\mathfrak{C}_{/C,\kappa}$ is the category of relatively κ-compact morphisms to C).

2.3.1.2 Sheaves of categorical ∞-operads

Lemma 2.3.1.2.1. *Let $\mathfrak{T}$ be an $(\infty,1)$-topos and let $\mathfrak{O}$ be an algebraic pattern. There is an equivalence*

$$\mathfrak{Seg}_{\mathfrak{O}}(\mathfrak{T}) \simeq (\infty,1)-\mathfrak{CAT}^{\lim}(\mathfrak{T}^{op}, \mathfrak{Seg}_{\mathfrak{O}}(\infty-\mathfrak{Grpd})), \tag{2.39}$$

where $(\infty,1)-\mathfrak{CAT}^{\lim}$ denotes the locally full sub-$(\infty,2)$-category of $(\infty,1)-\mathfrak{CAT}$ whose objects are the complete $(\infty,1)$-categories and whose 1-arrows are the limit-preserving $(\infty,1)$-functors.

Proof. By proposition 2.3.1.1.9, we may as in [Lur09, Remark 6.3.5.17] identify $\mathfrak{T}$ with the $(\infty,1)$-category of limit-preserving $(\infty,1)$-functors $\mathfrak{T}^{op} \to \infty-\mathfrak{Grpd}$. Thus we have equivalences

$$\mathfrak{Seg}_{\mathfrak{O}}(\mathfrak{T}) \simeq \mathfrak{Fun}^{\mathrm{Segal},\lim}(\mathfrak{O} \times \mathfrak{T}^{op}, \infty-\mathfrak{Grpd}) \simeq \mathfrak{Fun}^{\lim}(\mathfrak{T}^{op}, \mathfrak{Seg}_{\mathfrak{O}}(\infty-\mathfrak{Grpd})), \tag{2.40}$$

where for notational simplicity we have used $\mathfrak{Fun}^{\lim}(\mathfrak{A},\mathfrak{B})$ to denote the $(\infty,1)$-categories (thus far denoted $(\infty,1)-\mathfrak{CAT}^{\lim}(\mathfrak{A},\mathfrak{B})$) of limit-preserving ∞-functors $\mathfrak{A} \to \mathfrak{B}$, adapted to the notation $\mathfrak{Fun}^{\mathrm{Segal},\lim}(\mathfrak{P} \times \mathfrak{A}, \mathfrak{B})$ (where $\mathfrak{P}$ is the underlying $(\infty,1)$-category of an algebraic pattern) for the bifunctors which preserve limits in their second variable and are Segal in the first. □

Definition 2.3.1.2.2 (Categorical ∞-operad in an ∞-topos)**.** A categorical ∞-operad in $\mathfrak{T}$ is a limit-preserving $(\infty,2)$-functor $\mathfrak{T}^{op} \to \mathfrak{Seg}_{\mathfrak{O}}((\infty,1)-\mathfrak{Cat})$.

Remark 2.3.1.2.3. By lemma 3.1.1.1.1, the $(\infty,1)$-category of internal groupoids in $\mathfrak{T}$ is equivalent to $\mathfrak{T}$. A categorical ∞-operad in $\mathfrak{T}$ is the same thing as an operad internal to categories in $\mathfrak{T}$.

Example 2.3.1.2.4. When the $(\infty,1)$-topos $\mathfrak{T}$ is hypercomplete, so can be written as an $(\infty,1)$-category $\mathfrak{HSh}_\tau(\mathfrak{S})$ of hypersheaves on an $(\infty,1)$-site $(\mathfrak{S},\tau)$, we have a further equivalence between (categorical) ∞-operads in $\mathfrak{T}$ and hypersheaves of (categorical) ∞-operads on $(\mathfrak{S},\tau)$.

Proposition 2.3.1.2.5. *Fix κ a sufficiently large regular cardinal, and for any object $Z \in \mathfrak{T}$, let $\mathfrak{T}^\kappa_{/Z}$ denote the full sub-$(\infty,1)$-category of $\mathfrak{T}_{/Z}$ spanned by the κ-compact morphisms. The construction $Z \mapsto \mathfrak{Cospan}(\mathfrak{T}^\kappa_{/Z})^{\amalg}$ defines a categorical ∞-operad $\mathfrak{Cospan}(\mathfrak{T}^\kappa_{/\_})^{\amalg}$ in $\mathfrak{T}$.*

Proof. By proposition 2.3.1.1.19, the class of κ-compact morphisms admits a classifying object $\mathfrak{T}^{\kappa}_{/-}$, which is so that $\mathfrak{T}(Z, \mathfrak{T}^{\kappa}_{/-})$ is equivalent to $\iota_0 \mathfrak{T}^{\kappa}_{/Z}$. Since both functors $(-)^{\mathrm{op}}$ and $\mathfrak{Span}$ admit left-adjoints, they preserve limits, so that we obtain a sheaf of categorical ∞-operads. □

Remark 2.3.1.2.6. In order to free ourselves from the κ-compactness hypothesis, $\mathcal{C}ospan(\mathfrak{T}_{/-})^{\amalg}$ should in fact be constructed as a locally internal Segal Ω-object in $\mathfrak{T}$ as in [Joh02, Theorem B2.2.2].

2.3.2 Gluing the brane actions

Lemma 2.3.2.0.1 (Functoriality of brane actions, [MR18, §2.1.3])**.** *Let* $f\colon \mathscr{O} \to \mathscr{P}$ *be a map of hapaxunital ∞-operads. Then* f *induces an ∞-functor* $\mathfrak{B}f$ *of discrete cartesian fibrations from* $\pi_{\mathscr{O}}\colon \mathfrak{B}\mathscr{O} \to \mathfrak{Env}\mathfrak{Tw}(\mathscr{E}nv(\mathfrak{O}))^{\otimes}$ *to* $\pi_{\mathscr{P}}\colon \mathfrak{B}\mathscr{P} \to \mathfrak{Env}\mathfrak{Tw}(\mathscr{E}nv(\mathfrak{P}))^{\otimes}$.

Proof. Since $\mathfrak{Env}^{\otimes}(-)$, $\mathfrak{Tw}(-)^{\otimes}$ and $-^{2}$ are (covariant) $(\infty,1)$-functors, we immediately obtain a map of fibrations from $\mathrm{ev}_{0,\mathscr{O}}$ to $\mathrm{ev}_{0,\mathscr{P}}$, and simply need to check that its underlying ∞-functor sends the sub-$(\infty,1)$-category $\mathfrak{B}\mathscr{O}$ to $\mathfrak{B}\mathscr{P}$.

But since f is a morphism of ∞-operads □

Theorem 2.3.2.0.2. *Let* $\mathcal{O}$ *be a unital ∞-operad in an* $(\infty,1)$*-topos* $\mathfrak{T}$*. There is a lax morphism of* $\mathfrak{T}$*-sheaves of categorical ∞-operads* $\mathcal{O} \to \mathcal{C}ospan(\mathfrak{T}_{/-})^{\amalg}$.

Proof. We follow the same arguments as in [MR18, Proposition 2.2.4]. We again are reduced to constructing a morphism of monoidal $(\infty,2)$-categories in $\mathfrak{T}$

$$\mathfrak{Tw}^{\mathfrak{T}^{\mathrm{op}}}\big(\mathscr{E}nv^{\mathfrak{T}^{\mathrm{op}}}(\mathcal{O})\big) \to \mathfrak{T}_{/-}{}^{\mathrm{op}\,\amalg} \tag{2.41}$$

which we recast as

$$\mathfrak{Env}^{\mathfrak{T}^{\mathrm{op}}}\Big(\mathfrak{Tw}^{\mathfrak{T}^{\mathrm{op}}}\big(\mathscr{E}nv^{\mathfrak{T}^{\mathrm{op}}}(\mathcal{O})\big)\Big) \to \mathfrak{T}_{/-}{}^{\mathrm{op}}, \tag{2.42}$$

and finally as a cartesian $(\infty,1)$-functor between their Grothendieck constructions

$$\begin{array}{ccc} \int \mathfrak{Env}^{\mathfrak{T}^{\mathrm{op}}}\Big(\mathfrak{Tw}^{\mathfrak{T}^{\mathrm{op}}}\big(\mathscr{E}nv^{\mathfrak{T}^{\mathrm{op}}}(\mathcal{O})\big)\Big) & \longrightarrow & \int \mathfrak{T}_{/-}{}^{\mathrm{op}} \\ & \searrow \quad \mathfrak{T} \quad \swarrow & \end{array} \tag{2.43}$$

SubConstruction 2.3.2.0.2.1. Any cartesian fibration $p\colon \mathfrak{F} \to \mathfrak{B}$ admits a **dual** cocartesian fibration $p^{\vee}\colon \mathfrak{F}^{\vee} \to \mathfrak{B}^{\mathrm{op}}$ classifying the same $(\infty,1)$-functor $\mathfrak{B}^{\mathrm{op}} \to (\infty,1)\!-\!\mathfrak{Cat}$; this fact is studied rigorously, through two different constructions, in [Lur19, Proposition 14.4.2.4] and [BGN18, Therorem 5.3]. By [Lur19, Construction 14.4.2.1], the dual cartesian fibration $\mathfrak{F}^{\vee\mathrm{op}} \to \mathfrak{B}$ is determined by the universal property (or duality relation) that, for $\mathfrak{X} \to \mathfrak{B}$ any $(\infty,1)$-category over $\mathfrak{B}$, the $(\infty,1)$-groupoid $(\infty,1)\!-\!\mathfrak{Cat}_{/\mathfrak{B}}(\mathfrak{X}, \mathfrak{F}^{\vee\mathrm{op}})$ is a certain full sub-∞-groupoid of $(\infty,1)\!-\!\mathfrak{Cat}(\mathfrak{X} \times_{\mathfrak{B}} \mathfrak{F}, \infty\!-\!\mathfrak{Grpd})$.

In particular here, the fibration $\int \mathfrak{T}_{/-}{}^{\mathrm{op}} \to \mathfrak{T}$ is evidently dual to $\int \mathfrak{T}_{/-} \to \mathfrak{T}$; hence the datum of a morphism as in equation (2.43) is in fact equivalent to an $(\infty,1)$-functor

$$\int \mathfrak{Env}^{\mathfrak{T}^{\mathrm{op}}}\Big(\mathfrak{Tw}^{\mathfrak{T}^{\mathrm{op}}}\big(\mathscr{E}nv^{\mathfrak{T}^{\mathrm{op}}}(\mathcal{O})\big)\Big) \underset{\mathfrak{T}}{\times} \int \mathfrak{T}_{/-} \to \infty\!-\!\mathfrak{Grpd}, \tag{2.44}$$

and so to a discrete cartesian fibration over its source satisfying the conditions (of fibrewise representabilty) necessary for subConstruction 2.3.2.0.2.1, which we now endeavour to build.

The composition

$$\mathfrak{Op}(\mathcal{T}) \subset \mathfrak{Opd}^{\mathcal{T}^{op}} \xrightarrow{\mathfrak{B}^{\mathcal{T}^{op}}} \mathfrak{Cart}^{\mathcal{T}^{op}} \xrightarrow{\mathrm{ev}_0{}^{\mathcal{T}^{op}}} (\infty,1)\text{–}\mathfrak{Cat}^{\mathcal{T}^{op}} \tag{2.45}$$

produces an $(\infty,1)$-functor $(\mathrm{ev}_0 \circ \mathfrak{B})^{\mathcal{T}^{op}} : \mathfrak{Op}(\mathcal{T}) \to \mathfrak{Cat}(\mathcal{T})$. For any ∞-operad $\mathcal{O}$ in $\mathcal{T}$, its construction $(\mathrm{ev} \circ \mathfrak{B})^{\mathcal{T}^{op}} \mathcal{O}$ is canonically endowed with a transformation $\mathfrak{B}\mathcal{O} \to \mathfrak{Env}^{\mathcal{T}^{op}}(\mathfrak{Tw}^{\mathcal{T}^{op}}(\mathscr{Env}^{\mathcal{T}^{op}}(\mathcal{O})))$ whose transition maps preserve the cartesian arrows. One can then check that $\int (\mathrm{ev} \circ \mathfrak{B})^{\mathcal{T}^{op}} \mathcal{O} \to \int \mathfrak{Env}^{\mathcal{T}^{op}}(\mathfrak{Tw}^{\mathcal{T}^{op}}(\mathscr{Env}^{\mathcal{T}^{op}}(\mathcal{O})))$ is again a discrete cartesian fibration.

SubConstruction 2.3.2.0.2.2. For $p \colon \mathcal{F} \to \mathfrak{B}$ a cartesian fibration, the $(\infty,1)$-functor $\mathcal{F}^2 \to \mathcal{F} \times_{\mathfrak{B}} \mathfrak{B}^2$ mapping $(\xi \xrightarrow{f} \psi)$ to the pair $(\psi, p\xi \xrightarrow{pf} p\psi)$ admits a section (taking $(\psi, g \colon X \to p\psi)$ to the inverse image $g^*\psi \to \psi$ of ψ along g), which establishes an equivalence with the full sub-$(\infty,1)$-category of $\mathcal{F}^2$ spanned by the p-cartesian arrows.

We can finally define the required discrete cartesian fibration as the fibre product

$$\begin{array}{ccc} \mathfrak{B}_{\mathcal{T}}\mathcal{O} & \longrightarrow & \int (\mathrm{ev} \circ \mathfrak{B})^{\mathcal{T}^{op}} \mathcal{O} \\ \downarrow & & \downarrow \\ \int \mathfrak{Env}^{\mathcal{T}^{op}}(\mathfrak{Tw}^{\mathcal{T}^{op}}(\mathscr{Env}^{\mathcal{T}^{op}}(\mathcal{O}))) \times_{\mathcal{T}} \int \mathcal{T}_{/-} & \longrightarrow & \int \mathfrak{Env}^{\mathcal{T}^{op}}(\mathfrak{Tw}^{\mathcal{T}^{op}}(\mathscr{Env}^{\mathcal{T}^{op}}(\mathcal{O}))). \end{array} \tag{2.46}$$

It can then be checked that $\mathfrak{B}_{\mathcal{T}}\mathcal{O}$ satisfies the conditions listed above, so defines the requested lax morphism of categorical ∞-operads in $\mathcal{T}$. □

Construction 2.3.2.0.3. Let X be an object of $\mathcal{T}$. There is a natural transformation $\mathcal{T}_{/-}{}^{op} \Rightarrow \mathcal{T}_{/-}$ of $(\infty,1)$-functors $\mathcal{T} \to (\infty,1)\text{–}\mathfrak{Cat}$ (*not* $(\infty,2)$-functors, as $\mathcal{T}_{/-}{}^{op}$ is a functor to the co-dual $(\infty,2)$-category) whose component at $Z \in \mathcal{T}$ is the internal hom ∞-functor $\mathcal{M}or_{/Z}(-, X \times Z) \colon \mathcal{T}_{/Z}{}^{op} \to \mathcal{T}_{/Z}$, which preserves limits. This produces a morphism of categorical ∞-operads in $\mathcal{T}$

$$\mathcal{M}or_{/-}(\bullet, X \times -) \colon \mathcal{C}ospan(\mathcal{T}_{/-})^{\amalg} \to \mathcal{S}pan(\mathcal{T}_{/-})^{\times}. \tag{2.47}$$

Corollary 2.3.2.0.4. *Let X be an object of $\mathcal{T}$. There is a lax morphism $\mathcal{O} \to \mathcal{S}pan(\mathcal{T}_{/-})^{\times}$.*

Proof. The morphism is the composition

$$\mathcal{O} \xrightarrow{\mathfrak{B}_{\mathcal{T}}\mathcal{O}} \mathcal{C}ospan(\mathcal{T}_{/-})^{\amalg} \xrightarrow{\mathcal{M}or_{/-}(\bullet, X \times -)} \mathcal{S}pan(\mathcal{T}_{/-})^{\times}. \tag{2.48}$$

□

Remark 2.3.2.0.5. As in the proof of theorem 2.3.2.0.2, the morphism is given by a discrete cocartesian fibration $\mathfrak{B}_{\mathcal{T}}(\mathcal{O}, X)$ over $\int \mathfrak{Env}^{\mathcal{T}^{op}}(\mathfrak{Tw}^{\mathcal{T}^{op}}(\mathscr{Env}^{\mathcal{T}^{op}}(\mathcal{O}))) \times_{\mathcal{T}} \int \mathcal{T}_{/-}$. Its fibre over $(Z \in \mathcal{T}; \sigma, Y \to Z)$ is $\mathcal{T}_{/Z}(Y, \mathcal{M}or_{/Z}(\mathrm{Ext}(\sigma), X \times Z))$.

Construction 2.3.2.0.6. As in [MR18, §2.4], it is straightforward to see that the arguments giving the gluing of brane actions and those constructing graded brane actions combine to give brane actions for graded ∞-operads in an $(\infty,1)$-topos.

Part II

Derived moduli stacks for Gromov–Witten theory

CHAPTER

3

DERIVED MAPPING STACKS AND THE QUANTUM LEFSCHETZ PRINCIPLE

In chapter 4 we will initiate the study of quasimap theory, which concerns morphisms from algebraic curves into a target stack. Modulating them will thus require constructing stacks of morphisms and being able to control their algebro-geometric properties.

For completeness, we include in this chapter 3 a review of the basic notions of derived algebraic geometry. Derived algebraic geometry, standing at the intersection of topology and algebra, can be described as concerning the study of $(\infty, 1)$-topoi equipped with models of geometric theories, for example $(\infty, 1)$-topoi with a structure sheaf of local $\mathscr{E}_\infty$-rings or of strict henselian local $\mathscr{E}_\infty$-rings. Given a model of the bigger theory of $\mathscr{E}_\infty$-rings, one can take its spectrum to obtain an appropriately structured $(\infty, 1)$-topos, and through this construction characterise any structured $(\infty, 1)$-topos by its functor of points restricted to affine spectra. This allows one to reformulate the added structure and geometric conditions as only geometric conditions on the functor of points, the existence of an appropriate covering called an atlas.

This is the point of view, developed in [TV08], that we will use here. After recalling in section 3.1 the foundations of higher stacks, the derived commutative algebra needed to define their geometry, and their link with classical and "virtual" algebraic geometry, we will in section 3.2 turn more specifically to derived moduli stacks of maps. The only new results in section 3.1 are those in subsection 3.1.3 concerning a generalisation of the results of [MR18] and [STV15] on virtual structure sheaves to the relative version, the virtual pullbacks of [Man12a].

We begin section 3.2 by generalities about moduli stacks of maps, and the approach to maps with varying source using a moduli stack of algebraic stacks. Then in subsection 3.2.2 we describe an application to moduli of maps to targets which are described as the zero locus of a section of a vector bundle. This involves general results about the structure of such zero loci, and produces a categorification and geometrisation of the quantum Lefschetz principle, heretofore only established in G-theory.

3.1 Elements of spectral algebraic geometry

3.1.1 Derived stacks and their geometric presentations

3.1.1.1 Sheaves in $(\infty, 1)$-topoï

Lemma 3.1.1.1.1 ([Lur09, Theorem 6.1.6.8, Proposition 6.1.3.19]). *Let $\mathfrak{E}$ be an $(\infty,1)$-topos. Then $\mathfrak{E}$ satisfies the ∞-categorical* ***Giraud axioms****, that is it is an effective presentable $(\infty,1)$-category whose colimits are universal and whose coproducts are disjoint.*

Theorem 3.1.1.1.2 ([Lur09, Proposition 6.1.5.3, Proposition 6.1.3.10]). *Let $\mathfrak{E}$ be an $(\infty,1)$-category satisfying the Giraud axioms. Then there exists a small finitely complete $(\infty,1)$-category $\mathfrak{C}$ and a left exact accessible localisation $\infty\text{−}\mathfrak{Grpd}^{\mathfrak{C}^{op}} \to \mathfrak{E}$.*

Conversely, if $\mathfrak{E}$ admits such a presentation, it is an $(\infty,1)$-topos.

Thus, in order to understand $(\infty,1)$-topoï, one must understand left exact accessible localisation of presheaf $(\infty,1)$-categories.

Definition 3.1.1.1.3 (Strong saturation). Let $\mathfrak{E}$ be a cocomplete $(\infty,1)$-category. A class $\mathscr{W}$ of arrows of $\mathfrak{E}$ is **strongly saturated** if it satisfies the two-out-of-three property, is stable under pushouts, and the full subcategory of $\mathfrak{Arr}(\mathfrak{E}) = \mathfrak{E}^2$ is stable under small colimits.

By [Lur09, Remark 5.5.4.7], any class of morphisms is contained in a smallest strongly saturated class. A class whose strong saturation is $\mathscr{W}$ is said to **generate** $\mathscr{W}$.

Lemma 3.1.1.1.4 ([Lur09, Proposition 5.5.4.2 (3)]). *A localisation is accessible if and only if the class of morphisms it inverts is strongly saturated and generated by a small set of morphisms.*

Definition 3.1.1.1.5 (Connective morphism). Let $n \in \mathbb{N}$. A morphism $f\colon X \to Y$ in an $(\infty,1)$-topos $\mathfrak{E}$ is said to be n**-connective**, recursively, if it is an effective epimorphism and its diagonal $\Delta_f\colon X \to X \times_Y X$ is $(n-1)$-connective (where any morphism is considered (-1)-connective).

A morphism f is ∞-connective if it is n-connective for every $n \in \mathbb{N}$.

Definition 3.1.1.1.6 (Cotopological localisations). A localisation $\mathscr{L}$ is **cotopological** if it only inverts ∞-connective morphisms: for any morphism u in $\mathfrak{E}$, if $\mathscr{L}u$ is an equivalence then u is ∞-connective.

Definition 3.1.1.1.7 (Topological localisation). A localisation $\mathscr{L}$ is **topological** if there is a non-∞-connective arrow f such that $\mathscr{L}(f)$ is ∞-connective.

Proposition 3.1.1.1.8 ([Lur09, Proposition 6.5.2.19]). *Let $\mathfrak{E}$ be an $(\infty,1)$-topos. Any accessible left exact localisation $\mathfrak{E} \rightleftarrows \mathfrak{E}''$ factors as $\mathfrak{E} \underset{i}{\overset{\mathscr{L}}{\rightleftarrows}} \mathfrak{E}' \underset{i'}{\overset{\mathscr{L}'}{\rightleftarrows}} \mathfrak{E}''$ with $\mathscr{L}$ a topological localisation and $\mathscr{L}'$ a cotopological localisation.*

We have given a definition of cotopological and topological localisations in terms of the internal homotopy theory of an $(\infty,1)$-topos. These notions can also be given more purely topos-theoretic characterisations.

Lemma 3.1.1.1.9 ([Lur09, Proposition 6.5.2.16]). *A localisation $\mathscr{L}\colon \mathfrak{E} \to \mathfrak{E}[\mathscr{W}^{-1}]$ is cotopological if and only if it inverts no monomorphisms, that is: for any monomorphism u in $\mathfrak{E}$, if $\mathscr{L}u$ is an equivalence then u is an equivalence.*

Proposition 3.1.1.1.10 (Anel–Biedermann–Finster–Joyal). *A localisation is topological if and only if the class $\mathcal{W}$ of morphisms it inverts is stable under colimits, strongly saturated, and generated by a (small) class of monomorphisms (that is if it is a topological localisation in the sense of [Lur09, Definition 6.2.1.4]).*

Example 3.1.1.1.11 (Sheaves). Let $\mathfrak{C}$ be a small $(\infty,1)$-category. A sieve on an object $C \in \mathfrak{C}$ is a subobject of $よ_{\mathfrak{C}}(C)$. One can then define a Grothendieck topology on $\mathfrak{C}$ as determined by an assignment of covering sieves, by analogy with the 1-categorical case, which turns out to be equivalent to giving a Grothendieck topology on the category $\mathrm{Ho}\,\mathfrak{C}$. An $(\infty,1)$-site is a small $(\infty,1)$-category equipped with a Grothendieck topology.

Let $(\mathfrak{C},\tau)$ be an $(\infty,1)$-site. Define a τ-local equivalence in $\infty{-}\mathfrak{Grpd}^{\mathfrak{C}^{op}}$ to be a monomorphism of presheaves $\mathcal{F} \hookrightarrow よ_{\mathfrak{C}}C$ with representable target which, as a sieve on C, is covering. A τ-sheaf on $\mathfrak{C}$ is a presheaf which is local with respect to the τ-local equivalences. By [Lur09, Proposition 6.2.2.7], the full sub-$(\infty,1)$-category $\mathfrak{Sh}_\tau(\mathfrak{C}) \hookrightarrow \infty{-}\mathfrak{Grpd}^{\mathfrak{C}^{op}}$ on the τ-sheaves, that is the localisation of $\infty{-}\mathfrak{Grpd}^{\mathfrak{C}^{op}}$ at the τ-local equivalences, is a topological localisation.

Proposition 3.1.1.1.12 ([Lur09, Proposition 6.2.2.17]). *Let $\mathfrak{C}$ be a small $(\infty,1)$-category. The set of equivalence classes of topological localisations of $\infty{-}\mathfrak{Grpd}^{\mathfrak{C}^{op}}$ is in bijection with that of Grothendieck topologies on $\mathfrak{C}$.*

The hypercompletions of sheaf $(\infty,1)$-topoi can be understood concretely in a similar way.

Construction 3.1.1.1.13 (Hypercoverings). Let $(\mathfrak{C},\tau)$ be an $(\infty,1)$-site. A **τ-hypercovering** of $C \in \mathfrak{C}$ is an augmented simplicial object $\mathcal{F}_\bullet$ in $\infty{-}\mathfrak{Grpd}^{\mathfrak{C}^{op}}$ whose augmentation is $よ_C$ such that for every $[n] \in \Delta$ the map $\mathcal{F}_n \to \mathrm{cosk}_{n-1}(\mathcal{F}_\bullet)_n$ is a τ-covering. The hypercover is said to be **effective** if its codescent object is $よ_C$.

An object $\mathcal{X} \in \infty{-}\mathfrak{Grpd}^{\mathfrak{C}^{op}}$ is a **τ-hypersheaf** if it is local for the codescent objects of effective τ-hypercovers. The $(\infty,1)$-topos of hypersheaves is denoted $\mathfrak{HSh}_\tau(\mathfrak{C})$.

Proposition 3.1.1.1.14 ([TV05, Theorem 3.8.3]). *Let $\mathfrak{C}$ be a small $(\infty,1)$-category. There is a bijective correspondence between Grothendieck topologies on $\mathfrak{C}$ and hypercomplete left-exact localisations of $\infty{-}\mathfrak{Grpd}^{\mathfrak{C}^{op}}$.*

3.1.1.2 Geometric stacks and principal bundles

Let $\mathfrak{T}$ be a hypercomplete $(\infty,1)$-topos, which can thus be written as $\mathfrak{T} = \mathfrak{HSh}_\tau(\mathfrak{C})$ for some $(\infty,1)$-site $(\mathfrak{C},\tau)$. We wish to consider the objects of $\mathfrak{C}$ (or rather the presheaves they represent) as the basic geometric objects in $\mathfrak{T}$, and restrict our study from $\mathfrak{T}$ to the sheaves which are geometric enough in that they can be suitably covered by geometric objects. The topology τ can be too rigid a specification for this, so the more flexible notion of geometric context is used.

Definition 3.1.1.2.0 (Geometric context). A **geometric context** is a class $\mathtt{P}$ of morphisms stable by composition, equivalences and base-change, such that

- any morphism in a covering family is in $\mathtt{P}$,
- being in $\mathtt{P}$ is a local property (for any $f\colon X \to Y$ in $\mathtt{P}$, if there exists a covering family $\{\rho_i \colon U_i \to X\}_i$ of X such that all the composites $f\rho_i$ are in $\mathtt{P}$, then f is in $\mathtt{P}$), and
- for any X, Y, the natural morphisms $X, Y \rightrightarrows X \amalg^h Y$ are in $\mathtt{P}$.

By the effectivity condition of the Giraud axioms, the effective epimorphisms in an $(\infty,1)$-topos are particularly well-behaved, and in fact allow for the parameterisation of torsors under internal groupoids. Before going further with geometry, we will thus discuss principal bundles in an $(\infty, 1)$-topos, following [NSS15], which deals mainly with internal groups, that is groupoids $G_\bullet$ whose object of objects G_0 is terminal, but whose constructions and proofs generalise straightforwardly to the case of general groupoids.

Definition 3.1.1.2.1 (Groupoid principal bundle)**.** Let $G_\bullet$ be a groupoid object in a cartesian $(\infty, 1)$-category $\mathfrak{C}$. An **action** of $G_\bullet$ on an object $P \in \mathfrak{C}$ is given by an anchor map $\alpha\colon P \to G_0$ and a groupoid object of the form

$$P \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; P \times_{G_0} G_1 \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; P \times_{G_0} G_2 \tag{3.1}$$

such that the canonical arrows $P \times_{G_0} G_i \to G_i$ define a morphism of internal groupoids to $G_\bullet$.

Let now $\mathfrak{E}$ be an $(\infty, 1)$-topos, and let $B \in \mathfrak{E}$. A **principal $G_\bullet$-bundle** over X is an object $P \in \mathfrak{E}$ equipped with a $G_\bullet$-action and a map $P \to X$ exhibiting X as the codescent object of the action groupoid.

Example 3.1.1.2.2. Any internal groupoid $G_\bullet$ defines tautologically a principal bundle over its codescent object, which will be denoted $\mathcal{B}\, G_\bullet$ (*via* the canonical quotient map $G_0 \to \mathcal{B}\, G_\bullet$).

Example 3.1.1.2.3. Any internal groupoid $G_\bullet$ acts on its object G_1 of arrows by translation or composition. Concretely, the action groupoid is the simplicial décalage of $G_\bullet$, together with its canonical map to $G_\bullet$.

The décalage $(\infty, 1)$-functor is left-adjoint to the forgetful $(\infty, 1)$-functor from split (augmented) simplicial objects to simplicial objects. By [RV21, Proposition 2.3.11], the colimit of a split simplicial object is given by its augmentation, so this action defines a principal $G_\bullet$-bundle over the object G_0 of objects.

Example 3.1.1.2.4. Let $X \to G_0$ be an object in $\mathfrak{E}_{/G_0}$. The trivial $G_\bullet$-bundle on X is the base-change $X \times_{G_0} G_{\bullet+1}$.

Remark 3.1.1.2.5 ([NSS15, Propositions 3.7, 3.12]). As in the classical definition, every principal $G_\bullet$-bundle $P \to X$ is indeed principal, as well as locally trivial.

The principality condition means that the morphism $P \times_{G_0} G_1 \to P \times_X P$ induced by the two maps $P \times_{G_0} G_1 \to P$ which constitute the first stage of the action groupoid is an equivalence. This is simply a consequence of the effectiveness of the $(\infty, 1)$-topos $\mathfrak{E}$, as the action groupoid must be equivalent to the higher kernel of its codescent object $P \to X$.

Meanwhile, local triviality means that there exists a G_0-object $U \to G_0$ and an effective epimorphism $U \twoheadrightarrow X$ with an equivalence of principal $G_\bullet$-bundles between the base-change $U \times_X P \to U$ and the trivial $G_\bullet$-principal bundle over U. Indeed, one may choose $U \to G_0$ to be the anchor map $P \to G_0$, with the effective epimorphism $P \twoheadrightarrow X$ the structure map of the $G_\bullet$-bundle structure. Then the argument used above to establish the principality property, extended from the first stage to the full action groupoid, shows that the base-change of P to U is trivial.

Lemma 3.1.1.2.6 ([NSS15, Proposition 3.8])**.** *Let $X \to \mathcal{B}\, G_\bullet$ be a morphism. Its fibre $G_0 \times_{\mathcal{B}\, G_\bullet} X \to X$ is endowed with a canonical principal $G_\bullet$-bundle structure.*

Proof. The groupoid encoding the action of $G_\bullet$ on $P := G_0 \times_{\mathcal{B}\, G_\bullet} X$ may be defined as the base-change of the groupoid $G_\bullet$ along $X \to \mathcal{B}\, G_\bullet$, with as anchor maps the canonical arrows:

$$\begin{array}{ccccccc}
X & \longleftarrow & X \underset{\mathcal{B}\, G_\bullet}{\times} G_0 & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} & X \underset{\mathcal{B}\, G_\bullet}{\times} G_1 & \substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow} & \cdots \\
\big\downarrow & \lrcorner & \big\downarrow & \lrcorner & \big\downarrow & & \\
\mathcal{B}\, G_\bullet & \twoheadleftarrow & G_0 & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} & G_1 & \substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow} & \cdots
\end{array} \quad . \tag{3.2}$$

By the pasting law for pullbacks, one finds at each stage of this groupoid an equivalence

$$X \underset{\mathcal{B}\,G_\bullet}{\times} G_n \simeq X \underset{\mathcal{B}\,G_\bullet}{\times} G_0 \underset{G_0}{\times} G_n = P \underset{G_0}{\times} G_n, \tag{3.3}$$

showing that equation (3.2) exhibits a $G_\bullet$-action on P.

Furthermore, by [Lur09, Proposition 6.2.3.15], effective epimorphisms in $\mathfrak{E}$ are stable by pullback. As the arrow $G_0 \twoheadrightarrow \mathcal{B}\,G_\bullet$ is an effective epimorphism by definition, its pullback $P \to X$ is one as well. □

Proposition 3.1.1.2.7 ([NSS15, Proposition 3.13]). *Let $P \to X$ be a principal $G_\bullet$-bundle. The square*

$$\begin{array}{ccc} P & \longrightarrow & G_0 \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{B}\,G_\bullet \end{array} \tag{3.4}$$

is cartesian.

Corollary 3.1.1.2.8 ([NSS15, Theorem 3.17], [TV08, Proposition 1.3.5.3]). *The quotient map $G_0 \to \mathcal{B}\,G_\bullet$ is the universal $G_\bullet$-bundle, in that pullback induces an equivalence of $(\infty,1)$-categories*

Remark 3.1.1.2.9 ([NSS15, Proposition 3.16]). It follows in particular that the $(\infty,1)$-category of principal $G_\bullet$-bundles is actually an ∞-groupoid, that is that any morphism of principal bundles is invertible.

We may now come back to defining geometric conditions. We work in a hypercomplete $(\infty,1)$-topos $\mathfrak{T} = \mathfrak{HSh}_\tau(\mathfrak{C})$, whose objects we will also refer to as **stacks**.

Definition 3.1.1.2.10 (Geometric stack). Let $\mathtt{P}$ be a geometric context.

- A stack is (-1)**-geometric** if it is representable.
- A morphism of stacks $X \to Y$ is **in** (-1)-$\mathtt{P}$ if for any representable W and any morphism $W \to Y$, the pullback $W \times_Y X$ is representable and the canonical morphism $W \times_Y X \to X$ is in $\mathtt{P}$.

Now assume, for $n \geqslant -1$, that we have defined notions of n-geometric stack and of morphisms of stacks in n-$\mathtt{P}$. We will also say that a morphism of stacks $X \to Y$ is n**-representable** if for any representable W and any morphism $W \to Y$, the pullback $W \times_Y X$ is an n-representable stack.

- An internal groupoid is n-$\mathtt{P}$ if G_0 and G_1 are n-geometric and the map $s\colon G_1 \to G_0$ is in n-$\mathtt{P}$.
- A stack is $(n+1)$**-geometric** if it carries a universal $G_\bullet$-bundle, that is if it is equivalent to the classifying stack $\mathcal{B}\,G_\bullet$, for some n-$\mathtt{P}$ groupoid $G_\bullet$. In that case, by [TV08, Proposition 1.3.4.2. (2)], G_0 can be taken to be a disjoint union of representable stacks.
- A morphism of stacks $X \to Y$ is $(n+1)$-$\mathtt{P}$ if for any representable W and $W \to Y$, $X \times_Y W$ is n-geometric and is covered by representables U_α such that $U_\alpha \to X$ is $\mathtt{P}$.

Hence one may say that the study of n-geometric stacks is the study of equivariant structures in $(n-1)$-geometric stacks, *i.e.* the study of $(n+1)$-fold iterated equivariant structures in representable stacks.

Remark 3.1.1.2.11. The involution $G_1 \xrightarrow{\simeq} G_1$ making an internal category $G_\bullet$ into a groupoid induces an equivalence between the arrows s and t; this is the reason why it is enough to require only that s be in n-P to make $G_\bullet$ an n-P groupoid, as it will follow that all other face maps are also in n-P.

Proposition 3.1.1.2.12 ([TV08, Proposition 1.3.4.2]). *A stack X is n-geometric (for $n \geqslant 0$) if and only if its diagonal morphism $X \to X \times X$ is $(n-1)$-geometric and there exists an epimorphism $U \to X$ where U is a disjoint union of representables U_α and each $U_\alpha \to X$ is $(n-1)$-P.*

Such an epimorphism $\coprod_\alpha U_\alpha \to$ is called a n**-atlas** for X.

Proof. If $X \simeq \mathcal{B}\, G_\bullet$, the quotient map $G_0 \to \mathcal{B}\, G_\bullet$ furnishes the desired effective epimorphism.

Conversely, if X has an n-atlas $U \to X$, the effectivity condition dictates that the corresponding groupoid presentation should be its higher kernel $U_{/X}^{\times(\bullet+1)}$. □

3.1.2 Derived commutative algebra

3.1.2.1 Stable $(\infty,1)$-categories and their K-theory

Definition 3.1.2.1.1 ((Co)fibres). Let $\mathfrak{C}$ be an $(\infty,1)$-category and $f\colon X \to Y$ an arrow of $\mathfrak{C}$. If $\mathfrak{C}$ has a terminal object $*$, a **cofibre** of f is a colimit of the span $* \xleftarrow{!_X} X \xrightarrow{f} Y$, that is a pushout as below-left.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle !_X}\downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & \mathrm{cofib}(f) \end{array} \qquad \begin{array}{ccc} \mathrm{fib}(f) & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ \emptyset & \xrightarrow[!^Y]{} & Y \end{array} \tag{3.5}$$

If $\mathfrak{C}$ has an initial object $\emptyset$, a **fibre** of f is a limit of the cospan $\emptyset \xrightarrow{!^Y} Y \xleftarrow{f} X$, that is a pullback as indicated in the cartesian square above-right.

Notation 3.1.2.1.2 ((Co)fibred sequences). Let $\mathfrak{C}$ be an $(\infty,1)$-category with a zero object (meaning an object which is both terminal and initial), denoted 0. A **triangle** in $\mathfrak{C}$ is a coherent square whose bottom left corner is 0, as in equation (3.5).

Definition 3.1.2.1.3 (Stable $(\infty,1)$-category). An $(\infty,1)$-category is **stable** if it is pointed, every morphism has a fibre and a cofibre, and a triangle is exact if and only if it is coexact (*i.e.* every morphism is the fibre of its cofibre, as well as the cofibre of its fibre).

Lemma 3.1.2.1.4 ([Lur17, Proposition 1.1.4.1]). *An $(\infty,1)$-functor between stable $(\infty,1)$-categories is exact if and only if it is left exact, if and only if it is right exact.*

We denote $\mathfrak{St}$ the locally full sub-$(\infty,2)$-category of $(\infty,1)-\mathfrak{CAT}$ spanned by the stable $(\infty,1)$-categories and the exact $(\infty,1)$-functors between them.

Definition 3.1.2.1.5 (Stabilisation). Let $\mathfrak{C}$ be a finitely complete $(\infty,1)$-category. A **stabilisation** of $\mathfrak{C}$ is a stable $(\infty,1)$-category $\mathrm{Stab}(\mathfrak{C})$ endowed with a left-exact $(\infty,1)$-functor $\Omega^\infty_{\mathfrak{C}}\colon \mathrm{Stab}(\mathfrak{C}) \to \mathfrak{C}$ such that, for any stable $(\infty,1)$-category $\mathfrak{A}$, postcomposition with $\Omega^\infty_{\mathfrak{C}}$ induces an equivalence $\mathfrak{St}(\mathfrak{A}, \mathrm{Stab}(C)) \xrightarrow{\simeq} \mathfrak{Cat}^{\mathrm{l.ex.}}(\mathfrak{A}, \mathfrak{C})$.

In other words, if every finitely complete $(\infty,1)$-category admits a stabilisation, the assignment $\mathfrak{C} \mapsto \mathrm{Stab}(\mathfrak{C})$ should produce a right-$(\infty,2)$-adjoint to the inclusion $\mathfrak{St} \hookrightarrow \mathfrak{Cat}^{\mathrm{l.ex.}}$, with counit Ω^∞.

Lemma 3.1.2.1.6 ([Lur17, Corollary 1.4.2.27], [RV21, Proposition 4.4.4, Proposition 4.4.11, Theorem 4.4.12])**.** *Let $\mathfrak{C}$ be a pointed $(\infty, 1)$-category. The constructions of suspensions and loop space objects induce an adjunction $\Sigma_{\mathfrak{C}} \colon \mathfrak{C} \rightleftarrows \mathfrak{C} \colon \Omega_{\mathfrak{C}}$.*

Furthermore, $\mathfrak{C}$ is stable if and only if it is finitely cocomplete and $\Sigma_{\mathfrak{C}}$ is an equivalence, if and only if it is finitely complete and $\Omega_{\mathfrak{C}}$ is an equivalence.

Hence, if $\mathfrak{C}$ is a pointed finitely complete $(\infty, 1)$-category, the problem of constructing a stabilisation of $\mathfrak{C}$ is that of universally inverting the endo-$(\infty, 1)$-functor $\Omega_{\mathfrak{C}}$.

Proposition 3.1.2.1.7 ([Lur17, Proposition 1.4.2.24, Corollary 1.4.2.23])**.** *For every finitely complete $(\infty, 1)$-category $\mathfrak{C}$, the pointed finitely complete $(\infty, 1)$-category*

$$\mathfrak{C}^{\text{🔔}} = \varprojlim(\cdots \to \mathfrak{C}^{*/} \xrightarrow{\Omega_{\mathfrak{C}}} \mathfrak{C}^{*/} \xrightarrow{\Omega_{\mathfrak{C}}} \mathfrak{C}^{*/}) \tag{3.6}$$

is stable, and the canonical projection $(\infty, 1)$-functor $\Omega^{\infty}_{\mathfrak{C}} \colon \mathfrak{C}^{\text{🔔}} \to \mathfrak{C}$ exhibits $\mathfrak{C}^{\text{🔔}}$ as a stabilisation of $\mathfrak{C}$.

Notation 3.1.2.1.8 (t-structures and homotopy objects)**.** A **t-structure** on a stable $(\infty, 1)$-category $\mathfrak{A}$ is defined to be a t-structure on the triangulated category $\operatorname{Ho}\mathfrak{A}$.

Lemma 3.1.2.1.9 ([Lur17, Proposition 1.4.3.4])**.** *Let $\mathfrak{C}$ be a locally presentable $(\infty, 1)$-category. Let $\mathfrak{C}^{\text{🔔}}_{\leqslant -1}$ denote the full sub-$(\infty, 1)$-category of $\mathfrak{C}^{\text{🔔}}$ on the spectrum objects X such that $\Omega^{\infty}_{\mathfrak{C}}(X)$ is terminal in $\mathfrak{C}$. Then $\mathfrak{C}^{\text{🔔}}_{\leqslant -1}$ determines an accessible t-structure on $\mathfrak{C}^{\text{🔔}}$.*

Definition 3.1.2.1.10 (Connectivity)**.** A spectrum object $X \in \mathfrak{C}^{\text{🔔}}$ is said to be **connective** if it lies in the full sub-$(\infty, 1)$-category $\mathfrak{C}^{\text{🔔}}_{\geqslant 0}$.

Before turning to commutative algebra, we make a small digression about the K-theory of stable $(\infty, 1)$-categories, a decategorification which will allow the link between derived geometry and virtual phenomena in classical algebraic geometry.

Construction 3.1.2.1.11 (Non-commutative motives)**.** A Verdier localising sequence of stable $(\infty, 1)$-categories is a sequence of adjunctions

$$\mathfrak{A} \underset{\mathscr{R}}{\overset{i}{\rightleftarrows}} \mathfrak{B} \underset{j}{\overset{\mathscr{L}}{\rightleftarrows}} \mathfrak{C} \tag{3.7}$$

such that the composite $\mathscr{L}i$ is a zero object, the transformations $\mathrm{id}_{\mathfrak{A}} \overset{\eta}{\Rightarrow} \mathscr{R}i$ and $\mathscr{L}j \overset{\varepsilon}{\Rightarrow} \mathrm{id}_{\mathfrak{C}}$ are equivalences, and the sequence $i\mathscr{R} \overset{\varepsilon}{\Rightarrow} \mathrm{id}_{\mathfrak{B}} \overset{\eta}{\Rightarrow} j\mathscr{L}$ is a (co)fibre sequence in the stable $(\infty, 1)$-category $\mathfrak{St}(\mathfrak{B}, \mathfrak{B})$.

A localising invariant with values in a stable $(\infty, 1)$-category $\mathfrak{T}$ is an $(\infty, 1)$-functor $\mathscr{Q} \colon \iota_1 \mathfrak{St} \to \mathfrak{T}$ which preserves zero objects and takes localisation sequences to (co)fibre sequences in $\mathfrak{T}$. The $(\infty, 1)$-category of noncommutative motives is defined as the universal localising invariant $\mathscr{U} \colon \iota_1 \mathfrak{St} \to \mathfrak{NMot}$. More precisely, it is constructed as the full sub-$(\infty, 1)$-category of $(\infty\text{–}\mathfrak{Grpd}^{\text{🔔}})^{\iota_1 \mathfrak{St}^{\mathrm{op}}}$ on the localising invariants.

For any stable $(\infty, 1)$-category $\mathfrak{A}$, its K-theory spectrum is defined as $K(\mathfrak{A}) = \mathfrak{NMot}(\mathscr{U}(\infty\text{–}\mathfrak{Grpd}^{\text{🔔}}), \mathscr{U}\mathfrak{A})$. It is shown in [BGT13] that this recovers other usual definitions of K-theory.

Remark 3.1.2.1.12 (Zeroth K-group)**.** The zeroth homotopy group $K_0(\mathfrak{A}) := \pi_0(K(\mathfrak{A}))$ is characterised by a decategorification of the universal property of $K(\mathfrak{A})$. Say that a function φ from

$\mathrm{Obj}(\mathfrak{A})$ to (the underlying set of) an abelian group G is additive if for any fibred sequence $A \to B \to C$ in $\mathfrak{A}$ one has $\varphi(B) = \varphi(A) + \varphi(C)$. Then the function $\mathrm{Obj}(\mathfrak{A}) \to K_0(\mathfrak{A})$ sending an object to its class is additive, and is initial for this property.

Theorem 3.1.2.1.13 (of the heart, [Bar15, Theorem 6.1]). *Let $\mathfrak{S}$ be a stable $(\infty,1)$-category endowed with a bounded t-structure. Then $K(\mathfrak{S}) \simeq K(\mathfrak{S}^{\heartsuit})$.*

3.1.2.2 Derived commutative algebras and infinitesimal calculus

Theorem 3.1.2.2.1 ([Lur17, Corollary 4.8.2.19]). *The $(\infty,1)$-category $\infty\!-\!\mathfrak{Grpd}^{\Omega}$ of spectra admits a unique symmetric monoidal structure which preserves colimits in each variables and has as unit the sphere spectrum $\mathbb{S}$.*

Definition 3.1.2.2.2. An $\mathscr{E}_\infty$**-ring spectrum** is an $\mathscr{E}_\infty$-algebra in the symmetric monoidal $(\infty,1)$-category of spectra.

Construction 3.1.2.2.3 (Modules). There is ([Lur17, Definition 4.2.1.7]) an operad $\mathscr{AlgMod}$ with two colours $\mathfrak{a}$ and $\mathfrak{m}$, such that the full suboperad on the colour $\mathfrak{a}$ is equivalent to the associative ∞-operad $\mathscr{E}_1 \simeq \mathscr{A}_\infty$. Its algebras are identified with pairs of an $\mathscr{A}_\infty$-algebra and a module over it. If the $\mathscr{A}_\infty$-algebra A has in fact a structure of $\mathscr{E}_\infty$-algebras, the $(\infty,1)$-categories of left and right modules over it coincide.

Definition 3.1.2.2.4. Let $p\colon \mathfrak{E} \to \mathfrak{C}$ be a locally presentable fibration of $(\infty,1)$-categories. A **stable envelope** of p is a fibration $u\colon \mathfrak{Stab}(\mathfrak{E}/\mathfrak{C}) \to \mathfrak{E}$ such that pu is a locally presentable fibration, u carries pu-cartesian arrows to p-cartesian arrows, and for every $C \in \mathfrak{C}$ the ∞-functor $\mathfrak{Stab}(\mathfrak{E}/\mathfrak{C})_C \to \mathfrak{E}_C$ is a stabilisation $\Sigma^\infty_{\mathfrak{E}_C}$.

A **tangent bundle** for a locally presentable $(\infty,1)$-category $\mathfrak{C}$ is a stable envelope $\mathfrak{T}_{\mathfrak{C}}$ of the codomain fibration $\mathfrak{C}^2 \xrightarrow{\mathrm{ev}_1} \mathfrak{C}$.

Proposition 3.1.2.2.5 ([Lur17, Theorem 7.3.4.18]). *Let $\mathscr{O}$ be a coherent ∞-operad and let $\mathfrak{V}$ be an $\mathscr{O}$-monoidal $(\infty,1)$-category. Then a tangent bundle $\mathfrak{T}_{\mathscr{O}-\mathfrak{Alg}(\mathfrak{V})}$ exists and there is a canonical equivalence $\mathfrak{T}_{\mathscr{O}-\mathfrak{Alg}(\mathfrak{V})} \xrightarrow{\simeq} \mathscr{O}\!-\!\mathfrak{Alg}(\mathfrak{V}) \underset{\mathscr{O}-\mathfrak{Alg}(\mathfrak{V})^{\mathscr{O}}}{\times} \mathscr{O}\!-\!\mathfrak{Mod}(\mathfrak{V})^{\mathscr{O}}$ of locally presentable fibrations over $\mathscr{O}\!-\!\mathfrak{Alg}(\mathfrak{V})$.*

Construction 3.1.2.2.6 (Cotangent complex). Let $\mathfrak{C}$ be a locally presentable $(\infty,1)$-category. By [Lur17, Proposition 7.3.2.6, Remark 7.3.2.15], the $(\infty,1)$-functor $\mathfrak{T}_{\mathfrak{C}} \to \mathfrak{C}^2 \xrightarrow{\mathrm{ev}_1} \mathfrak{C}$ admits a left-adjoint. It is called the **absolute cotangent complex** $(\infty,1)$-functor, and denoted $C \mapsto \mathbb{L}_C$.

If $C \to D$ is a morphism in $\mathfrak{C}$, by [Lur17, Remark 7.3.3.2], its **relative cotangent complex** can be defined as the cofibre $\mathbb{L}_{D/C} = \mathrm{cofib}(\mathbb{L}_C \to \mathbb{L}_D)$.

The cotangent complex is the main tool needed to speak of differential calculus and in particular of smoothness for $\mathscr{E}_\infty$-algebras. To interpret correctly the numerical criteria for these definitions, it will be necessary to restrict to connective algebras.

Notation 3.1.2.2.7. We say that an $\mathscr{E}_\infty$-ring spectrum is **connective** if its underlying spectrum is. If $\Bbbk$ is a base connective $\mathscr{E}_\infty$-ring spectrum, we will write $\mathfrak{dAlg}_{\Bbbk}$ for the $(\infty,1)$-category of connective $\Bbbk$-algebras, which we nickname "derived $\Bbbk$-algebras".

Definition 3.1.2.2.8. Let $\varphi\colon A \to B$ be a morphism of $\mathscr{E}_\infty$-ring spectra. We say that φ is

- **finitely presented** if $\mathscr{E}_\infty\!-\!\mathfrak{Alg}_A(B,-)$ commutes with filtered colimits,
- **formally smooth** if $\mathbb{L}_{B/A}$ has Tor-amplitude $\leqslant 0$,

- **formally étale** if $\mathbb{L}_{B/A}$ vanishes,
- **smooth** (resp. **étale**) if it is finitely presented and formally smooth (resp. and formally étale).

Remark 3.1.2.2.9 (Spectral and derived algebra). A morphism $\varphi\colon A \to B$ of connective $\mathscr{E}_\infty$-ring spectra is said to be **strong** if, for all $n \in \mathbb{N}$, the canonical map of (classical) $\pi_0 B$-modules $\pi_0(\pi_n A \otimes_{\pi_0 A} \pi_0 B) \to \pi_n B$ is an isomorphism.

By [Gre17, Proposition 2.2.11, Lemma 2.2.10], for any connective $\mathscr{E}_\infty$-ring spectrum A, the free A-algebra on one generator $A\{x\}$ is flat if and only if A is a $\mathbb{Q}$-algebra, if and only if $\pi_0 A$ is a $\mathbb{Q}$-algebra.

Let $\Bbbk$ be a fixed base connective $\mathscr{E}_\infty$-ring spectrum. We take the Grothendieck topology on $\mathfrak{dAlg}_{\Bbbk}{}^{op}$ to be generated by the étale (surjective) coverings. This defines a notion of derived stacks of $\Bbbk$.

There are now two geometricity conditions which can be imposed, corresponding to two geometric contexts. For the context of étale morphisms, the geometric derived stacks are called **Deligne–Mumford derived stacks** over $\Bbbk$. For the context of smooth morphisms, the geometric derived stacks are called **Artin derived $\Bbbk$-stacks**. We also talk of algebraic derived $\Bbbk$-stacks.

Proposition 3.1.2.2.10 ([Lur19, Proposition 3.5.4.2]). *Let X be a Deligne–Mumford derived stack. The $(\infty, 1)$-category of points of the $(\infty, 1)$-topos $X_{\text{ét}}$ is equivalent to the $(\infty, 1)$-category of geometric points of X.*

3.1.2.3 Formal algebraicity and $\mathscr{L}_\infty$-algebroids

Definition 3.1.2.3.1 (Formally algebraic derived stack). Let X be a derived stack.

- X is **nilcomplete** (or **convergent**) if for any $A \in \mathfrak{dAlg}_{\Bbbk}$, $X(A) \to \varprojlim_n X(\tau_{\leqslant n} A)$ is an equivalence.
- X is **infinitesimally cohesive** if for any cartesian square as below-left in $\mathfrak{dAlg}_{\Bbbk}$

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow \varphi \\ B' & \xrightarrow[\psi]{} & B \end{array} \qquad\qquad \begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & \lrcorner ? & \downarrow X(\varphi) \\ X(B') & \xrightarrow[X(\psi)]{} & X(B) \end{array} \tag{3.8}$$

  such that $\pi_0\varphi$ and $\pi_0\psi$ are surjections with nilpotent kernels, the induced square of ∞-groupoids above-right is cartesian.
- X is formally algebraic if it is nilcomplete, infinitesimally cohesive, and admits a pro-cotangeant complex (meaning that the $(\infty, 1)$-functor of derivations is pro-representable).

Theorem 3.1.2.3.2 ([TV08, Theorem 2.2.6.12],[Lur19, Theorem 18.1.0.2]). *A derived stack X is n-algebraic if and only if it is formally algebraic, admits a cotangeant complex, and its truncation $t_0(X)$ is an Artin $(n+1)$-stack.*

3.1.3 Virtual classes and pullbacks from derived thickenings

3.1.3.1 Definition from derived geometry

Let $f\colon X \to Y$ be a **quasi-smooth** morphism of derived stacks, that is its cotangent complex $\mathbb{L}_{f\colon X/Y}$ is of perfect (homotopical) Tor-amplitude smaller than 1, that is there is an integer $n \in \mathbb{N}$ such that the Tor-amplitude is concentrated in $[-n, 1]$.

Remark 3.1.3.1.1. By [GR17a, Chapter 4, Lemma 3.1.3], as the quasi-smooth morphism f is of finite Tor-amplitude, the pullback of quasicoherent sheaves f^* maps $\mathfrak{Coh}^b(Y)$ to $\mathfrak{Coh}^b(X)$. As we work in G-theory, which is the K-theory of the stable $(\infty, 1)$-category of bounded coherent sheaves, the notation f^* will be understood in this subsection to mean the restriction of the pullback operation to coherent sheaves.

Due to theorem 3.1.2.1.13 and [Lur19, Corollary 2.5.9.2 with $n = 0$], the closed embedding $\jmath_X \colon t_0 X \hookrightarrow X$ induces an isomorphism $\jmath_{X,*} \colon G_0(t_0 X) \xrightarrow{\simeq} G_0(X)$ in G-theory, with inverse $(\jmath_{X,*})^{-1}(\mathcal{G}) = \sum_{i \geqslant 0} (-1)^i [\pi_i(\mathcal{G})]$.

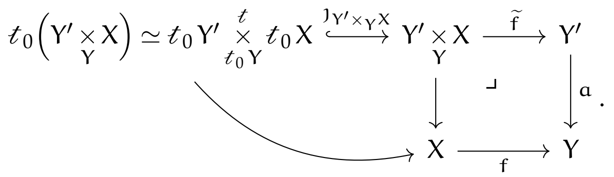
It is therefore natural to define the virtual pullback along $t_0 f$ to be given by the actual pullback along f, intertwined with these isomorphisms.

However we wish to consider the virtual pullback as a bivariant class, that is defined as a collection of maps $G_0(Y') \to G_0(X \times_Y S) = G_0(X \times^t_Y Y')$ indexed by all $t_0 Y$-schemes $Y' \to t_0 Y$, or more generally by all derived Y-schemes $Y' \to Y$. Then the virtual pullback we defined should be the map corresponding to the $t_0 Y$-scheme $\mathrm{id}_{t_0 Y} \colon t_0 Y = t_0 Y$.

We recall that we use the notation $\times^t$ (a fibre product decorated by t) to differentiate the strict (1- or 2-categorical) fibre products of classical stacks from the implictly ∞-categorical fibre products of derived stacks.

Definition 3.1.3.1.2. The **bivariant virtual pullback** along f is the collection, indexed by all Y-schemes $a \colon Y' \to Y$, of maps $(t_0 f)^{!,a}_{\mathrm{DAG}} \colon G_0(t_0 Y') \to G_0(t_0(Y' \times_Y X))$ defined as follows.

For a morphism of schemes $a \colon Y' \to Y$, we have the diagram

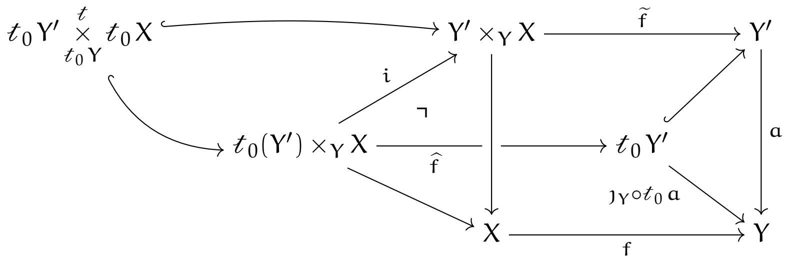


(3.9)

Then we set $(t_0 f)^{!,a}_{\mathrm{DAG}} \coloneqq (\jmath_{Y' \times_Y X,*})^{-1} \circ \widetilde{f}^* \circ \jmath_{Y',*}$.

Lemma 3.1.3.1.3. *The virtual pullback only depends on* $t_0 a \colon t_0 Y' \to t_0 Y$. *That is, for any* $a_1, a_2 \colon Y'_1, Y'_2 \to Y$ *with* $t_0 a_1 = t_0 a_2$, *the virtual pullbacks* $f^{!,a_1}_{\mathrm{DAG}}$ *and* $f^{!,a_2}_{\mathrm{DAG}}$ *induced by* a_1 *and* a_2 *are equal.*

Proof. For any $a \colon Y' \to Y$, we compare the virtual pullbacks induced by a and $t_0 Y' \xrightarrow{t_0(a)} t_0 Y \xrightarrow{\jmath_Y} Y$.



(3.10)

The back square is cartesian and its side $\jmath_{Y'}$ is a closed immersion and thus proper, so the base-change formula gives $\widetilde{f}^* \circ \jmath_{Y',*} = i_* \widehat{f}^*$. Commutativity of the leftmost triangle implies that $i_* \jmath_{t_0 Y' \times_Y X,*} = \jmath_{Y' \times_Y X,*}$, and as both closed immersions involved induce isomorphisms in G-theory, we have $(\jmath_{Y' \times_Y X,*})^{-1} i_* = (\jmath_{t_0 Y' \times_Y X,*})^{-1}$. Putting the ingredients together, we finally obtain that

$$f^{!,a}_{\mathrm{DAG}} := (\jmath_{Y'\times_Y X,*})^{-1}\widetilde{f}^*\jmath_{Y',*} = (\jmath_{Y'\times_Y X,*})^{-1} i_*\widehat{f}^* = (\jmath_{t_0 Y'\times_Y X,*})^{-1}\widehat{f}^* =: f^{!,\jmath_Y\circ t_0 a}_{\mathrm{DAG}}. \tag{3.11}$$

□

Remark 3.1.3.1.4 (Functoriality). The virtual pullbacks satisfy obvious functoriality properties. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two composable arrows, and let $a\colon Z' \to Z$ be a Z-scheme. We have the commutative diagram

$$\begin{array}{ccccccc} t_0(Z'\times_Z X) & & t_0(Z'\times_Z X) & & & & \\ & \searrow^{\jmath_{Z'\times_Z X}} & & \searrow^{\jmath_{Z'\times_Z Y}} & & & \\ & & Z'\times_Z X & \xrightarrow{\widetilde{f}} & Z'\times_Z Y & \xrightarrow{\widetilde{g}} & Z' \\ & & \downarrow & & \downarrow b & & \downarrow a \\ & & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z \end{array} \tag{3.12}$$

It follows by associativity of fibre products that

$$\begin{aligned}(t_0 f)^{!,b}_{\mathrm{DAG}} \circ (t_0 g)^{!,a}_{\mathrm{DAG}} &= (\jmath_{(Z'\times_Z Y)\times_Y X,*})^{-1} \circ \widetilde{f}^* \circ (\jmath_{Z'\times_Z Y,*}) \circ (\jmath_{Z'\times_Z Y,*})^{-1} \circ \widetilde{g}^* \circ \jmath_{Z',*} \\ &= (\jmath_{Z'\times_Z X,*})^{-1} \circ \widetilde{gf}^* \circ \jmath_{Z',*} =: (t_0(gf))^{!,a}_{\mathrm{DAG}}.\end{aligned} \tag{3.13}$$

3.1.3.2 Comparison with the construction from obstruction theories

Construction 3.1.3.2.1 (Virtual pullbacks from perfect obstruction theories)**.** Let $g\colon V \to W$ be a morphism of Artin stacks of Deligne–Mumford type (*i.e.* relatively DM) endowed with a perfect obstruction theory $\varphi\colon E \to \mathbb{L}_{g\colon V/W}$, inducing the closed immersion $\varphi^\vee\colon \mathfrak{C}_{g\colon V/W} \hookrightarrow \mathfrak{E}$, where $\mathfrak{E} = t_0(\mathbb{V}_V(E[1]^\vee))$ is the vector bundle (Picard) stack associated with E and $\mathfrak{C}_g$ is the intrinsic normal cone of g (constructed in [BF97]). As in [MR18] we define a derived thickening $\mathbb{R}^\varphi V$ of V as the derived intersection

$$\begin{array}{ccc} \mathbb{R}^\varphi V & \xrightarrow{q} & \mathfrak{C}_g \\ p\downarrow & \lrcorner & \downarrow \varphi^\vee \\ V & \xrightarrow[0_{\mathfrak{E}}]{} & \mathfrak{E} \end{array}. \tag{3.14}$$

Note that the arrow p is a retract of $\jmath_V$, and provides a splitting of the induced perfect obstruction theory $\jmath_V^*\mathbb{L}_{\mathbb{R}^\varphi V} \to \mathbb{L}_V$. We may use it to define a map of derived stacks $\mathbb{R}^\varphi g\colon \mathbb{R}^\varphi V \xrightarrow{p} V \xrightarrow{f} W$ which is a derived thickening of g.

We also recall the construction of the virtual pullback $g^!_\varphi$, or $g^!_{\mathrm{POT}}$, from the perfect obstruction φ, defined in [Man12a] for Chow homology then [Qu18] for G_0-theory.

Let $a\colon W' \to W$ and write $g'\colon V' \to W'$ the base-change of g. Recall that one may define a deformation space $\mathfrak{D}_{V'}W'$ over $\mathbb{P}^1_k$, with general fibre W' giving the open immersion $j\colon W'\times\mathbb{A}^1_k \hookrightarrow \mathfrak{D}_{V'}W'$, and special fibre $\mathfrak{C}_{g'}$ giving the complementary closed immersion $i\colon \mathfrak{C}_{g'}\times\{\infty\} \hookrightarrow \mathfrak{D}_{V'}W'$. It follows that there is an exact sequence of abelian groups $G_0(\mathfrak{C}_{g'}) \to G_0(\mathfrak{D}_{V'}W') \to G_0(W'\times\mathbb{A}^1) \to 0$ (coming from the fibred sequence of G-theory spectra). Furthermore, as (by excess intersection) i^*i_* is equivalent to tensoring by the symmetric algebra on the conormal bundle of $\mathfrak{C}_{g'}$ in $\mathfrak{D}_{V'}W'$ and as the latter is trivial, we have $i^*i_* = 0$, inducing a map $G_0(W'\times\mathbb{A}^1) \to G_0(\mathfrak{C}_{g'})$:

concretely, any section $j^{*,-1}$ of j^* gives the same map when post-composed with i^* so we do have a well-defined map $i^*j^{*,-1}$. The specialisation map $\mathrm{sp}\colon G_0(W') \to G_0(\mathfrak{C}_{g'})$ is then defined by precomposing it by $\mathrm{pr}^*\colon G_0(W') \to G_0(W' \times \mathbb{A}^1)$. Finally, the cartesian square defining V' induces by [Man12a, Proposition 2.26] a closed immersion $c\colon \mathfrak{C}_{g'} \hookrightarrow a^*\mathfrak{C}_g = V' \times_V \mathfrak{C}_g$, and the virtual pullback $g^{!,a}_\varphi$ along g is constructed as the composite

$$g^{!,a}_\varphi\colon G_0(W') \xrightarrow{\mathrm{sp}} G_0(\mathfrak{C}_{g'}) \xrightarrow{c_*} G_0(a^*\mathfrak{C}_g) \xrightarrow{(a^*\varphi^\vee)_*} G_0(a^*\mathfrak{E} = V' \times_V \mathfrak{E}) \xrightarrow{0^*_{a^*\mathfrak{E}}} G_0(V'). \tag{3.15}$$

Lemma 3.1.3.2.2. *The virtual pullback $(t_0\mathbb{R}^\varphi g)^!_{\mathrm{DAG}}$ as defined above for the map $\mathbb{R}^\varphi g$ coincides with the virtual pullback $g^!_\varphi$ of [Man12a; Qu18]: for any $a\colon W' \to W$, we have $(t_0\mathbb{R}^\varphi g)^{!,a}_{\mathrm{DAG}} = g^{!,a}_\varphi\colon G_0(W') \to G_0(V')$.*

Proof. We adapt the results of [Jos10, Proposition 3.5] to the more general case of a morphism that need not be a regular embedding.

Let again $a\colon W' \to W$ and write $g'\colon V' \to W'$ the base-change of g. We now review our construction of the virtual pullback from derived thickenings from the point of view of the perfect obstruction theory. The map $g^{!,a}_{\mathrm{DAG}}$ of definition 3.1.3.1.2 is computed in the following way: we define a derived thickening $\mathbb{R}^\varphi V'$ of $V' = V \times^t_W W'$ as $V' \times_{\mathfrak{E}} \mathfrak{C}_g$; note that we have $\mathbb{R}^\varphi V' = V' \times_V \mathbb{R}^\varphi V$ and writing $p'\colon \mathbb{R}^\varphi V' \to V'$ we obtain a derived thickening $\mathbb{R}^\varphi g' = g' \circ p'\colon \mathbb{R}^\varphi V' \to W'$ of g'. Then $g^{!,a}_{\mathrm{DAG}}$ is the pullback along $\mathbb{R}^\varphi g'$ followed by the inverse of $\jmath_{\mathbb{R}^\varphi V',*}$.

We also note that the fibred product $V' \times_{a^*\mathfrak{E}} (a^*\mathfrak{C}_g)$ is the base-change of $V \times_{\mathfrak{E}} \mathfrak{C}_g$ along $a'\colon V' \to V$, so the square

$$\begin{array}{ccc} \mathbb{R}^\varphi V' & \xrightarrow{q'} & a^*\mathfrak{C}_g \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle a^*\varphi^\vee} \\ V' & \xrightarrow[0_{a^*E}]{} & a^*\mathfrak{E} \end{array} \tag{3.16}$$

is cartesian. As p' is proper, we have $0^*_{a^*\mathfrak{E}}(a^*\varphi^\vee)_* = p'_*q'^{,*}$; concomitantly, as p' is a retract of $\jmath_{\mathbb{R}^\varphi V'}$ we have in G-theory $(\jmath_{\mathbb{R}^\varphi V',*})^{-1} = p'_*$. We conclude that the virtual pullback of [Qu18] coincides with $(\jmath_{\mathbb{R}^\varphi V',*})^{-1} \circ q'^{,*} \circ c_* \circ \mathrm{sp}$, and thus it only remains to check that the latter part specialises to $(\mathbb{R}^\varphi g')^* = p'^{,*} \circ g'^{,*}$. But the deformation space $\mathfrak{D}_{V'}W'$ provides exactly an interpolation between $g'\colon V' \to W'$ and $V' \hookrightarrow \mathfrak{C}_{g'}$, so by transporting this comparison along the $\mathbb{A}^1$-invariance of G-theory the lemma is proved. □

Recall that for any quasi-smooth morphism $f\colon X \to Y$ of derived Artin stacks, by [STV15, Proposition 1.2] the canonical map $\varphi\colon j^*_X\mathbb{L}_f \to \mathbb{L}_{t_0 f}$ is a perfect obstruction theory.

Proposition 3.1.3.2.3. *Let $f\colon X \to Y$ be a quasi-smooth relatively DM map of derived Artin stacks. The virtual pullback $(t_0 f)^!_{\mathrm{DAG}}$ defined with derived geometry is equal to $(t_0\mathbb{R}^\varphi t_0 f)^!_{\mathrm{DAG}}$, and thus to the virtual pullback $(t_0 f)^!_\varphi$ of [Man12a; Qu18], induced by the obstruction theory $\varphi\colon j^*_X\mathbb{L}_f \to \mathbb{L}_{t_0 f}$.*

Proof. The proof is similar to the one given in [MR18, Proposition 4.3.2] for the comparison of the virtual classes defined from perfect obstruction theories and derived geometry, which mainly followed [LS12]: one constructs a deformation to the normal bundle of the closed immersion $\jmath_X\colon t_0 X \hookrightarrow X$, and finally uses that G-theory is $\mathbb{A}^1$-invariant. □

We shall henceforth simply write $(t_0 f)^!$ for the virtual pullback along f.

Example 3.1.3.2.4 (Virtual classes). Suppose $Y = \mathrm{Spec}(k)$ so $f\colon X \to \mathrm{Spec}(k)$ is the structure morphism. The virtual structure sheaf of $t_0 X$ is $\left[\mathcal{O}^{\mathrm{vir}}_{t_0 X}\right] = f^{!,\mathrm{id}_X}([\mathcal{O}_{\mathrm{Spec}(k)}]) = (\jmath_{X,*})^{-1}([\mathcal{O}_X])$.

Example 3.1.3.2.5. Suppose that the classical map g is already a quasi-smooth immersion, so that $\mathrm{id}_{\mathbb{L}_g}$ is a perfect obstruction theory. Then the virtual pullback is given by the Gysin pullback $g^!$, studied in details for example in [Jos10].

Remark 3.1.3.2.6 (Virtual pullbacks in generalised motivic homology theories). Our construction of virtual pullbacks only relies on the fact that G-theory is insensitive to the non-reduced structure, and the identification with the classical definition requires simply the specialisation morphism and, more generally, the $\mathbb{A}^1$-invariance. These ingredients are present in motivic homotopy theory (by construction for the $\mathbb{A}^1$-invariance, and by [Kha19a, Corollary 3.2.9] for the insentivity to derived structures), so the virtual pullbacks in motivic cohomology theories also admit the derived geometric interpretation.

In fact such virtual pullbacks were constructed for motivic Borel–Moore homology with coefficients in any étale motivic spectrum in [Kha19c, Construction 3.4] from the virtual pullbacks canonically associated with a quasi-smooth derived enhancement (through its derived deformation space).

3.2 Derived moduli of maps

3.2.1 Moduli problems for morphisms of stacks

3.2.1.1 Representability

Definition 3.2.1.1.1. Let $\mathcal{T} = \mathfrak{HSh}_\tau(\mathcal{S})$ be a hypercomplete $(\infty, 1)$-topos. The internal hom between two objects $X \to B, Y \to B$ in a slice $\mathcal{T}_{/B}$ is denoted $\mathcal{M}or_{/B}(X, Y)$ and called the **mapping (derived) B-stack**.

It is determined, as a sheaf, by

$$\mathcal{M}or_{/B}(X, Y)(Z) = \mathcal{T}_{/B}(X \times_B Z, Y). \tag{3.17}$$

Lemma 3.2.1.1.2 ([MR18], [HP14, Proposition 5.1.10]). *Let B be a spectral Artin stack, and let C and V be spectral Artin B-stacks. Consider the universal mapping diagram*

$$\begin{array}{ccccc} & & C \times_B \mathcal{M}or_B(C, V) & & \\ & \swarrow^{p} & & \searrow^{\mathrm{ev}} & \\ \mathcal{M}or_B(C, V) & & & & V \\ & \searrow & & \swarrow & \\ & & B & & \end{array} \tag{3.18}$$

There is a canonical equivalence $\mathbb{T}_{\mathcal{M}or_B(C,V)/B} \simeq p_* \mathrm{ev}^* \mathbb{T}_{V/B}$ in $\mathfrak{QCoh}(\mathcal{M}or_B(C, V))$.

Remark 3.2.1.1.3. The cotangent complex, on the other hand, is twisted by the dualising complex.

Theorem 3.2.1.1.4 ([HP14, Theorem 5.1.1]). *Let M be a 1-Artin derived stack. Let $C \to M$ be a formally proper (in the sense of [HP14, Definition 1.1.3]) M-stack of finite Tor-amplitude n, and let $V \to M$ be a locally almost finitely presented Artin derived M-stack with quasi-affine diagonal. The $\mathcal{M}or_M(C, V)$ is a locally almost finitely presented $(n + 1)$-Artin derived stack over M.*

We shall also be concerned with certain substacks of the derived stacks of morphisms, parameterising appropriate kinds of maps.

Lemma 3.2.1.1.5 ([TV08, Corollary 2.2.2.10]). *Let* X *be an algebraic derived stack. The base-change* $(\infty,1)$*-functor* $\jmath_X^*\colon \mathfrak{dSt}^{\mathrm{aff}}_{/X} \to \mathfrak{dSt}^{\mathrm{aff}}_{/t_0 X}$ *induces an equivalence between the full sub-*$(\infty,1)$*-categories spanned by the Zariski open immersions.*

Example 3.2.1.1.6 (Substack of representable morphisms). We let $\mathcal{M}or^{\mathrm{rep}}_{/B}(X,Y)$ denote the substack of $\mathcal{M}or_{/B}(X,Y)$ parameterising only those morphisms which are representable, or geometric. By the methods of [Ols06, Corollary 1.6], it is an open substack.

Definition 3.2.1.1.7 (Stable moduli of maps). Fix a source $C \to B$. A **stable substack** is a subfunctor $\mathscr{M}$ of $\mathcal{M}or_B(C,-)$ such that, for any closed immersion $Z \hookrightarrow X$ of B-derived stacks, the canonical map $\mathscr{M}(Z) \to \mathscr{M}(X) \times_{\mathcal{M}or_B(C,X)} \mathcal{M}or_B(C,Z)$ is an equivalence.

Example 3.2.1.1.8 (Maps with degree). Let M be a commutative monoid with indecomposable zero. An M-valued **degree** function for $C \to B$ is a function associating functorially with each family of maps $f\colon C \to V$ a section $\deg(f)\colon B \to B \times M$.

For any fixed $\beta \in B$, the condition that $\deg(f) = \beta$ determines a stable substack.

3.2.1.2 Maps with varying source

From the definition of an $(\infty,1)$-topos, the assignment $S \mapsto \iota_0 \mathfrak{dSt}_{/S}$ defines a derived stack.

It is shown in [Lur19, section 19.4] and [PY20, section 3] that this derived stack is formally algebraic.

3.2.2 Maps to zero loci of vector bundles

Any quasi-smooth derived scheme is Zariski-locally presented as the (derived) zero locus of a section of a vector bundle on some smooth scheme. The Lefschetz hyperplane theorem then gives a way of understanding the cohomology of such a zero locus from the data of that of the ambient scheme and of the vector bundle. The quantum Lefschetz principle, similarly, gives the quantum cohomology, that is the Gromov–Witten theory, of the zero locus from that of the ambient scheme and the Euler class of the vector bundle.

Let X be a smooth projective variety and let E be a vector bundle on X, and consider the abelian cone stack $\mathbb{R}^0 p_* \mathrm{ev}^* E$ on $\overline{\mathscr{M}}_{g,n}(X,\beta)$, where $\mathrm{ev}\colon \overline{\mathscr{C}}_{g,n}(X,\beta) \to \overline{\mathscr{M}}_{g,n}(X,\beta)$ is the canonical evaluation map (corresponding by the isomorphism $\overline{\mathscr{C}}_{g,n}(X,\beta) \simeq \overline{\mathscr{M}}_{g,n+1}(X,\beta)$ to evaluation at the $(n+1)$th marking) and $p\colon \overline{\mathscr{C}}_{g,n} \to \overline{\mathscr{M}}_{g,n}$ is the projection. Let s be a regular section of E and $i\colon Z \hookrightarrow X$ be its zero locus. An inspection of the moduli problems (see the proof of corollary 3.2.2.3.4) reveals that the disjoint union, over all classes $\gamma \in A_1 Z$ mapped by i_* to β, of the moduli stacks of stable maps to Z of degree γ coincides with the zero locus of the induced section $\mathbb{R}^0 p_* \mathrm{ev}^* s$ of $\mathbb{R}^0 p_* \mathrm{ev}^* E$. The natural question, leading to the quantum Lefschetz theorem, is whether this identification remains true at the "virtual" level, which was conjectured by Cox, Katz and Lee in [CKL01, Conjecture 1.1]. It was indeed proved in [KKP03] for Chow homology, and the statement was lifted in [Jos10] to G_0-theory, that under assumptions on E the Gromov–Witten theory of Z is equivalent to that of X twisted by the Euler class of E, in that the following holds.

Theorem 3.2.2.0.1 ([KKP03; Jos10]). *For any* $\gamma \in A_1 Z$ *such that* $i_*\gamma = \beta$, *let* $u_\gamma\colon \overline{\mathscr{M}}_{0,n}(Z,\gamma) \hookrightarrow \overline{\mathscr{M}}_{0,n}(X,\beta)$ *denote the closed immersion. Suppose* E *is convex, that is* $\mathbb{R}^1 p_*(C,\mu^*E) = 0$ *for any stable map* $\mu\colon C \to X$ *from a rational (*i.e. *genus-0) stable curve* $C \xrightarrow{p} S$ *(so that the cone* $\mathbb{R}^0 p_* \mathrm{ev}^* E$ *is a vector bundle). Then*

$$\sum_{i_*\gamma=\beta} u_{\gamma,*} \left[\overline{\mathscr{M}}_{0,n}(Z,\gamma)\right]^{\mathrm{vir}} = \left[\overline{\mathscr{M}}_{0,n}(X,\beta)\right]^{\mathrm{vir}} \smile c_{\mathrm{top}}(\mathbb{R}^0 p_* \mathrm{ev}^* E) \in A_\bullet\left(\overline{\mathscr{M}}_{0,n}(X,\beta)\right), \quad (3.19)$$

and

$$\sum_{i_*\gamma=\beta} u_{\gamma,*}\left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathscr{M}}_{0,n}(Z,\gamma)}\right] = \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathscr{M}}_{0,n}(X,\beta)}\right] \otimes \lambda_{-1}(\mathbb{R}^0 p_* \operatorname{ev}^* E) \in G_0\left(\overline{\mathscr{M}}_{0,n}(X,\beta)\right). \tag{3.20}$$

It was shown in [Coa+12] that the quantum Lefschetz principle as stated in (3.19) can be false when the vector bundle E is not convex (or as soon as g is greater than 0). The reason for this is that $\mathbb{R}^0 p_* \operatorname{ev}^* E$ no longer equals $\mathbb{R} p_* \operatorname{ev}^* E$ and the twisting Euler class should be corrected by taking into account the term $\mathbb{R}^1 p_* \operatorname{ev}^* E$: in other words, one should use the full derived pushforward and view the induced cone as a *derived* vector bundle $\mathbb{R} p_* \operatorname{ev}^* E$; this will require viewing our moduli stacks through the lens of derived geometry.

In this note we use this pilosophy to undertake the task of relaxing the hypotheses on theorem 3.2.2.0.1 and lifting it to a categorified (and a geometric) statement, by which we mean that:

- we will give a formula at the level of a derived ∞-category of quasicoherent sheaves,
- we will not need to fix the genus to 0,
- we will not need to assume that E is convex, or in fact a classical vector bundle (*i.e.* it can come from any object of the ∞-category $\mathfrak{Perf}(\mathcal{O}_X)$),
- we will not need to assume that the section is regular, as we can allow the target to be any derived scheme rather than a smooth scheme.

We note however that only the categorified form of the formula will hold in full generality, as the usual convexity (and genus) hypotheses are still needed to ensure coherence conditions so as to decategorify to G_0-theory.

The main result of this subsection addresses the question of similarly understanding the virtual statement of the quantum Lefschetz principle as a derived geometric phenomenon, and of deducing an expression for the "virtual structure sheaf" of $\coprod_\gamma \overline{\mathscr{M}}_{g,n}(Z,\gamma)$, understanding along the way the appearance of the Euler class of the bundle. In the remainder of this introduction, in order to distinguish notationally the classical formulæ, we shall write $\mathbb{R}u\colon \coprod_\gamma \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma) \hookrightarrow \mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)$ the canonical closed immersion (beware that $\mathbb{R}u$ is not a right derived functor, but simply a morphism of derived stacks which is a thickening of u).

Theorem 3.2.2.0.2 (Categorified quantum Lefschetz principle, see corollary 3.2.2.3.4 and proposition 3.2.2.2.1). *Let* X *be a derived scheme,* $\mathcal{E} \in \mathfrak{Perf}(\mathcal{O}_X)$*, and* s *a section of* $\mathbb{V}_X(\mathcal{E})$ *with zero locus* $Z = X \times^{\mathbb{R}}_{\mathbb{V}_X(\mathcal{E})} X$*. Write* $\jmath\colon \mathbb{E}^\vee := (\mathbb{R} p_* \operatorname{ev}^* \mathcal{E})^\vee \to \mathcal{O}_{\overline{\mathscr{M}}_{g,n}(X,\beta)}$ *the cosection (of modules) corresponding to* $\mathbb{R} p_* \operatorname{ev}^* s$*. There is an equivalence*

$$(\mathbb{R}u)_* \mathcal{O}_{\coprod_\gamma \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma)} \simeq \mathcal{O}_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)} \otimes \mathcal{S}ym\,(\operatorname{cofib}(\jmath))/\{t\} = \mathcal{S}ym\,(\operatorname{cofib}(\jmath))/\{t\} \tag{3.21}$$

in $\mathfrak{QCoh}\left(\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)\right)$*, where* $\mathcal{S}ym\,(\operatorname{cofib}(\jmath))$ *has a canonical* $\mathcal{O}_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)}\{t\}$*-algebra structure.*

We first notice that, in this categorified statement and unlike in the G-theoretic one, the Euler class of $\mathbb{E}^\vee$ is refined to one taking into account the section s. Nonetheless this is indeed a categorification of theorem 3.2.2.0.1, as we will explain in corollary 3.2.2.2.6 and corollary 3.2.2.3.9. When s is the zero section, meaning that $\jmath$ is the zero morphism, then $\mathcal{S}ym(\operatorname{cofib}(\jmath)) = \mathcal{S}ym(\mathbb{E}^\vee[1]) \otimes \mathcal{O}_{\mathbb{A}^1}$, with $\mathcal{S}ym(\mathbb{E}^\vee[1]) = \bigwedge^\bullet(\mathbb{E}^\vee)$ so that in that case we do recover a categorified Euler class. In particular, passing to the G_0 groups will indeed provide an identification of the cofibres of any and all sections, and hence give back equation (3.20); this is corollary 3.2.2.3.9.

The theorem will in fact come as a corollary of a geometric statement, as a translation of the fact that Euler classes (also known, in the categorified setting, as Koszul complexes) represent zero loci of sections. Indeed, we will show that the moduli stack $\coprod_\gamma \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma) = \mathcal{S}pec_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)}\left((\mathbb{R}u)_*\mathcal{O}_{\coprod_\gamma \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma)}\right)$ satisfies the universal property of the zero locus of $\mathbb{R}p_*\,\mathrm{ev}^*\,s$, meaning that (per corollary 3.2.2.3.4, the *geometric quantum Lefschetz principle*) it features in the cartesian square

$$\begin{array}{ccc} \coprod_{i_*\gamma=\beta} \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma) & \xrightarrow{\mathbb{R}u_1} & \mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta) \\ {\scriptstyle \mathbb{R}u_2}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathbb{R}p_*\,\mathrm{ev}^*\,s} \\ \mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta) & \xrightarrow[0_{\mathbb{E}}]{} & \mathbb{E}|_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)} \end{array} . \tag{3.22}$$

The formula equation (3.21) for its relative function ring will then be a consequence of the general result proposition 3.2.2.2.1 describing zero loci of sections of vector bundles.

Remark 3.2.2.0.3. The geometric and categorified quantum Lefschetz principles are not only valid for (derived) schematic targets, but also for orbifold Gromov–Witten theory. In fact X and Z can be allowed to be derived algebraic stacks, and $\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta) \subset \mathcal{M}or_{/\mathfrak{M}^{tw}_{g,n}}(\mathfrak{C}^{tw}_{g,n}, X\times\mathfrak{M}^{tw}_{g,n})$ (where $\mathfrak{C}^{tw}_{g,n} \to \mathfrak{M}^{tw}_{g,n}$ denotes the universal twisted curve) can be any open substack corresponding to a quasimap stability condition, as used for example in [CJW19]. For notational simplicity however (and as quasimap theory for general stacks, *i.e.* non global quotients, is not yet fully developed), we have decided to fix the choice of the Gromov–Witten stability condition.

The original proof of the quantum Lefschetz principle in [KKP03] also consisted of applying an excess intersection formula to a geometric (or homological) statement, here the fact that the embedding u satisfies the compatibility condition implying that Gysin pullback along it preserves the virtual class. The situation was shed light upon in [Man12a], where it was shown that, using relative perfect obstruction theories (POTs), one can construct *virtual pullbacks*, which always preserve virtual classes. The embedding u being regular, its own cotangent complex can be used as a POT to construct a virtual pullback, which evidently coincides with the Gysin pullback.

Here we will use the results of subsection 3.1.3 to show that, since the virtual pullbacks may be understood as coming from derived geometric pullbacks of coherent sheaves, our statement for the embedding of derived moduli stacks does imply the quantum Lefschetz formula for the virtual classes (and in fact its standard proof).

3.2.2.1 Derived vector bundles

Definition 3.2.2.1.1 (Abelian cones and vector bundles)**.** Let X be a derived Artin k-stack. The **total space** of a quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ is the derived stack $\mathbb{V}_X(\mathcal{M})$ described by the $(\infty,1)$-functor of points mapping an X-derived stack $\phi\colon T \to X$ to the ∞-groupoid

$$\mathbb{V}_X(\mathcal{M})(T) := \mathfrak{QCoh}(T)\big(\mathcal{O}_T, \phi^*\mathcal{M}\big). \tag{3.23}$$

We call **abelian cone** over X any X-stack equivalent to the total space $\mathbb{V}_X(\mathcal{M})$ of a quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$. We shall say that $\mathbb{V}_X(\mathcal{M})$ is a **perfect** cone if $\mathcal{M}$ is perfect (equivalently, dualisable), and a **vector bundle** if $\mathcal{M}$ is locally free of finite rank (as defined in [Lur19, Notation 2.9.3.1]).

Lemma 3.2.2.1.2 ([TV07, Sub-lemma 3.9],[AG14, Theorem 5.2])**.** *Suppose $\mathcal{M}$ is of perfect Tor-amplitude contained in $[a,b]$, where $(a,b) \in (-\mathbb{N}) \times \mathbb{Z}$. Then the derived stack $\mathbb{V}_X(\mathcal{M})$ is $(-a)$-geometric and strongly of finite presentation.*

Remark 3.2.2.1.3. If $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module, by [Lur19, Proposition 2.9.2.3] we may take a Zariski open cover $\coprod_i U_i \to X$ with $\mathcal{M}|_{U_i}$ free of rank r_i. We deduce from this (or from [Lur17, Remark 7.2.4.22] and [Lur19, Remark 2.9.1.2]) that any locally free module has Tor-amplitude concentrated in degree 0, and it will follow from proposition 3.2.2.1.10 that any vector bundle is smooth over its base.

Remark 3.2.2.1.4. If $\mathcal{M}$ is dualisable, with dual $\mathcal{M}^\vee$, then as pullbacks commute with taking duals we have for any $\phi\colon T \to X$

$$\begin{aligned} \mathbb{V}_X(\mathcal{M})(\phi) &= \mathfrak{QCoh}(T)\big(\phi^*\mathcal{M}^\vee, \mathcal{O}_T\big) \\ &= \mathfrak{Alg}(\mathcal{O}_X)\big(\mathcal{S}ym_{\mathcal{O}_X}(\mathcal{M}^\vee), \phi_*\mathcal{O}_T\big) = \mathcal{S}pec_X(\mathcal{S}ym_{\mathcal{O}_X}(\mathcal{M}^\vee))(\phi) \end{aligned} \tag{3.24}$$

where $\mathcal{S}pec_X$ denotes the non-connective relative spectrum $(\infty,1)$-functor. Hence the restriction of $\mathbb{V}_X$ to $\mathfrak{Perf}(\mathcal{O}_X)$ is naturally equivalent to the composite $\mathcal{S}pec_X \circ \mathcal{S}ym_{\mathcal{O}_X} \circ (-)^\vee$. In particular, if $\mathcal{M}$ is a connective module then $\mathbb{V}_X(\mathcal{M})$ is a relatively coaffine stack (while if $\mathcal{M}$ is co-connective $\mathbb{V}_X(\mathcal{M})$ is an affine derived X-scheme).

Note however that the $(\infty,1)$-functor $\mathcal{S}pec_X$ only becomes fully faithful when restricted to either connective $\mathcal{O}_X$-algebras (as this restriction is equivalent to the Yoneda embedding thereof), but not when acting on general $\mathcal{O}_X$-algebras in degrees of arbitrary positivity.

Warning 3.2.2.1.5 (Terminology). Note that our convention for derived perfect cones is dual to that used in (among others) [Toë14] (and dating back to EGA2), which defines the total space of a quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ as the X-stack whose sheaf of sections is $\mathcal{M}^\vee$, *i.e.* what we denote $\mathbb{V}_X(\mathcal{M}^\vee)$.

Example 3.2.2.1.6.

i. If X is a classical Deligne–Mumford stack and $\mathcal{M}$ is of perfect Tor-amplitude in $[0,1]$, the truncation $t_0(\mathbb{V}_X(\mathcal{M}[1]^\vee))$ is the abelian cone Picard stack $\mathcal{H}^1/\mathcal{H}^0(\mathcal{M}^\vee)$ of [BF97, Proposition 2.4].

ii. By [TV08, Proposition 1.4.1.6], $\mathbb{V}_X(\mathbb{T}_X) = TX = \mathcal{M}or(k[\varepsilon], X)$ is the tangent bundle stack of X. More generally, using $k[\varepsilon_n]$ where ε_n is of (homotopical) degree n we have the shifted tangent bundle $T[-n]X \simeq \mathbb{V}_X(\mathbb{T}_X[-n])$. Dually, one also defines the shifted cotangent stack $T^\vee[n]X = \mathbb{V}_X(\mathbb{L}_X[n])$.

Construction 3.2.2.1.7. For any derived stack X, the $(\infty 1)$-functor $\mathbb{V}_X$ gives a link between two functorial (in X) constructions. On the one hand we have the $(\infty,1)$-functor $(-)_{\text{ét}}\colon \mathfrak{dSt}_k \to (\infty,1)-\mathfrak{Cat}$ mapping a derived k-stack X to its étale $(\infty,1)$-topos $X_{\text{ét}}$ and a map of derived stacks $f\colon X \to Y$ to the direct image f_* of the induced geometric morphism, mapping a sheaf $\mathcal{F}$ on $\mathfrak{dSt}_{k,/X}$ to the sheaf $f_*\mathcal{F}\colon (U \to Y) \mapsto \mathcal{F}(U \times_Y X \to X)$.

On the other hand, we have the $(\infty,1)$-functor $\mathfrak{QCoh}(-)$ mapping a derived k-stack X to the stable $(\infty,1)$-category $\mathfrak{QCoh}(X)$, and a map $f\colon X \to Y$ to $\mathcal{M} \mapsto f_*\mathcal{M}$ (where the direct image sheaf is considered an $\mathcal{O}_Y$-module through $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$). Then for any $\mathcal{M} \in \mathfrak{QCoh}(X)$, we obtain the functor of points of its total space, $\mathbb{V}_X(\mathcal{M})$, which is an étale sheaf on $\mathfrak{dSt}_{k,/X}$.

Lemma 3.2.2.1.8. *Let $\mathfrak{dSt}_k^{(\text{f.coh.d.})}$ denote the wide and locally full sub-$(\infty,1)$-category whose 1-arrows are the morphisms of finite cohomological dimension (see [HP14, Definition A.1.4, Lemma A.1.6]). The $(\infty,1)$-functors $\mathbb{V}_X\colon \mathfrak{QCoh}(X) \to X_{\text{ét}}$ assemble into a natural transformation $\mathbb{V}\colon \mathfrak{QCoh}(-) \Rightarrow (-)_{\text{ét}}$ of ∞-functors $\mathfrak{dSt}_k^{(\text{f.coh.d.})} \to \infty - \mathfrak{Cat}$.*

Proof. We must show that, for any $f\colon X \to Y$ and any $\mathcal{M} \in \mathfrak{QCoh}(X)$, we have $f_*(\mathbb{V}_X(\mathcal{M})) =$

$\mathbb{V}_Y(f_*\mathcal{M})$. For any $\phi\colon U \to Y$, the base change along f will take place in the cartesian square

$$\begin{array}{ccc} X \times_Y U & \xrightarrow{X\times_Y \phi} & X \\ {\scriptstyle f\times_Y U}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[\phi]{} & Y \end{array} . \tag{3.25}$$

Then we have $\mathbb{V}_Y(f_*\mathcal{M})(U) = \mathfrak{QCoh}(U)\big(\mathcal{O}_U, \phi^* f_*\mathcal{M}\big)$ while

$$\begin{aligned} f_*(\mathbb{V}_X(\mathcal{M}))(U) &= \mathfrak{QCoh}(X\times_Y U)\big(\mathcal{O}_{X\times_Y U}, (X\times_Y \phi)^*\mathcal{M}\big) \\ &\simeq \mathfrak{QCoh}(X\times_Y U)\big((f\times_Y U)^*\mathcal{O}_U, (X\times_Y \phi)^*\mathcal{M}\big) \\ &\simeq \mathfrak{QCoh}(U)\big(\mathcal{O}_U, (f\times_Y U)_*(X\times_Y \phi)^*\mathcal{M}\big). \end{aligned} \tag{3.26}$$

By the base-change property of [HP14, Proposition A.1.5 (3)], the two coincide. □

Remark 3.2.2.1.9. By [Toë12, Theorem 2.1] , if $f\colon X \to Y$ is quasi-smooth and proper then f_* sends perfect $\mathcal{O}_X$-modules to perfect $\mathcal{O}_Y$-modules.

Finally, we shall use the following well-known description of the cotangent complex of a perfect cone.

Proposition 3.2.2.1.10 ([AG14, Theorem 5.2]). *Let $\mathcal{M}$ be a perfect $\mathcal{O}_X$-module, and write $\pi\colon \mathbb{V}_X(\mathcal{M}) \to X$ the structure morphism. Then $\mathbb{L}_{\pi\colon \mathbb{V}_X(\mathcal{M})/X} \simeq \pi^*\mathcal{M}^\vee$.*

Proof. The equivalence is established fibrewise in [Lur17, Proposition 7.4.3.14]. □

3.2.2.2 Excess intersection formula

In this section, we work with the closed embedding $u\colon T \hookrightarrow M$ of derived stacks defined as the zero locus of a section s of a perfect cone $\mathcal{S}pec_M \mathcal{S}ym_{\mathcal{O}_M}(\mathcal{F}^\vee)$ on M: we fix a perfect $\mathcal{O}_X$-module $\mathcal{F}$ and a morphism of algebras $s^\sharp\colon \mathcal{S}ym_{\mathcal{O}_M}(\mathcal{F}^\vee) \to \mathcal{O}_M$, corresponding to the cosection $\tilde{s}\colon \mathcal{F}^\vee \to \mathcal{O}_M$ of the module $\mathcal{F}^\vee$.

Proposition 3.2.2.2.1. *The derived M-stack T may be recovered as the relative spectrum of the quotient $\mathcal{O}_M$-algebra*

$$u_*\mathcal{O}_T \simeq \mathcal{S}ym_{\mathcal{O}_M}(\mathrm{cofib}(\tilde{s})) \underset{\mathcal{O}_M\{t\}}{\otimes} \mathcal{O}_M =: \mathcal{S}ym_{\mathcal{O}_M}(\mathrm{cofib}(\tilde{s}))/(t-1), \tag{3.27}$$

*where the structure map $\epsilon_{\mathcal{O}_M}\colon \mathcal{O}_M\{t\} \to \mathcal{O}_M$ is the quotient arrow $\mathcal{O}_M\{t\} \to \mathcal{O}_M\{t\}/(t-1) \simeq \mathcal{O}_M$ mapping t to 1 (*i.e. *corresponding to the identity morphism of $\mathcal{O}_M$-modules $\mathrm{id}_{\mathcal{O}_M}\colon \mathcal{O}_M \to \mathcal{O}_M$).*

More generally, the monad u_*u^* identifies with tensoring by $\mathcal{S}ym_{\mathcal{O}_M}(\mathrm{cofib}(\tilde{s}))/(t-1)$.

Proof. From the canonical fibre sequence $\mathcal{F}^\vee \xrightarrow{\tilde{s}} \mathcal{O}_M \to \mathrm{cofib}(\tilde{s})$ we obtain, by application of the $(\infty,1)$-functor $\mathcal{S}ym_{\mathcal{O}_M}$, an $\mathcal{O}_M\{t\}$-algebra structure $\mathcal{O}_M\{t\} := \mathcal{S}ym_{\mathcal{O}_M}(\mathcal{O}_M) \to \mathcal{S}ym_{\mathcal{O}_M}(\mathrm{cofib}(\tilde{s}))$. As $\mathcal{S}ym_{\mathcal{O}_M}$ is a left-adjoint it preserves colimits (by [RV21, Theorem 2.4.2]) whence the latter term, image by $\mathcal{S}ym_{\mathcal{O}_M}$ of the $\mathcal{O}_M$-module $0 \oplus_{\mathcal{F}^\vee} \mathcal{O}_M^{\oplus 1} =: \mathrm{cofib}(\tilde{s})$, is the pushout of algebras (so by [Lur17, Proposition 3.2.4.7] the tensor product) $\mathcal{O}_M \otimes_{\mathcal{S}ym_{\mathcal{O}_M}(\mathcal{F}^\vee)} \mathcal{O}_M\{t\}$.

By definition, the algebra $\mathcal{S}ym_{\mathcal{O}_M}(\mathrm{cofib}(\widetilde{s})) \otimes_{\mathcal{O}_M\{t\}} \mathcal{O}_M$ under consideration fits in the left pushout square in the diagram

$$\begin{array}{ccccc} \mathcal{S}ym_{\mathcal{O}_M}(\mathrm{cofib}(\widetilde{s}))/(t-1) & \longleftarrow & \mathcal{S}ym_{\mathcal{O}_M}(\mathrm{cofib}(\widetilde{s})) & \longleftarrow & \mathcal{O}_M \\ \uparrow & \lrcorner & \uparrow & \lrcorner & \uparrow \\ \mathcal{O}_M & \xleftarrow[\epsilon_{\mathcal{O}_M}]{} & \mathcal{O}_M\{t\} & \xleftarrow[\mathcal{S}ym(\widetilde{s})]{} & \mathcal{S}ym_{\mathcal{O}_M}(\mathcal{F}^\vee). \end{array} \tag{3.28}$$

From the previous discussion the right square is also cocartesian, so that the bigger diagram is also a pushout square. We now observe that the lower composite identifies with $s^\sharp$ (since the map $\epsilon_{\mathcal{O}_M}$ is the counit of the adjunction $\mathcal{S}ym_{\mathcal{O}_M} \dashv \mathit{frgt}$), so that the big pushout square computes the function $\mathcal{O}_M$-algebra of the zero locus of s.

Finally, both u_* and u^* are left-adjoints, so by the homotopical Eilenberg–Watts theorem of [Hov15] (see also [GR17a, Chapter 4, Corollary 3.3.5]) their composite u_*u^* is equivalent to tensoring by $u_*u^*\mathcal{O}_M$. □

Remark 3.2.2.2.2 (Geometric interpretation). Let $\overline{s}\colon \mathbb{A}^1_M \to \mathbb{V}_M(\mathcal{F})$ be the linearisation of s, obtained as the image of $\widetilde{s}$ by $\mathbb{V}_M$. The zero locus of $\overline{s}$ is $\mathbb{A}^1_M|_T \cup \mathbb{A}^0_{M\setminus T}$, so taking the fibre at any non-zero element λ of $\mathbb{A}^1_M$ recovers $T \times \{\lambda\} \cup \emptyset \simeq T$.

Remark 3.2.2.2.3 (Koszul complexes). Suppose $\mathcal{F}$ is locally free. Then, passing to a Zariski open cover $\coprod U_i \to M$, we may assume as in remark 3.2.2.1.3 that $\mathcal{F}|_{U_i}$ is free of rank r_i. Write $\widetilde{s}|_{U_i} = (s_\ell)_{1\leqslant \ell\leqslant r}$ in coordinates. Then we recover the Koszul complex $\bigotimes_{\ell=1}^r \mathrm{cofib}(s_\ell)$, as studied for instance in [KR19, §2.3.1] or [Vez11].

Recall that the exterior algebra of the quasicoherent $\mathcal{O}_M$-module $\mathcal{F}$ is $\bigwedge^\bullet \mathcal{F} := \mathcal{S}ym_{\mathcal{O}_M}(\mathcal{F}[1]) = \bigoplus_{n\geqslant 0}(\bigwedge^n \mathcal{F})[n]$.

Corollary 3.2.2.2.4 (Excess intersection formula)**.** *For any quasicoherent $\mathcal{O}_T$-module $\mathcal{M}$ that is the restriction (along u^*) of an $\mathcal{O}_M$-module, there is an equivalence*

$$u^*u_*\mathcal{M} = \mathcal{M} \otimes_{\mathcal{O}_T} \bigwedge\nolimits^\bullet \mathcal{F}^\vee|_T. \tag{3.29}$$

Proof. The $(\infty,1)$-functor u^* is a left-adjoint (since u is a closed immersion) so it preserves colimits, among which in particular cofibres. By definition, we are given an equivalence $u^*\widetilde{s} \simeq u^*0 = 0$, so the image by u^* of equation (3.27) takes the form $\mathcal{S}ym(\mathrm{cofib}\,0)/(t-1)$. By definition of the zero morphism, we may decompose the pushout $\mathrm{cofib}\,0$ as the composite of two amalgamated sums:

$$\begin{array}{ccccc} \mathcal{F}^\vee[1] \oplus \mathcal{O}_M & \longleftarrow & \mathcal{F}^\vee[1] & \xleftarrow{\;!\;} & 0 \\ \uparrow & \lrcorner & \uparrow{\scriptstyle !} & \lrcorner & \uparrow{\scriptstyle !} \\ \mathcal{O}_M & \xleftarrow{\;!\;} & 0 & \xleftarrow{\;!\;} & \mathcal{F}^\vee \end{array}, \qquad (\text{lower composite } \mathcal{F}^\vee \xrightarrow{0} \mathcal{O}_M) \tag{3.30}$$

so that $\mathcal{S}ym_{\mathcal{O}_M}(\mathrm{cofib}\,0) = \mathcal{S}ym_{\mathcal{O}_M}(\mathcal{F}^\vee[1] \oplus \mathcal{O}_M) = \mathcal{S}ym_{\mathcal{O}_M}(\mathcal{F}[-1]^\vee) \otimes_{\mathcal{O}_M} \mathcal{O}_M\{t\}$, and "quotienting by $(t-1)$" gives back $\mathcal{S}ym_{\mathcal{O}_M}(\mathcal{F}[-1]^\vee)$. As u^* has a structure of monoidal $(\infty,1)$-functor, this extends to any $\mathcal{O}_T$-module $\mathcal{M}$ in the image of u^*.

Of course, this result can also be obtained more directly from the fact that the leftmost diagram below is the image by $\mathcal{S}pec_M \mathcal{S}ym_{\mathcal{O}_M}$ of the rightmost one:

$$\begin{array}{ccc} T & \xrightarrow{u} & M \\ {\scriptstyle u}\downarrow & \lrcorner & \downarrow{\scriptstyle 0_{\mathbb{V}_M(\mathcal{F})}} \\ M & \xrightarrow[0_{\mathbb{V}_M(\mathcal{F})}]{} & \mathbb{V}_M(\mathcal{F}) \end{array} \qquad \begin{array}{ccc} u_*\mathcal{O}_T & \longleftarrow & 0 \\ \uparrow & \lrcorner & \uparrow{\scriptstyle !} \\ 0 & \xleftarrow[!]{} & \mathcal{F}^\vee \end{array} \quad . \tag{3.31}$$

□

Remark 3.2.2.2.5 (Lie-theoretic interpretation). The excess intersection formula can also be seen as coming from the study of the $\mathscr{L}_\infty$-algebroid associated with the closed embedding u. Indeed, we are studying the geometry of a closed sub-derived stack $T \subset M$, which can be understood through that of its formal neighbourhood $\widehat{M}_T = M \times_{M_{dR}} T_{dR}$. This is a formally algebraic derived stackwhich is a formal thickening of T. By [GR17b, Chapter 5, Theorem 2.3.2], the $(\infty, 1)$-category of formal thickenings of T is equivalent to that of groupoid objects in formally algebraic derived stacks over T (*via* the $(\infty, 1)$-functor sending a thickening $T \to \mathcal{F}$ to its simplicial kernel), and following the philosophy of formal moduli problems it can be considered as a model for the $(\infty, 1)$-category of $\mathscr{L}_\infty$-algebroids.

We have the sequence of adjunctions $u^* \dashv u_* \dashv u^!$, implying that the comonad u^*u_* is left-adjoint to the monad $u^!u_*$. Let us write $T \xrightarrow{\widehat{u}} \widehat{M}_T \xrightarrow{p} M$ the factorisation of u, so that $u^!u_* = \widehat{u}^!p^!p_*\widehat{u}_*$. Note that $p\colon M \times_{M_{dR}} T_{dR} \to M$ is the canonical projection, and as both T_{dR} and M_{dR} are étale over $\operatorname{Spec} k$ it is also an étale morphism, and we recover $\widehat{u}^!\widehat{u}_*$. Following [GR17b, Chapter 8, 4.1.2], the monad $u^!u_*$ becomes the universal enveloping algebra of the $\mathscr{L}_\infty$-algebroid associated with u, endowed with the Poincaré–Birkhoff–Witt filtration. As the $(\infty, 1)$-functor of assciated graded is conservative when restricted to (co)connective filtrations, we only need an expression for the associated graded of the PBW filtration. The result is then nothing but the PBW isomorphism of [GR17b, Chapter 9, Theorem 6.1.2] stating that for any regular embedding of derived stacks $u\colon T \hookrightarrow M$, the monad $\widehat{u}^!\widehat{u}_*$ on $\mathfrak{Coh}^b(T)$ is equivalent to tensoring by $\mathcal{S}ym_{\mathcal{O}_T}(\mathbb{T}_{\widehat{u}})$, and $\mathbb{T}_{\widehat{u}} = \mathbb{T}_u$ since p is étale. Passing back to the adjoint, we do obtain that u^*u_* is equivalent to tensoring with $\mathcal{S}ym_{\mathcal{O}_T}(\mathbb{T}_u^\vee)$.

A similar equivalence between the Hopf comonad u^*u_* and tensoring by the jet algebra (the dual of the universal enveloping algebra) of $\mathbb{T}_u$ was established in [CCT14, Theorem 1.3] using the model of dg-Lie algebroids for $\mathscr{L}_\infty$-algebroids (see [CG18, Proposition 4.3, Theorem 4.11] for a precise statement of the equivalence between dg-Lie algebroids and formally algebraic derived stacks as models for $\mathscr{L}_\infty$-algebroids). However this approach does not provide the PBW theorem needed to identify the jet algebra of $\mathbb{T}_u$ with $\mathcal{S}ym(\mathbb{L}_u)$.

Finally, it is easy to see from proposition 3.2.2.1.10 that the base-change property of cotangent complexes and the fibre sequence associated with the composition $\varpi \circ s = \mathrm{id}$ imply $\mathbb{L}_u = u^*\mathbb{L}_s = u^*s^*\mathbb{L}_\varpi[1] = u^*\mathcal{F}^\vee[1]$.

Then, when u is quasi-smooth so that $u_*\mathcal{O}_T$ is coherent, conservativity of the restriction of u^* to $\mathfrak{Coh}^b(M)_T$ gives another reason for the equivalence $u_*\mathcal{O}_Z \simeq \mathcal{S}ym(\operatorname{cofib}(\widetilde{s}))/\{t\}$.

Although it is not possible to directly relate s and the zero section at the geometric level and to obtain an expression of $u_*\mathcal{O}_T$ in terms of the Euler class of $\mathcal{F}^\vee$, passing to G-theory a homotopy between the maps they induce always does exist, and hence we recover the classical formulation of the quantum Lefschetz hyperplane formula.

We recall the notation of the G-theoretic Euler class of a locally free $\mathcal{O}_M$-module $\mathcal{G}$ of finite rank: $\lambda_{-1}(\mathcal{G}) \coloneqq [\bigwedge^\bullet \mathcal{G}] = \sum_{i \geqslant 0} [\bigwedge^i \mathcal{G}[i]] = \sum_i (-1)^i [\bigwedge^i \mathcal{G}] \in G_0(M)$.

Corollary 3.2.2.2.6 ([Kha19b, Lemma 2.1]). *Suppose $\mathcal{F}$ is a vector bundle. There is an equality of* G*-theory operators*

$$u_* u^* = (-) \otimes \lambda_{-1}(\mathcal{F}^\vee)\colon \mathrm{G}_0(M) \to \mathrm{G}_0(M). \tag{3.32}$$

Proof. We first note that, by definition, $\mathcal{F}$ being locally free of finite rank means that it is (flat-locally) almost perfect, which makes it bounded, and flat, which makes it of Tor-amplitude concentrated in $[0]$ and implies that $\mathcal{F}^\vee[1]$ has Tor-amplitude in $[-1,0]$ so that its symmetric algebra is still bounded and thus in $\mathfrak{Coh}^b(M)$, defining an element of $\mathrm{G}_0(M)$.

By [Kha19b, Lemma 1.3], the fibre sequence $\mathcal{O}_M \to \operatorname{cofib}(\widetilde{s}) \to \mathcal{F}^\vee[1]$ implies that $[\mathcal{S}ym^n_{\mathcal{O}_M}(\operatorname{cofib}\widetilde{s})] = \oplus_{i=0}^n [\mathcal{S}ym^{n-i}(\mathcal{O}_M) \otimes \mathcal{S}ym^i(\mathcal{F}^\vee[1])]$ for all $n \geqslant 0$. By the $\mathbb{A}^1$-invariance of G-theory we may remove the symmetric algebra of $\mathcal{O}_M$, which gives the result: the map $M \to \mathbb{A}^1_M$ selecting the fibre 1 is a section of the projection $\mathbb{A}^1_M \to M$, so it becomes invertible in an $\mathbb{A}^1$-invariant setting. □

3.2.2.3 Identification of the derived moduli stacks

Let X be a derived stack and $\mathcal{E} \in \mathfrak{Perf}(\mathcal{O}_X)$ a perfect $\mathcal{O}_X$-module, giving the perfect cone $E = \mathbb{V}_X(\mathcal{E})$. Let s be a section of E, and denote $Z = X \times_{s,E,0} X \subset X$ its (derived) zero locus.

For a fixed morphism $\pi\colon \mathfrak{C} \to \mathfrak{M}$ of derived k-stacks *proper and of finite cohomological dimension*, we consider the universal map from a base-change of $\mathfrak{C}$ over the derived mapping $\mathfrak{M}$-stack $\mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$:

$$\begin{array}{ccc} \mathfrak{C} \underset{\mathfrak{M}}{\times} \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) & \xrightarrow{\mathrm{ev}} & X \\ \Big\downarrow{\scriptstyle p} & & \\ \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) & & \end{array} . \tag{3.33}$$

Let $\mathbb{E} := p_* \mathrm{ev}^* E = \mathbb{V}_{\mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})}(p_* \mathrm{ev}^* \mathcal{E})$ be the induced abelian (and perfect by remark 3.2.2.1.9 if π is quasi-smooth) cone over $\mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$, and $\sigma := p_* \mathrm{ev}^* s$ its induced section. Write also $0_{\mathbb{E}}\colon \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \to \mathbb{E}$ for the zero section.

Theorem 3.2.2.3.1. *There is an equivalence of $\mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$-derived stacks*

$$\mathcal{M}or_{\mathfrak{M}}(\mathfrak{C}, Z \times \mathfrak{M}) \simeq \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \underset{\sigma, \mathbb{E}, 0_{\mathbb{E}}}{\times} \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}), \tag{3.34}$$

that is the diagram

$$\begin{array}{ccc} \mathcal{M}or_{\mathfrak{M}}(\mathfrak{C}, Z \times \mathfrak{M}) & \xrightarrow{u_1} & \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \\ {\scriptstyle u_2}\Big\downarrow & \lrcorner & \Big\downarrow{\scriptstyle \sigma = p_* \mathrm{ev}^* s} \\ \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) & \xrightarrow[0_{\mathbb{E}}]{} & \mathbb{E} = p_* \mathrm{ev}^* E \end{array} \tag{3.35}$$

is cartesian.

The theorem will follow directly from some formal results.

We apply example 1.1.2.2.9 to the $(\infty,1)$-category $\mathfrak{dSt}_k$, which as an $(\infty,1)$-topos is cartesian closed and thus self-enriched, and we find that $\mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, Z \times \mathfrak{M})$ is equivalent to the fibre product

$$\mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \underset{\mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M})}{\times} \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}), \tag{3.36}$$

with structure morphisms induced by s and the zero section of E. Hence, in order to prove theorem 3.2.2.3.1 we only need to identify the two derived stacks over which the fibre products are taken (as well as the two pairs of structure maps), the derived stack of maps to the abelian cone E and the induced cone $\mathbb{E} = p_* \operatorname{ev}^* E$.

Remark 3.2.2.3.2. In our context of a cartesian closed $(\infty, 1)$-category, the internal hom $(\infty, 1)$-functor is further characterised as a right adjoint to taking cartesian product, so the fact that it preserves limits follows more directly from [RV21, Theorem 2.4.2].

Proposition 3.2.2.3.3. *There is an equivalence of* $\mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$*-derived stacks*

$$\mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M}) \simeq \mathbb{E}. \tag{3.37}$$

Proof. Let $a \colon S \to \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$ be an $\mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$-stack, with corresponding family $C_a = a^* \mathfrak{C} = S \times_{\mathfrak{M}} \mathfrak{C} \to S$ (where we implicitly push the structure maps forward along $\mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \to \mathfrak{M}$). Note that, as $p \colon \mathfrak{C} \times_{\mathfrak{M}} \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \to \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$ is just projection onto the first factor, we have

$$\begin{aligned} p^{-1}(a) &= S \times_{\mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})} \left(\mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \times_{\mathfrak{M}} \mathfrak{C} \right) \\ &= S \times_{\mathfrak{M}} \mathfrak{C} =: C_a, \end{aligned} \tag{3.38}$$

as seen in the cartesian diagram

$$\begin{array}{ccccc} C_a = S \underset{\mathfrak{M}}{\times} \mathfrak{C} & \xrightarrow{\widetilde{a}} & \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \underset{\mathfrak{M}}{\times} \mathfrak{C} & \longrightarrow & \mathfrak{C} \\ \big\downarrow & \lrcorner & \big\downarrow p & \lrcorner & \big\downarrow \\ S & \xrightarrow[a]{} & \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) & \longrightarrow & \mathfrak{M} \end{array} . \tag{3.39}$$

By lemma 3.2.2.1.8, as $\pi \colon \mathfrak{C} \to \mathfrak{M}$ was supposed of finite cohomological dimension and morphisms of finite cohomological dimension are stable by base-change, we have

$$\begin{aligned} \mathbb{E}(a) &= \mathbb{V}_{\mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})}(p_* \operatorname{ev}^* \mathcal{E})(a) \\ &= p_* \mathbb{V}_{\mathfrak{C} \times_{\mathfrak{M}} \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})}(\operatorname{ev}^* \mathcal{E})(a) \\ &= \mathbb{V}_{\mathfrak{C} \times_{\mathfrak{M}} \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})}(\operatorname{ev}^* \mathcal{E})(C_a) \\ &= \mathfrak{Perf}(\mathcal{O}_{C_a})\big(\mathcal{O}_{C_a}, \widetilde{a}^* \operatorname{ev}^* \mathcal{E}\big) = \mathfrak{dSt}_{/X}(C_a, E), \end{aligned} \tag{3.40}$$

where $\operatorname{ev} \circ \widetilde{a} \colon C_a \to X$ is the map from a family of curves to X classified by a.

Meanwhile, we have by definition

$$\begin{aligned} & \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M})(a) \\ = & \operatorname{Map}_{\mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})}(S, \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M})) \\ \simeq & \mathfrak{dSt}_{/\mathfrak{M}}(S, \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M})) \underset{\mathfrak{dSt}_{/\mathfrak{M}}(S, \mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}))}{\times} \{a\}. \end{aligned} \tag{3.41}$$

Indeed, from[Lur09, Lemma 6.1.3.13] the standard categorical arguments[1] show that for any morphism $p \colon M' \to M$ in an $(\infty, 1)$-category and any cospan $S \to M' \leftarrow T$ over M' we have

[1] suggested to the author by Benjamin Hennion

$\mathfrak{dSt}_{/M'}(S,T) \simeq \mathfrak{dSt}_{/M}(S,T) \times_{\mathfrak{dSt}_{/M}(S,M')} \{p\}$; and we can compute

$$\begin{aligned}&\mathcal{M}or_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M})(a)\\ \simeq&\mathfrak{dSt}_{/\mathfrak{M}}(S \times_{\mathfrak{M}} \mathfrak{C}, E \times \mathfrak{M}) \underset{\mathfrak{dSt}_{/\mathfrak{M}}(S \times_{\mathfrak{M}} \mathfrak{C}, X \times \mathfrak{M})}{\times} \{a\}\\ =&\mathfrak{dSt}_{/X \times \mathfrak{M}}(C_a, E \times \mathfrak{M}) = \mathfrak{dSt}_{/X}(C_a, E).\end{aligned} \tag{3.42}$$

□

This completes the proof of theorem 3.2.2.3.1. □

We will also write $p_{g,n} = p$, $\mathrm{ev}_{g,n} = \mathrm{ev}$ and $\mathbb{E}_{g,n} = \mathbb{E} = (p_{g,n})_* \mathrm{ev}^*_{g,n} E$.

Corollary 3.2.2.3.4 (Geometric quantum Lefschetz principle). *Fix a class $\beta \in A_1X$. There is an equivalence of $\mathcal{M}or_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, Z \times \mathfrak{M}_{g,n})$-derived stacks*

$$\coprod_{i_*\gamma=\beta} \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma) \simeq \mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta) \underset{\mathbb{E}|_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)}}{\times} \mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta). \tag{3.43}$$

Proof. Note first that, as Zariski-open immersions are stable by pullbacks, both $\coprod_{i_*\gamma=\beta} \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma)$ and $\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta) \times_{\mathbb{E}} \mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)$ are open sub-derived stacks of $\mathcal{M}or_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, Z \times \mathfrak{M}_{g,n})$, so by [STV15, Proposition 2.1] to show that they are equal it is enough to show that their truncations define identical substacks of $t_0\mathcal{M}or_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, Z \times \mathfrak{M}_{g,n})$.

As a ring is discrete if and only if the ∞-groupoids of morphisms toward it are, the truncation of such a derived mapping stack with (discrete) source flat over the discrete base is $\mathcal{M}ap_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, t_0(Z \times \mathfrak{M}_{g,n}))$ (see [TV08, Theorem 2.2.6.11, hypothesis (1)]), and similarly $t_0\left(\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta) \times_{\mathbb{E}} \mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)\right) = \overline{\mathscr{M}}_{g,n}(X,\beta) \times^s_{t_0\mathbb{E}} \overline{\mathscr{M}}_{g,n}(X,\beta)$. In addition, the truncation $(\infty,1)$-functor commutes with colimits so $t_0\left(\coprod_{i_*\gamma=\beta} \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma)\right) = \coprod_{i_*\gamma=\beta} \overline{\mathscr{M}}_{g,n}(t_0Z,\gamma)$.

We now compare the two stacks pointwise. For any $S \to \mathfrak{M}_{g,n}$ (with corresponding prestable genus-g curve $C_S \to S$), we have that $\left(\coprod_{i_*\gamma=\beta} \overline{\mathscr{M}}_{g,n}(t_0Z,\gamma)\right)(S) = \coprod_{i_*\gamma=\beta} \overline{\mathscr{M}}_{g,n}(t_0Z,\gamma)(S)$ is tautologically the disjoint union (over $\gamma \in i_*^{-1}(\beta)$) of the groupoids of S-indexed families of stable maps from C_S to t_0Z of class γ, and

$$\left(\overline{\mathscr{M}}_{g,n}(X,\beta) \underset{t_0\mathbb{E}}{\overset{t}{\times}} \overline{\mathscr{M}}_{g,n}(X,\beta)\right)(S) \simeq \overline{\mathscr{M}}_{g,n}(X,\beta)(S) \underset{t_0\mathbb{E}|_{\overline{\mathscr{M}}_{g,n}(X,\beta)}(S)}{\overset{t}{\times}} \overline{\mathscr{M}}_{g,n}(X,\beta)(S) \tag{3.44}$$

with $t_0\mathbb{E}|_{\overline{\mathscr{M}}_{g,n}(X,\beta)}(S) = \hom(C_S, E)$. An object of the latter 2-fibre product consists of a pair of stable maps f_1, f_2 from C_S to X and an automorphism φ of C_S such that $s \circ f_1 = 0 \circ f_2 \circ \varphi$, while an automorphism $(f_1, f_2, \varphi) \simeq (f_1, f_2, \varphi)$ is given by a pair of automorphisms of C_S compatible with all the data; or automorphisms of the two stable maps compatible with φ (so that when φ is not id_{C_S} the notion of automorphism is more rigid than the usual automorphisms of a single stable map). In particular, the obvious functor $\coprod_{i_*\gamma=\beta} \overline{\mathscr{M}}_{g,n}(t_0Z,\gamma)(S) \to \left(\overline{\mathscr{M}}_{g,n}(X,\beta) \times^t_{t_0\mathbb{E}} \overline{\mathscr{M}}_{g,n}(X,\beta)\right)(S)$ sending a stable map $f\colon C_S \to Z$ to $(i_{Z\hookrightarrow X} \circ f, i_{Z\hookrightarrow X} \circ f, \mathrm{id}_{C_S})$ is clearly fully faithful, and in fact an equivalence. □

We may now apply proposition 3.2.2.2.1 to deduce a proof of theorem 3.2.2.0.2. the results of subsection 3.1.3. In corollary 3.2.2.3.9, we we can also recover from this and corollary 3.2.2.2.6 the classical (virtual) quantum Lefschetz formula.

Remark 3.2.2.3.5. Further evidence for this geometric form of the quantum Lefschetz principle can also be found by comparing the tangent complexes. Let us write temporarily $M(X)$ and $M(Z)$ for the moduli stacks of stable maps $\mathbb{R}\overline{\mathscr{M}}_{g,b}(X,\beta)$ and $\coprod_{i_*\gamma=\beta}\mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma)$, and $M'(Z)$ for the zero locus $M(X)\times_{\mathbb{E}} M(X)$. The universal property of the latter stack induces a canonical morphism denoted $\Upsilon\colon M(Z)\to M'(Z)$ such that $u'_i\circ\Upsilon = u_i$ for $i=1,2$ where $u_i\colon M(Z)\hookrightarrow M(X)$ and $u'_i\colon M'(Z)\hookrightarrow M(X)$ are the canonical arrows (as in equation (3.35)).

We know from remark 3.2.1.1.3 that $\mathbb{T}_{M(X)/\mathfrak{M}_{g,n}} \simeq \pi_{g,n,*}\operatorname{ev}^*_{g,n}\mathbb{T}_X|_{M(X)}$. There is a fibre sequence $i_1^*\mathbb{L}_X\to\mathbb{L}_Z\to\mathbb{L}_{i_1\colon Z/X}$, and as Z sit by definition in a cartesian square we have that $\mathbb{L}_{i_1} = i_2^*\mathbb{L}_{X/E} = \mathcal{E}^\vee[1]|_Z$ (where once again we have written $i_{1,2}\colon Z\hookrightarrow X$ the two canonical inclusions). As both pushforward and pullback preserve fibre sequences, we obtain finally that $\mathbb{T}_{M(Z)/\mathfrak{M}_{g,n}}$ is the fibre of the morphism $\pi_{g,n,*}\operatorname{ev}^*_{g,n}\mathbb{T}_X|_{M(Z)}\to \pi_{g,n,*}\operatorname{ev}^*_{g,n}\mathcal{E}|_{M(Z)}$.

Following the same logic, writing $M'(Z)$ for the zero locus, we see that $\mathbb{T}_{M'(Z)}$ is the fibre of $\mathbb{T}_{M(X)} = \pi_{g,n,*}\operatorname{ev}^*_{g,n}\mathbb{T}_X|_{M(X)}\to \pi_{g,n,*}\operatorname{ev}^*_{g,n}\mathcal{E}|_{M'(Z)}$. But we have seen that $\Upsilon^*\circ u_i'^{,*} = u_i^*$ so it is clear that $\Upsilon^*\mathbb{T}_{M'(Z)}\simeq\mathbb{T}_{M(Z)}$.

As it is sufficient and necessary for a morphism of derived stacks to be an equivalence that it induce an isomorphism on the truncation and that its (co)tangent complex vanish, this is another way of proving theorem 3.2.2.3.1.

Example 3.2.2.3.6. Let $(X, f\colon X\to\mathbb{A}^1)$ be a Landau–Ginzburg model, from which we deduce the perfect cone $T^\vee X = \mathbb{V}_X(\mathbb{L}_X)$ and section $d_{dR}f$, whose zero locus is by definition the critical locus $\mathbb{R}\operatorname{Crit}(f)$ (which is the intersection of two Lagrangians in a 0-shifted symplectic derived stack and thus carries a canonical (-1)-shifted symplectic form). Then the derived moduli stack of stable maps to $\mathbb{R}\operatorname{Crit}(f)$ is the zero locus of the induced section of

$$\pi_*\operatorname{ev}^* T^\vee X = \mathbb{V}_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)}(\pi_*(\operatorname{ev}^*\mathbb{L}_X)) \tag{3.45}$$

But notice that

$$T^\vee\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)\simeq\mathbb{V}_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)}((\pi_*\operatorname{ev}^*\mathbb{T}_X)^\vee)\simeq\mathbb{V}_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)}(\pi_!\operatorname{ev}^*\mathbb{L}_X) \tag{3.46}$$

where $\pi_!\colon\mathcal{F}\mapsto\pi_*(\mathcal{F}^\vee)^\vee\simeq\pi_*(\mathcal{F}\otimes\omega_\pi)$ is the left adjoint to π^* (by [Lur19, Proposition 6.4.5.3]), so $\mathbb{R}\overline{\mathscr{M}}_{g,n}(\mathbb{R}\operatorname{Crit}(f),\beta)$ cannot be expected to carry a (-1)-shifted symplectic structure if (g,n) differs from $(0,1)$ or $(1,0)$.

It is also possible to go the other way, that is to obtain a Landau–Ginzburg model from our general setting. If $\varpi\colon E^\vee\to X$ is the dual of the perfect cone with section s, then the section ϖ^*s of ϖ^*E can be paired with the tautological section t of $\varpi^*E^\vee$, defining a function $w_s = \langle s,t\rangle$ on the total space $E^\vee$. By [Isi12, Corollary 3.8], if X is smooth, there is an equivalence $\mathfrak{Coh}^b(Z)\simeq\mathfrak{Sing}(\mathbb{R}\operatorname{Zero}(w_s)/\mathbb{G}_m)$ with the $\mathbb{G}_m$-equivariant dg-category of singularities of $\mathbb{R}\operatorname{Zero}(w_s)$ (where $\mathbb{G}_m$ acts by rescaling on the fibres of $E^\vee$). However we only have $Z=\mathbb{R}\operatorname{Crit}(w_s)$ if Z is smooth (see [CJW19, Lemma 2.2.2] in the regular and underived case).

To conclude, we explain how to recover from the categorified quantum Lefschetz principle a virtual statement in G-theory.

Proposition 3.2.2.3.7. *With the notations of subsection 3.2.2.2, if $\mathcal{F}$ is a vector bundle then* $(t_0 u)_*\left[\mathcal{O}_T^{\mathrm{vir}}\right] = \left[\mathcal{O}_M^{\mathrm{vir}}\right]\otimes\lambda_{-1}(\pi_0\mathcal{F}^\vee)$.

Proof. By naturality of the transformation $\jmath$, we have $(t_0u)_* = (\jmath_{M,*})^{-1}u_*\jmath_{T,*}$ so that $(t_0u)_*(t_0u)^! = (\jmath_{M,*})^{-1}u_*u^*\jmath_{M,*} = (\jmath_{M,*})^{-1}(\jmath_{M,*}(-)\otimes\lambda_{-1}(\mathcal{F}^\vee))$ by corollary 3.2.2.2.6. Hence $(t_0u)_*\left[\mathcal{O}_T^{\mathrm{vir}}\right] =$

$(t_0 u)_* (t_0 u)^! \left[\mathcal{O}_M^{\mathrm{vir}}\right] = (\jmath_{M,*})^{-1}(\lambda_{-1}(\mathcal{F}^\vee))$.

By [Lur19, Corollary 25.2.3.3], as $\mathcal{F}^\vee$ is flat over $\mathcal{O}_M$ so are its exterior powers $\bigwedge^n(\mathcal{F}^\vee)$. In particular, by [TV08, Proposition 2.2.2.5. (4)] they are strong $\mathcal{O}_M$-modules, meaning that $\pi_i(\bigwedge^n \mathcal{F}^\vee) \simeq \pi_i(\mathcal{O}_M) \otimes_{\pi_0(\mathcal{O}_M)} \pi_0(\bigwedge^n \mathcal{F}^\vee)$ for all natural integers i and n, and we conclude that

$$\begin{aligned}(t_0 u)_* \left[\mathcal{O}_T^{\mathrm{vir}}\right] &= \sum_{i \geqslant 0} (-1)^i \sum_{n \geqslant 0} (-1)^n \left[\pi_i\left(\bigwedge^n \mathcal{F}^\vee\right)\right] \\ &= \sum_{i \geqslant 0} (-1)^i [\pi_i(\mathcal{O}_M)] \otimes \sum_{n \geqslant 0} (-1)^n \left[\bigwedge^n \pi_0(\mathcal{F}^\vee)\right]\end{aligned} \tag{3.47}$$

as required. □

Remark 3.2.2.3.8. In the setting of the quantum Lefschetz principle, the only cases in which $\mathbb{E}_{g,n}$ is a vector bundle are when E is convex, that is $\mathbb{R}^1 p_* f^* \mathcal{E} = 0$ for any stable map $(p\colon C \to S, f\colon C \to X)$ from a rational curve C, and thus the genus is $g = 0$, which is the setting in which the quantum Lefschetz principle is already known. We conclude that it is not possible to relax the hypotheses for the quantum Lefschetz principle in G_0-theory, and that the more general version is thus only valid in its categorified form.

One may also notice that as the cotangent complex of u is $\mathscr{p}_* \operatorname{ev}^* \mathcal{E}^\vee[1]$, which has (homotopical) Tor-amplitude in $[0, 2]$ (in fact $[1, 2]$) unless the above conditions are satisfied, so that u is not quasi-smooth and the virtual pullback along it cannot be defined.

Corollary 3.2.2.3.9. *If* $\mathbb{E}_{0,n} = \mathscr{p}_{0,n;*} \operatorname{ev}_{0,n}^* E$ *is a vector bundle (that is if* E *is convex), the* G_0*-theoretic quantum Lefschetz formula of theorem 3.2.2.0.1 holds:*

$$(t_0 u)_* \sum_{i_* \gamma = \beta} \left[\mathcal{O}_{\mathscr{M}_{0,n}(Z,\gamma)}^{\mathrm{vir}}\right] = \left[\mathcal{O}_{\mathscr{M}_{0,n}(X,\beta)}^{\mathrm{vir}}\right] \otimes \lambda_{-1}(\pi_0 \mathscr{p}_{0,n;*} \operatorname{ev}_{0,n}^* \mathcal{E}^\vee). \tag{3.48}$$

□

CHAPTER

4

QUASI-STABLE MAPS AND THEIR GEOMETRIC FIELD THEORY

Let X be a (derived) scheme. In [STV15] were constructed derived moduli stacks $\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)$ of stable maps into X of class $\beta \in A_1X$, genus $g \in \mathbb{N}$ and with n markings, in the following way. There is a moduli stack $\mathfrak{M}_{g,n}$ of prestable curves of genus g with n markings, which has a universal curve $\mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$. Then $\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)$ is the open substack of the mapping derived stack $\mathcal{M}or_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$ determined by imposing a stability condition which must be fulfilled by the maps. These stacks are however not particularly well behaved: in particular, the universal curve $\mathfrak{C}_{g,n}$ does not have a convenient description: unlike in the case of the moduli stacks of stable curves $\overline{\mathscr{M}}_{g,n}$, the forgetful morphism $\mathfrak{M}_{g,n+1} \to \mathfrak{M}_{g,n}$ is *not* the universal curve. Since the brane action produces the object of extensions which is computed as a fibre of this forgetful morphism, this means that the brane action for this operad would not have interesting geometric content.

A variant used in [MR18] is the collection of moduli stacks $\mathfrak{M}^X_{g,n}$ of prestable curves whose components are decorated by effective curve classes in X, introduced in [Cos06]. They can be used in place of $\mathfrak{M}_{g,n}$ to define the moduli stack of stable maps into X just as well, but they have better formal properties. In particular, the universal curve is given by the morphism $\mathfrak{M}^X_{g,n+1} \to \mathfrak{M}^X_{g,n}$ forgetting the last marking.

The stacks of curves assemble into a modular operad, whose genus-0 part is reduced and hence gives a brane action from the construction of [MR18]. The target of a brane action is the object of extensions of the identity of the unique colour, extensions which are defined as the fibre of the morphism forgetting the last input. Hence, in the case of the operad $\mathfrak{M}^X_0$ of genus-0 curves, one recovers the universal curve and, by imposing the stability condition, the derived moduli stacks of stable maps, in the spans

$$
\begin{array}{ccc}
 & \coprod_\beta \mathbb{R}\overline{\mathscr{M}}_{0,n+1}(X,\beta) & \\
 {\scriptstyle (\mathrm{Stab},\mathrm{ev}_1,\dots,\mathrm{ev}_n)} \swarrow & & \searrow {\scriptstyle \mathrm{ev}_{n+1}} \\
\overline{\mathscr{M}}_{0,n+1} \times X^n & & X
\end{array}
\tag{4.1}
$$

giving the brane action.

In this final chapter, we will turn to the case where the target X is allowed to be a 1-Deligne-Mumford stack. In that case, as noted in [CCK15], the Gromov–Witten stability condition can be generalised to a family of "quasimap" stability conditions parameterised by a positive rational number (and in fact a choice of polarisation of X). To be more precise, the quasimap stability conditions were developed in [CCK15] for quotient stacks, but using the more general notion of stable loci from the "Beyond GIT" program of [Hal18] one can formulate it for more general polarised algebraic stacks. We will present in subsection 4.2.1.1 the definitions in this general context, although we do not yet refine the algebraicity results of [CCK15].

In addition to generalising the notion of stability, passing to the stacky setting adds some new technical changes to the result studied in [MR18].

The first of these is that, as was noticed in [AV02], for stacky targets, schematic prestable curves are no longer sufficient to obtain a proper moduli space, and the source curves for the stable maps must also be allowed to develop stacky structures, which come in the form of gerbes banded by cyclotomic groups μ_r at the markings. It is not possible to glue together two marked points which contain gerbes of different orders, and so the operad of moduli stacks of stacky curves will no longer be monochromatic, but will have its colours indexed by the strictly positive integers (the orders of gerbes). This is why the construction of a Gromov–Witten (or quasimap) geometric field theory for stacks requires the more general hapaxunital brane action of chapter 2. Indeed, here the collection of colours does not form a discrete set, but rather each colour r, indexing marked gerbes of order r, has as group of automorphisms $\mathcal{B}\,\mu_r$: it ensues that only the colour 1 is unital. This operadic technical point reflects a geometric phenomenon: when working with moduli stacks of stacky curves, the forgetful morphism giving the universal curve cannot forget any markings, but only the *schematic* ones, of order 1. This ensures that, by taking the hapaxunital extensions, the brane action once again recovers the universal curve.

The second phenomenon, also a consequence of the stacky structure of the curves, is that as the markings are now gerbes, the morphisms evaluating a stable map at a marking no longer land in the target X but in a stack parameterising gerbes in X, typically called its cyclotomic inertia stack. In subsection 4.2.2.2, we will explain how our brane action recovers the cyclotomic loop stack (the natural derived thickening of the cyclotomic inertia stack) as a consequence of the operadic structure of $\mathfrak{M}_0^X$, without having to manually stipulate it for the evaluations. More precisely, we will show as theorem 4.2.2.2.7 that the derived moduli stacks of quasi-stable maps $\mathbb{R}\mathcal{Q}^{\mathcal{L}}_{0,n}(X,\beta)$ appear as the apex of correspondences exhibiting a lax $(\overline{\mathscr{M}}_{0,n})_n$-algebra structure on the rigidified cyclotomic loop stack.

4.1 Source and targets for quasimaps

In this preliminary section, we prepare the definition of quasi-stable maps by defining their sources and the conditions on their targets. As explained above, the sources of quasi-stable maps must be stacky curves, so we begin in subsection 4.1.1.1 by defining them and collecting results on their local structure, and follow in subsection 4.1.1.2 by a study of their moduli stacks. As for the target, we shall need to use a good notion of polarisation to define stability conditions. In subsection 4.1.2.2 we review the notion of stable loci and stratifications on algebraic stacks from [Hal18], mainly focusing on those determined by cohomologically-defined numerical invariants. Those can in principle be defined from any line bundles, but for further geometric applications it is convenient to require these line bundles to be ample, that is to define a polarisation of the stack X. In order to make sense of this notion we also review in subsection 4.1.2.1 the construction of projective homogeneous spectra in derived geometry from [Gre17] and [Hek21]. Nothing in this section 4.1 is original.

4.1.1 Moduli of balanced orbicurves

4.1.1.1 Stacky curves and their local structure

In view of proposition 3.1.2.2.10, it will be enough to define families of stacky curves on truncated stacks, or even on spectra of fields, and then define a general family of stacky curves to be a (flat) family whose geometric fibres are all stacky curves.

Definition 4.1.1.1.1 (Nodal curve)**.** Let S be a truncated stack. A **relative nodal curve** over S is a flat quasi-smooth relative algebraic space $C \to S$ of relative dimension 1 whose singular locus is unramified over S.

Suppose now S be a derived stack. A relative nodal curve over S is a flat morphism $C \to S$ such that, for any geometric point $s\colon \operatorname{Spec}\kappa \to S$, the fibre C_s (which, by flatness, is a truncated stack) is a nodal curve over κ.

Example 4.1.1.1.2. By [Ja20, Tag 0C59], when S is truncated then $C \to S$ is a relative nodal curve if and only if it is flat of finite presentation and for every nonempty fibre C_s at a geometric point $s\colon \operatorname{Spec}\kappa \to S$,

- C_s is equidimensional of dimension 1,
- every closed point c of C_s is either in the smooth locus or behaves like a node, or ordinary double point, with $\widehat{\mathcal{O}_{C,c}} \simeq \Omega[\![x,y]\!]/(xy)$.

Definition 4.1.1.1.3 (Stacky Curve)**.** Let S be a derived stack. A relative n**-marked stacky curve of genus** g over S is a flat morphism $C \to S$ along with closed sub-S-stacks $\Sigma_i \hookrightarrow C$, for $i \in [\![1,n]\!]$, referred to as **markings**, such that

- $C \to S$ is a flat proper tame Deligne–Mumford stack of finite presentation which is étale-locally a nodal curve,
- the stacks Σ_i are all disjoint and in the smooth locus of $C \to S$,
- each $\Sigma_i \to S$ is an étale 1-gerbe,
- the canonical morphism from C to its coarse moduli space is an equivalence on the (open) complement of the markings and the nodes.

Our first goal will be to construct a moduli stack parameterising families of stacky curves. Before that, we start by recalling their local structure in the non-schematic parts, the nodes and the markings.

Lemma 4.1.1.1.4 (Local structure at the nodes, [AV02, Theorem 4.4.1, Proposition 3.2.3])**.** *Let $C \to S$ be a stacky curve. There exists a surjective morphism $\coprod_\alpha U_\alpha \to C$ with, for each α, an action of a finite* *group Γ_α on U_α such that $[U_\alpha/\Gamma_\alpha] \to C$ is étale (in particular U_α is a pointed nodal curve). Furthermore, the action of Γ_α on the preimage of the smooth locus is free and, for any nodal point u of a geometric fibre U_s of a chart U, the stabiliser Γ_u is a cyclic group sending each branch of U_s to itself and acting on the tangent space of each branch by multiplication with a primitive root of unity (of order that of Γ_u).*

Remark 4.1.1.1.5 (Coordinate description and balancing condition). In coordinates, this means that in a formal neighbourhood of a node, C_s is isomorphic to $[\mathrm{Spec}(\kappa[x,y]/(xy))/\mu_r]$, where μ_r acts by $(x,y) \mapsto (\zeta x, \zeta' y)$ with ζ and ζ' primitive roots of unity.

The action of a nodal point's stabiliser group is said to be **balanced** if the roots of unity acting on each branch are the inverse to each other, that is if there exists a coordinate description in which $\zeta' = \zeta^{-1}$. All stacky curves will be required to be balanced, and accordingly this balancing will always be implicitly assumed from here on.

We now turn to the description of the markings of a stacky curve. For this, we need an intermediate construction.

Construction 4.1.1.1.6 (Root stacks, [AGV08, Appendix B],[Cad07]). 1. Let $\mathcal{L}$ be an invertible sheaf on X, corresponding to a morphism $\ulcorner\mathcal{L}\urcorner\colon X \to \mathcal{B}\,\mathbb{G}_m$, and let r be a positive integer. Consider the r-th power map $\mathbb{G}_m \to \mathbb{G}_m, \lambda \mapsto \lambda^r$; it induces $\mathcal{B}\,\mathbb{G}_m \to \mathcal{B}\,\mathbb{G}_m$, which in terms of the functor of points maps a line bundle $\mathcal{M}$ to $\mathcal{M}^{\otimes r}$.

The r**th root stack** of $\mathcal{L}$ is the fibre product

$$\begin{array}{ccc} \sqrt[r]{\mathcal{L}/X} := X \underset{\mathcal{B}\,\mathbb{G}_m}{\times} \mathcal{B}\,\mathbb{G}_m & \longrightarrow & \mathcal{B}\,\mathbb{G}_m \\ \big\downarrow & & \big\downarrow {\scriptstyle */(-)^r} \\ X & \xrightarrow[\ulcorner\mathcal{L}\urcorner]{} & \mathcal{B}\,\mathbb{G}_m . \end{array} \tag{4.2}$$

The arrow $\sqrt[r]{\mathcal{L}/X} \to \mathcal{B}\,\mathbb{G}_m$ classifies a line bundle $\widetilde{\mathcal{L}}$ on $\sqrt[r]{\mathcal{L}/X}$, and the commutativity of the square equation (4.2) corresponds to an equivalence $\widetilde{\mathcal{L}}^{\otimes r} \simeq \mathcal{L}$.

The Kummer exact sequence $0 \to \mu_r \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ induces by [BDR18, Corollary 4.6] a cofibre sequence $\mathcal{B}\,\mathbb{G}_m \to \mathcal{B}\,\mathbb{G}_m \to \mathcal{B}^2\,\mu_r$, and the composite map $\sqrt[r]{\mathcal{L}/X} \to \mathcal{B}\,\mathbb{G}_m \to \mathcal{B}^2\,\mu_r$ exhibits $\sqrt[r]{\mathcal{L}/X}$ as a μ_r-banded X-gerbe.

2. Let $(\mathcal{L}, s)$ be a pair of a line bundle with a section, given by $\ulcorner(\mathcal{L},s)\urcorner\colon X \to [\mathbb{A}^1/\mathbb{G}_m] =: \Theta$. For any positive integer r, the r**th root stack** of $(\mathcal{L}, s)$ is the fibre product

$$\begin{array}{ccc} \sqrt[r]{(\mathcal{L},s)} := X \underset{\Theta}{\times} \Theta & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \big\downarrow & & \big\downarrow {\scriptstyle (-)^r/(-)^r} \\ X & \xrightarrow[\ulcorner(\mathcal{L},s)\urcorner]{} & [\mathbb{A}^1/\mathbb{G}_m] \end{array} \tag{4.3}$$

along the rth power map $\Theta \to \Theta$.

As before, the map $\sqrt[r]{(\mathcal{L},s)} \to \Theta$ classifies a line bundle $\widetilde{\mathcal{L}}$ on $\sqrt[r]{(\mathcal{L},s)}$ with a section $\widetilde{s}$, and commutativity of the square corresponds to an equivalence $(\widetilde{\mathcal{L}}^{\otimes r}, \widetilde{s}^r) \simeq (\mathcal{L}, s)$. In particular, on the locus where $s = 0$, this makes $\widetilde{s}$ nilpotent of order r.

Proposition 4.1.1.1.7 (Structure of the markings, [AGV08, Theorem 4.2.1], [GS16, Theorem 1])**.** *Let* $(C \to S, \Sigma_1, \cdots, \Sigma_n)$ *be an* n*-pointed stacky curve, and assume that for all* $i \in [\![1,n]\!]$ *the index of the* i*th section is constant, with value* d_i*.*

There is a canonical isomorphism of S*-stacky curves* $C_{sm} \simeq \prod_{/|C|} \sqrt[d_i]{(\mathcal{O}_{|C|}(S_1, \sigma_1))/C}_{sm}$*, inducing the identity on* $|C|_{sm}$*.*

There is a canonical isomorphism of S*-gerbes* $\Sigma_i \simeq \sqrt[d_i]{N_{s_i\colon S\to|C|}/S}$ *(where* $N_{s_i\colon S\to|C|}/S$ *is the normal bundle of* S *embedded in* $|C|$ *via the section* s_i*), inducing a canonical* μ_{d_i}*-band on* Σ_i*.*

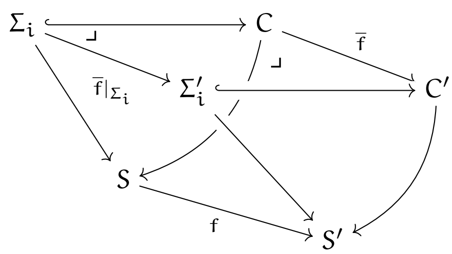
In particular, each marking $\Sigma_i \subset C$ defines a canonical morphism $S \to \mathcal{B}^2\,\mu_{d_i}$.

4.1.1.2 The moduli stacks of stacky curves

Definition 4.1.1.2.1 (Morphisms of stacky curves)**.** Let $(C \to S; \Sigma_1, \cdots, \Sigma_n)$ and $(C' \to S'; \Sigma'_1, \cdots, \Sigma'_n)$ be n-pointed stacky curves. A **morphism** $(C \to S; \Sigma_1, \cdots, \Sigma_n) \to (C' \to S'; \Sigma'_1, \cdots, \Sigma'_n)$ is a morphism $S \to S'$ along with an S-equivalence $C \xrightarrow{\simeq} C' \times_{S'} S$, equivalently given by a morphism $\overline{f}\colon C \to C'$ making the square

$$\begin{array}{ccc} C & \xrightarrow{\overline{f}} & C' \\ \downarrow & \lrcorner & \downarrow \\ S & \xrightarrow[f]{} & S' \end{array} \tag{4.4}$$

cartesian, such that for every $i \in [\![1,n]\!]$, the restriction $\overline{f}|_{\Sigma_i}\colon \Sigma_i \to C'$ factors through $\Sigma'_i \hookrightarrow C$ and the S-morphism $\Sigma_i \to \overline{f}^*\Sigma'_i = C \times_{C'} \Sigma'_i$ induced by $\overline{f}|_{\Sigma_i}$ is an equivalence. In other words, a morphism is given by a diagram



(4.5)

in which the two displayed squares (and hence, by the gluing law for pullbacks, the third one as well) are cartesian.

A 2**-morphism** of morphisms of stacky curves is a 2-morphism of the morphism of stacks $\overline{f}$ constituting the morphism of stacky curves.

In this way one obtains a 2-category (in fact, as is easily seen, a 2-groupoid) of n-pointed stacky curves as a locally full sub-2-category of $\mathfrak{dSt}^{2\times 2}$ on those squares

$$\begin{array}{ccc} \coprod_{i=1}^n \Sigma_i & \hookrightarrow & C \\ \downarrow & & \downarrow \\ S & = & S \end{array} \tag{4.6}$$

which define a twisted curve over S.

As the conditions defining stacky curves are stable under base-change, the projection to $\mathfrak{dSt}$ satisfies descent. The associated **derived moduli stack of stacky curves** (of genus g with n marked points) will be denoted $\mathfrak{M}_{g,n}$.

Theorem 4.1.1.2.2 ([Ols07, Theorem 1.10]). *The derived stack $\mathfrak{M}_{g,n}$ is 1-Artin.*

Proof. By the results of subsection 3.2.1.2, it is formally algebraic and has a cotangent complex. Furthermore, by [Ols07, Theorem 1.10], its truncation is an Artin 1-stack. The result then follows from the representabilty criterion of theorem 3.1.2.3.2. □

Remark 4.1.1.2.3. While we will only be interested in balanced stacky curves, one may also define a moduli stack of not necessarily balanced stacky curves, of which balanced ones form a clopen substack by[AV02, Proposition 8.1.1].

Proposition 4.1.1.2.4. *The derived stack $\mathfrak{M}_{g,n}$ is truncated.*

Proof. We need to check that the closed immersion $t_0\,\mathfrak{M}_{g,n} \hookrightarrow \mathfrak{M}_{g,n}$ is an equivalence. Since it is obviously an equivalence at the level of truncations, all that needs to be verified is that it is étale.

The same arguments as in [PY20] compute that the stalk at a point $\ulcorner(C \xrightarrow{p} T, (\Sigma_i))\urcorner \colon T \to \mathfrak{M}_{g,n}$ of the cotangeant complex of $\mathfrak{M}_{g,n}$ is

$$\ulcorner(C \xrightarrow{p} T, (\Sigma_i))\urcorner^* \mathbb{L}_{\mathfrak{M}_{g,n}} = \left(p_* \mathbb{T}_{C/T}(-\sum \Sigma_i)\right)^{\vee}[-1]. \tag{4.7}$$

Since C is a curve and quasi-smooth, the cotangent complex is indeed concentrated in degree 0, so that $\mathfrak{M}_{g,n}$ is smooth.

But $t_0\,\mathfrak{M}_{g,n}$ is known to be smooth as well, so the result follows. □

Remark 4.1.1.2.5. This means that $\mathfrak{M}_{g,n}$ is the extension to derived rings of its truncation, which is the moduli stack of twisted curves studied in [AGV08] and [Ols07]. This stack was usually denoted $\mathfrak{M}^{tw}_{g,n}$ to emphasise the stacky nature of the moduli problem it represents. Unlike in the cited works we omit this emphasis as the stacky theory is the natural context in which we place ourselves; however, due to proposition 4.1.1.2.4 we have not introduced any shift in notation to show that we work in the derived setting, as will on the contrary be done for the moduli stacks of quasi-stable maps.

In order to work with these stacks and the operad they comprise, it will be useful to introduce some refinements.

By similar arguments as [Ja20, Tag 0E6K], there is a decomposition of $\mathfrak{M}_{g,n}$ in clopen substacks

$$\mathfrak{M}_{g,n,(r_1,\dots,r_n)} \tag{4.8}$$

where r_i is the order of the gerbe at the ith marking.

Next, we introduce a graded version, adapted to a target for maps from curves, and due to [Cos06]. Let $(X, \mathcal{L})$ be a polarised 1-algebraic derived stack, and let $\mathrm{Eff}(X, \mathcal{L})$ denote its monoid of effective classes.

Construction 4.1.1.2.6. For any class $\beta \in \mathrm{Eff}(X, \mathcal{L})$, one can define a moduli stack $\mathfrak{M}_{g,n,\beta}$ of prestable stacky curves decorated with stable decompositions of β. If C is a prestable curve, each irreducible component $C_i \subset C$ is decorated with an element $\beta_i \in \mathrm{Eff}(X, \mathcal{L})$, in such a way that $\sum_i \beta_i = \beta$ and that $2g_i - 2 + n_i + \epsilon\beta_i(\mathcal{L}_0) > 0$.

The precise definition is carried out recursively as in [Cos06, page 569, before Proposition 2.0.2].

Proposition 4.1.1.2.7. *The forgetful map*

$$\mathfrak{M}_{g,n+1,(r_1,\dots,r_n,1),\beta} \to \mathfrak{M}_{g,n,(r_1,\dots,r_n),\beta} \tag{4.9}$$

exhibits $\mathfrak{M}_{g,n+1,(r_1,\dots,r_n,1),\beta}$ *as the universal curve* $\mathfrak{C}_{g,n,(r_1,\dots,r_n),\beta}$ *over* $\mathfrak{M}_{g,n,(r_1,\dots,r_n),\beta}$.

Proof. As in [Cos06, Proposition 2.1.1] (and [PY20] for the truncatedness of the universal curve: since the families of curves classified by $\mathfrak{M}_{g,n}$ are required to be flat, so is its universal curve). □

4.1.2 Stable loci of polarised Artin stacks

4.1.2.1 Polarisations on quasi-projective derived stacks

Definition 4.1.2.1.1. Let B be an ∞-groupoid. The $(\infty,1)$-category of B**-graded objects** in an $(\infty,1)$-category $\mathfrak{C}$ is the functor $(\infty,1)$-category $\mathfrak{C}^B$.

Suppose $\mathfrak{C}^\otimes$ is presentably symmetric monoidal and B is also endowed with an $\mathscr{E}_\infty$-monoid structure. The procedure of Day convolution then produces a symmetric monoidal structure on $\mathfrak{C}^B$. The $(\infty,1)$-category of X-graded algebras in $\mathfrak{C}^\otimes$ is $\mathscr{E}_\infty\text{—}\mathfrak{Alg}(\mathfrak{C}^B)$.

In the case where $\mathfrak{C}$ is the $(\infty,1)$-category of modules over some derived ring, it will be useful to have a more geometric interpretation of gradings.

Construction 4.1.2.1.2 (Monoid algebra)**.** By [Lur17], an $(\infty,1)$-category $\mathfrak{C}$ with products is equivalent to its $(\infty,1)$-category $\mathscr{E}_\infty\text{—}\mathfrak{Alg}_s(\mathfrak{C}^{op})^{op}$ of cocommutative cogebras. As seen in [Lur17, Example 6.2.4.13], the $(\infty,1)$-functor $\Sigma^\infty\colon \infty\text{—}\mathfrak{Grpd} \to \infty\text{—}\mathfrak{Grpd}^{\sqcap} \simeq \mathbb{S}\text{—}\mathfrak{Mod}$ left-adjoint to Ω^∞ has a symmetric monoidal structure, and so preserves cocommutative cogebras.

If G is an $\mathscr{E}_\infty$-monoid, its **monoid** $\mathbb{k}$**-algebra** is $\mathbb{k}[G] := \mathbb{k} \otimes_{\mathbb{S}} \Sigma^\infty G$, endowed with its commutative cogebra structure (in $\mathbb{k}\text{—}\mathfrak{Alg}$), which is grouplike when G is.

Due to the cogebra structure on $\mathbb{k}[G]$, its spectrum $\mathrm{Spec}(\mathbb{k}[G])$ has the structure of a commutative group object in derived schemes. We write $D_{\mathbb{k}}(G) := \mathrm{Spec}(\mathbb{k}[G])$ for the **diagonalisable group derived** $\mathbb{k}$**-scheme** corresponding to G.

Lemma 4.1.2.1.3 ([Gre17, Corollary 1.5.14])**.** *Let* X *be a Deligne–Mumford derived* $\mathbb{k}$*-stack and* G *an* $\mathscr{E}_\infty$*-monoid. There is an equivalence of* $(\infty,1)$*-categories*

$$\mathscr{E}_\infty\text{—}\mathfrak{Alg}(\mathfrak{QCoh}(X)^G) \simeq (\mathfrak{dSt}^{aff}_{/X})^{\mathcal{B}\,D_{\mathbb{k}}(G)} = D_{\mathbb{k}}(G)\text{—}\mathfrak{Mod}(\mathfrak{dSt}^{aff}_{/X}), \tag{4.10}$$

where $\mathfrak{dSt}^{aff}_{/X}$ *is the full sub-*$(\infty,1)$*-category of* $\mathfrak{dSt}_{/X}$ *on the relatively affine* X*-stacks.*

Proposition 4.1.2.1.4 ([Gre17, Proposition 1.6.5])**.** *Let* X *be a Deligne–Mumford derived stack and* G *an* $\mathscr{E}_\infty$*-monoid. Let* $\mathcal{A}$ *be a* G*-graded quasicoherent* $\mathcal{O}_X$*-algebra. There is an equivalence of* $(\infty,1)$*-categories*

$$\mathfrak{QCoh}\big(\mathcal{S}pec_X(\mathcal{A})/D_X(G)\big) \simeq \mathcal{A}\text{—}\mathfrak{Mod}\big(\mathfrak{QCoh}(X)^G\big). \tag{4.11}$$

In particular, taking $\mathcal{A}$ *to be* $\mathcal{O}_X$ *equipped with the trivial* G*-grading, we find* $\mathfrak{QCoh}(\mathcal{B}\,D_X(G)) \simeq \mathfrak{QCoh}(X)^G$.

Lemma 4.1.2.1.5 ([Gre17, Lemmata 2.5.1, 2.5.2])**.** *Let* G *be an* $\mathscr{E}_\infty$*-monoid, let* S *be a base Deligne–Mumford derived stack, and let* $\mathcal{A}$ *be a* G*-graded quasicoherent* $\mathcal{O}_X$*-algebra. The invariant subscheme* $(\mathcal{S}pec_X\,\mathcal{A})^{D_X(G)}$ *is equivalent to* $\mathcal{S}pec_X\,\mathcal{A}_e$ *where* $\mathcal{A}_e$ *is the component indexed by the unit* $e \in G$*. It is a closed immersion defined by the* ***irrelevant ideal*** $\pi_0\mathcal{A}_{\neq e}$*, and its complement is* $D_X(G)$*-invariant in* $\mathcal{S}pec_X\,\mathcal{A}$.

Definition 4.1.2.1.6. Let Σ denote the free $\mathscr{E}_\infty$-monoid on one generator. Let X be a Deligne–Mumford derived stack and let $\mathcal{A}$ be a Σ-graded quasicoherent $\mathcal{O}_X$-algebra. The (relative) Proj construction of $\mathcal{A}$ is the quotient

$$\mathcal{P}roj_X(\mathcal{A}) := \big(\mathcal{S}pec_X(\mathcal{A}) \setminus V(\pi_0\mathcal{A}_{\neq 0})\big)/\mathbb{G}_m. \tag{4.12}$$

Proposition 4.1.2.1.7 ([Hek21, Proposition 5.5.5])**.** *Let* S *be a base derived stack defined over* $\operatorname{Spec}\mathbb{Q}$*, so that* Σ*-grading coincides with* $\mathbb{N}$*-grading, and let* $\mathcal{B}_\bullet$ *be an* $\mathbb{N}$*-graded quasicoherent* $\mathcal{O}_S$*-algebra. Then* $\mathcal{P}roj_S\,\mathcal{B}_\bullet$ *represents the* $(\infty,1)$*-functor of points mapping* $\psi\colon Y \to S$ *to the full sub-*∞*-groupoid of* $\iota_0\big(\mathcal{O}_T{-}\mathfrak{Alg}^{\mathbb{N}-\mathrm{gr}}_{(\psi^*\mathcal{B}_\bullet)/}\big)$ *on those maps* $\psi^*\mathcal{B}_\bullet \to \mathcal{M}_\bullet$ *such that* $\mathcal{M}_\bullet \simeq \mathcal{S}ym\mathcal{M}_1$ *is freely generated by a line bundle and* $\psi^*\mathcal{B}_1\mathcal{M}_1$ *is surjective on* π_0 *groups.*

This suggests a definition of ampleness for line bundles.

Definition 4.1.2.1.8. Let $\varpi\colon X \to S$ be a relative derived algebraic space, defined over $\operatorname{Spec}\mathbb{Q}$. A line bundle $\mathcal{L}$ on X is **relatively ample** over S (or simply ϖ-ample) if the counit map $\varpi^*\varpi_*(\mathcal{S}ym\mathcal{L}) \to \mathcal{S}ym\mathcal{L}$ defines an open immersion $X \to \mathcal{P}roj_S(\varpi_*\mathcal{S}ym\mathcal{L})$.

If $\varpi\colon X \to S$ is now a relative Deligne–Mumford derived stack, a line bundle $\mathcal{L}$ on X is ϖ-ample if the induced line bundle on the relative coarse moduli space is relatively ample.

4.1.2.2 Stability conditions and Θ-stratifications

This section consists of reminders from [Hal18] and [Hal20] in order to define the stability locus of a polarised stack and formulate the stability condition.

Notation 4.1.2.2.1. Consider the action of $\mathbb{G}_m$ on $\mathbb{A}^1$. We shall let Θ denote the quotient stack

$$\Theta := \left[\mathbb{A}^1/\mathbb{G}_m\right]. \tag{4.13}$$

It is then the moduli stack for pairs of a line bundle and a section thereof.

We shall also make use, for X an Artin derived stack, of the mapping stacks:

$$\mathcal{F}ilt(X) := \mathcal{M}or(\Theta, X) \qquad \text{and} \qquad \mathcal{G}rad(X) := \mathcal{M}or(\mathcal{B}\,\mathbb{G}_m, X). \tag{4.14}$$

They fit in the following diagram:

$$\mathcal{G}rad(X) \underset{\mathrm{ev}_0}{\overset{\hookrightarrow}{\rightleftarrows}} \mathcal{F}ilt(X) \xrightarrow{\mathrm{ev}_1} X = \mathcal{M}or(*, X) \tag{4.15}$$

where ev_0 can be interpreted as the map of associated graded of a filtered stack, while ev_1 is the map of underlying object of the filtration.

Definition 4.1.2.2.2 (Θ-stratum)**.** A **Θ-stratum** in X is a clopen substack $S \subset \mathcal{F}ilt(X)$ such that $\mathrm{ev}_1|_S\colon S \to X$ is a closed immersion.

Definition 4.1.2.2.3 (Θ-stratification)**.** A **Θ-stratification** on X consists of a well-ordered set $(\Gamma, \leqslant)$ and

- for each $\gamma \in \Gamma$ an open substack $X_{\leqslant\gamma} \subset X$ such that, whenever $\gamma < \gamma'$ in Γ then $X_{\leqslant\gamma} \subset X_{\leqslant\gamma'}$,
- in each $X_{\leqslant\gamma}$ a Θ-stratum $S \subset \mathcal{F}ilt(X_{\leqslant\gamma})$ such that

$$X_{\leqslant\gamma} \setminus \mathrm{ev}_1(S_\gamma) = X_{<\gamma} := \bigcup_{\gamma'<\gamma} X_{\leqslant\gamma'}. \tag{4.16}$$

By [Hal20, Lemma 1.2.3], Θ-strata (and thus also Θ-stratifications) in X are the same thing as truncated Θ-strata (and Θ-stratifications) in $t_0 X$.

Lemma 4.1.2.2.4. *Let* $f\colon X' \to X$ *be a morphism of stacks. If* f *is representable (resp. a monomorphism, a closed immersion, an open immersion, smooth, étale) then so is the induced map* $\mathcal{F}ilt(X') \to \mathcal{F}ilt(X)$.

Construction 4.1.2.2.5. Let X be a stack endowed with a Θ-stratification. For every $\gamma \in \Gamma$, the clopen substack $S_\gamma \subset \mathcal{F}ilt(X_{\leqslant\gamma})$ defines an open substack of $\mathcal{F}ilt(X)$. In particular, we have an injection $\mathrm{IrrComp}(S_\gamma) \hookrightarrow \mathrm{IrrComp}(\mathcal{F}ilt(X))$.

Hence the Θ-stratification selects a set of irreducible components of $\mathcal{F}ilt(X)$, and an indexing of them by Γ.

Construction 4.1.2.2.6. Consider Σ a collection of irreducible components of $\mathcal{F}ilt(X)$, and $\mu\colon \Sigma \to \Gamma$ a locally constant map.

We first extend μ to

$$\begin{aligned} \mu\colon |\mathcal{F}ilt(X)| &\to \Gamma \cup \{-\infty\} \\ f &\mapsto \max(\{-\infty\} \cup \{\mu(\sigma) \mid f \in \sigma\}). \end{aligned} \tag{4.17}$$

We can then define a **stability function**

$$\begin{aligned} M^\mu\colon |X| &\to \Gamma \cup \{-\infty\} \\ x &\mapsto \sup\{\mu(f) \mid f(1) = x\}. \end{aligned} \tag{4.18}$$

From this data, we finally define

$$|X|_{\leqslant\gamma} = \{x \in |X| \mid M^\mu(x) \leqslant \gamma\} \tag{4.19}$$

and

$$|\mathcal{F}ilt(X)|_\gamma = \{f \in |\mathcal{F}ilt(X)| \mid f \text{ lies in } \Sigma,\ \mu(f) = M^\mu(f)\}. \tag{4.20}$$

Theorem 4.1.2.2.7 ([Hal18, Theorem 2.7])**.** *The subsets above define a Θ-stratification on* X *if and only if certain technical conditions are satisfied, chief among which the existence of a Harder–Narasimhan (*i.e. *maximally destabilising) filtration: for any geometric point* x *of* X*, writing* $\mathrm{Flag}(p) := \mathcal{F}ilt(X) \times_X \{p\}$*, the space* $|\mathrm{Flag}(p)|$ *contains a unique point* f *lying over an irreducible component in* Σ *with* $\mu(f) = M^\mu(p)$.

The numerical invariants we will be interested in will come from cohomology classes, and more specifically from the Chern classes of line bundles.

Lemma 4.1.2.2.8. *The Picard group of* Θ *is canonically isomorphic to* $\mathbb{Z}$.

We will work, as in [Hal18, §3.7], with a cohomology theory $H^\bullet$ taking coefficients in a subring A of an archimedean ordered field and which must satisfy $H^\bullet(\Theta^n) \simeq A[u_1, \dots, u_n]$ with each generator u_i in cohomological degree 2.

Proposition 4.1.2.2.9 (*cf.* [Hal18, Example 4.13])**.** *Let* $f\colon \Theta \to X$ *be a filtered point, with associated graded* $f(0)$*. Then, letting* $\ell = c_1(\mathcal{L})$*, we have that* $f^*\ell = -\mathrm{wt}(\mathcal{L}_{f(0)})u_2$.

Proof. We have $f^*\ell = c_1(f^*\mathcal{L}) \in H^2(\Theta, \mathbb{Q})$. But $f^*\mathcal{L}$ is a line bundle on Θ so is of the form $\mathcal{O}_\Theta\langle w\rangle$ for a weight w. This is exactly the opposite of the weight of the $\mathbb{G}_m$-sheaf $\mathcal{L}_{f(0)}$. □

Remark 4.1.2.2.10. For every $n \in \mathbb{N}_{>0}$, there is a ramified covering $\Theta \to \Theta, z \mapsto z^n$. A class $\gamma \in H^2(\Theta, \mathbb{Q})$ is scaled by n under the action of this covering. Hence there is a need to normalise.

Definition 4.1.2.2.11. A **cohomologically defined numerical invariant** is given in the following way. Say that $\beta \in H^4(X, \mathbb{Q})$ is **positive-definite** if for any $\varphi\colon \mathbf{B}\mathbb{G}_m \to X$ inducing a non-trivial homomorphism $\mathbb{G}_m \to \mathrm{Aut}_X(\varphi(*))$, the class $\varphi^*\beta$ is a positive multiple of the generator u^2 of $H^4(\Theta, \mathbb{Q})$.

Pick $\lambda \in H^2(X, \mathbb{Q})$ and $\beta \in H^4(X, \mathbb{Q})$ positive-definite. We may then set, for any $f \in \mathcal{F}ilt(X)$,

$$\mu(f) = \frac{f^*\lambda}{\sqrt{f^*\beta}} \in \mathbb{R}_{\mathrm{alg}}. \tag{4.21}$$

Note that the covering of order n scales $H^4(\Theta, \mathbb{Q})$ by n^2 so such numerical invariants are indeed invariant under these coverings.

Definition 4.1.2.2.12 (Θ-reductivity)**.** A stack X is Θ-reductive if $\mathrm{ev}_1 : \mathcal{F}ilt(X) \to X$ satisfies the valuative criterion for properness, that is is right orthogonal to the inclusion of the spectrum of the function field K of any discrete valuation ring R:

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \mathcal{F}ilt(X) \\ \downarrow & \exists! \nearrow & \downarrow \mathrm{ev}_1 \\ \operatorname{Spec} R & \longrightarrow & X \end{array} \tag{4.22}$$

Example 4.1.2.2.13.
- If V is affine and G reductive, then $[V/G]$ is Θ-reductive. This is not the case if V is projective, but the stack can then be included in its affine cone.
- The moduli stack of objects of an abelian category is Θ-reductive.

Theorem 4.1.2.2.14. *Let X be Θ-reductive and let μ be a cohomologically-defined numerical invariant. If μ satisfies an appropriate boundedness condition, then every μ-unstable point of X admits a $\mathcal{HN}$-filtration, unique up to the ramified coverings of Θ.*

Corollary 4.1.2.2.15. *If μ is a cohomologically defined numerical invariant on a Θ-reductive stack X, it defines a Θ-stratification on X if and only if it satisfies the following boundedness condition:*

For any $\xi\colon \operatorname{Spec} A \to X$, there exists a quasi-compact substack $X' \subset X$ such that, for all finite type $p \in (\operatorname{Spec} A)(\Bbbk)$ and $f \in \mathrm{Flag}(\xi(p))$ with $\mu(f) > 0$, there is $f' \in \mathrm{Flag}(\xi(p))$ such that $\mu(f') \geqslant \mu(f)$ and $\mathrm{gr}(f') \in X'$.

Example 4.1.2.2.16. If X is quasicompact, the condition is satisfied.

Proposition 4.1.2.2.17. *If X is Θ-reductive, then so is its semistable locus (relative to a cohomology class).*

The condition of Θ-reductivity can also be used (over $\mathbb{Q}$) to check for tameness.

Definition 4.1.2.2.18 (Seshadri completeness)**.** A morphism $f\colon X \to Y$ is $\mathbb{S}$**-complete** if for any DVR R, with maximal ideal (ϖ), it is right orthogonal to $\mathrm{ST}_R \setminus 0 \to \mathrm{ST}_R$ where $\mathrm{ST}_R = \operatorname{Spec}(R([s,t]/(st - \varpi)))/\mathbb{G}_m$, where the $\mathbb{G}_m$-action comes from a grading giving s weight 1 and t weight -1, and the point 0 is $\{s = t = 0\}$.

Theorem 4.1.2.2.19. *Let X be an algebraic stack locally of finite type with quasi-affine diagonal over a quasi-separated locally noetherian algebraic space defined over $\mathbb{Q}$. Then X admits a separated good moduli space if and only if it is Θ-reductive and $\mathbb{S}$-complete.*

4.2 Gromov–Witten action on the loop stack

4.2.1 The derived moduli stack of quasi-stable maps

4.2.1.1 Quasi-stable maps to polarised orbifolds

Let X be a quasiprojective 1-Artin derived stack, and pick a polarisation $\mathcal{L}_0$. Let also $\varepsilon \in \mathbb{Q}_{>0}$ be a positive rational number. We will sometimes think of $\mathcal{L} := \varepsilon \otimes \mathcal{L}_0 \in \mathrm{Pic}(X)$ as a rational polarisation of X.

Construction 4.2.1.1.1 (Degree of a polarisation)**.** Let C be a smooth irreducible schematic curve. Recall that the **degree** $\deg_C(\mathcal{M})$ of a line bundle $\mathcal{M}$ on C can be defined as the pairing $\int_{[C]} c_1(\mathcal{M}) \in \mathbb{Z}$ of the first Chern class of $\mathcal{M}$ with the fundamental class of C.

Now if C is more generally a stacky curve, the degree of $\mathcal{M} \in \mathrm{Pic}(C)$ is defined in the following way, following [AGV08, before Theorem 7.2.1]: let $\nu\colon \widetilde{C} \to C$ be the normalisation of C, so that the irreducible components of C give rise to connected components of $\widetilde{C}$. For each connected (and irreducible) component $\widetilde{C}_i$, pick a finite morphism $\varphi_i\colon D_i \to \widetilde{C}_i$ from a connected smooth schematic curve D_i, of degree d_i. Then one may set

$$\deg_C(\mathcal{M}) = \sum_i \frac{1}{d_i} \deg_{D_i} \varphi_i^* \nu^* \mathcal{M}. \tag{4.23}$$

Definition 4.2.1.1.2 (Class of a map)**.** Let $f\colon C \to X$ be a map from a stacky curve to X. Its **class** is the element of $\mathfrak{Grp}(\mathrm{Pic}(X), \mathbb{Q})$ mapping $\mathcal{M}$ to $\deg_C(f^*\mathcal{M})$. If f is representable, it in fact belongs to $\mathfrak{Grp}\left(\mathrm{Pic}(X), \frac{1}{e}\mathbb{Z}\right)$, where e is the least common multiple of the orders of the automorphism groups of geometric points of $t_0(X)$.

Remark 4.2.1.1.3. There is a morphism $A_1 X \to \mathfrak{Grp}(\mathrm{Pic}(X), \mathbb{Q})$ mapping a class $\beta \in A_1 X$ to the homomorphism $\mathcal{M} \mapsto \int_\beta c_1(\mathcal{M})$. By the projection formula, the class of a map $f\colon C \to X$ is the image under this morphism of the curve class $f_*[C] \in A_1 X$.

Definition 4.2.1.1.4 (Prestable quasimap)**.** Let $g, n \in \mathbb{N}$ and $\beta \in \mathfrak{Grp}(\mathrm{Pic}(X), \mathbb{Q})$. Consider an n-marked prestable curve $(C; \Sigma_1, \dots, \Sigma_n)$ of genus g and a map $C \to X$ of class β such that the generic point η_i of each irreducible component $C_i \subset C$ is mapped to the $\mathcal{L}$-stable locus of X. In other words, there are at most finitely many points mapped to the unstable locus; call those points **basepoints** of f.

We say that $(C; \Sigma_1, \dots, \Sigma_n; f)$ is a **pre-$\mathcal{L}$-quasistable map** of genus g with n markings and of class β to X if its basepoints are all disjoint from the special points of the curve.

Lemma 4.2.1.1.5 ([AGV08, Lemma 2.1.2])**.** *Let e be the least common multiple of the orders of the automorphism groups of geometric points of $t_0(X)$. The line bundle $t_0 \mathcal{L}^{\otimes e}$ is the pullback of a line bundle on the coarse moduli space $|t_0(X)|$.*

Definition 4.2.1.1.6 (Stable quasimap)**.** A pre-$\mathcal{L}$-quasistable map to X is **$\mathcal{L}$-quasistable** if

- the rational line bundle
$$\omega_{|C|,\log} \otimes (f^* \mathcal{L}_0^{\otimes e})^{\otimes \varepsilon / e} \tag{4.24}$$
is ample, where $\omega_{|C|,\log} = \omega_{|C|}(\sum_{i=1}^n |\Sigma_i|)$ is the logarithmic canonical sheaf of the marked curve,

- for every point x of C,
$$\varepsilon \cdot \ell(x) \leqslant 1, \tag{4.25}$$
where $\ell(x)$ is the order of contact of f at x.

Remark 4.2.1.1.7. By the numerical criterion for ampleness of line bundles on a curve, one finds that $\omega_{|C|,\log} \otimes (f^*\mathcal{L}_0^{\otimes e})^{\otimes \varepsilon/e}$ is ample if and only if for each irreducible component C_i of C we have $2g_i - 2 + n_i + \frac{\varepsilon}{e}\deg(e \cdot f^*\mathcal{L}_0|_{C_i}) > 0$.

Example 4.2.1.1.8. When $\varepsilon \geqslant 2$, we can see from remark 4.2.1.1.7 that the $\varepsilon \otimes \mathcal{L}_0$-stable quasimap theory of X is exactly the Gromov–Witten theory of the $\mathcal{L}_0$-stable locus.

The conditions for being an $\mathcal{L}$-quasistable map to X define an open substack $\mathcal{Q}^{\mathcal{L}}_{g,n}(X,\beta) \subset t_0\mathcal{M}or_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$, which is a stable substack in the sense of definition 3.2.1.1.7 (in fact, comparing remark 4.2.1.1.7 with the definition of the grading-stability for graded moduli of curves in construction 4.1.1.2.6, we see that it is even a stable substack of one defined by a degree function).

Theorem 4.2.1.1.9 ([CCK15, Theorem 2.7])**.** *Suppose* X *is a global quotient stack* $[V/G]$ *where* V *is an irreducible affine variety and* G *a reductive algebraic group. The stack* $\mathcal{Q}^{\mathcal{L}}_{g,n}(X,\beta)$ *is Deligne–Mumford.*

The derived moduli stack of quasimaps to X is the corresponding open $\mathbb{R}\mathcal{Q}^{\mathcal{L}}_{g,n}(X,\beta) \subset \mathcal{M}or_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$.

Lemma 4.2.1.1.10 ([MR18, Proposition 4.3.1])**.** *Suppose the* 1*-Artin derived stack* X *is quasi-smooth, and its* $\mathcal{L}$*-stable locus is a smooth* 1*-Deligne–Mumford stack. The derived stack* $\mathbb{R}\mathcal{Q}^{\mathcal{L}}_{g,n}(X,\beta)$ *is quasi-smooth and the perfect obstruction theory it induces on* $\mathcal{Q}^{\mathcal{L}}_{g,n}(X,\beta)$ *coincides with the one used (for example) in [CCK15, §2.4.5] for the classical construction of the virtual structure sheaf.*

Proof. It follows from the computation of the cotangent complex in remark 3.2.1.1.3 (and the fact that the restriction of $\mathrm{ev}\colon \mathcal{M}or_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n}) \to X$ to $\mathbb{R}\mathcal{Q}^{\mathcal{L}}_{g,n}(X,\beta)$ is the universal map) that the induced obstruction theories coincide. From this and the quasi-smoothness assumption on X we deduce that the cotangent complex has To-amplitude concentrated in degrees $[-1,2]$, and with the assumptions on the stable locus the arguments in the proof of [CKM14, Theorem 4.5.2] restrict it to $[0,1]$. □

It further follows from proposition 3.1.3.2.3 that the virtual structure sheaf defined from $\mathbb{R}\mathcal{Q}^{\mathcal{L}}_{g,n}(X,\beta)$ coincides with the usual virtual structure sheaf, that is that the geometry of $\mathbb{R}\mathcal{Q}^{\mathcal{L}}_{g,n}(X,\beta)$ recovers the "virtual geometry" of $\mathcal{Q}^{\mathcal{L}}_{g,n}(X,\beta)$.

4.2.1.2 The operad structure on the moduli stacks of stacky curves

As we have only constructed brane actions for ∞-operads and not modular ∞-operads in chapter 2, we will only obtain here the genus-0 quasimap invariants. However, the moduli stacks of stacky curves at play do form a modular operad, that we can describe here at little added cost, in hope of using it for higher-genus invariants in later work.

Construction 4.2.1.2.1 (Genus-graded modular graphs)**.** A **genus-graded modular graph** is a connected graph G equipped with a function $g\colon \mathrm{Vert}(G) \to \mathbb{N}$ associating with each vertex (or corolla) v of G an integer $g(v)$ called its genus.

The **total genus** of a genus-graded modular graph (G,g) is $\chi(G) + \sum_{v \in \mathrm{Vert}(G)} g(v)$. A genus-graded modular graph is said to be **stable** if it is connected and for every vertex v, letting $n(v)$ denote its number of adjacent edges, $2g(v) - 2 + n(v) > 0$.

A map of genus-graded modular graphs is a map of the underlying graphs compatible with the genus of its source and the total genus of its target.

This produces (by [HRY20, Proposition 4.16]) a category Υ_g of genus-graded modular graphs.

Example 4.2.1.2.2 (Genera on elementary graphs). The corolla $\star_n$ has a single vertex, with $n+1$ adjacent leaves. A genus-grading on a corolla is thus equivalent to the datum of a natural integer g, and the corolla $\star_n$ can be equipped with any genus-grading, giving a stable graph as long as $2g-1+n>0$. We let $\star_{(g,n)}$ denote the resulting genus-graded modular graph. Note that the unstable genus-graded corollas are exactly $\star_{(0,-1)}, \star_{(0,0)}, \star_{(0,1)}, \star_{(1,-1)}$.

The generic edge η has no vertex so admits a single genus grading.

Lemma 4.2.1.2.3. *The category Υ_g of genus-graded modular graphs admits an active-inert factorisation system, giving rise to the usual algebraic pattern structures $\Upsilon_g^\natural$ and $\Upsilon_g^\flat$ as in example 2.1.1.1.6 on its opposite.*

Here the "usual" patterns are the ones whose elementaries are the genus-graded corollas, denoted $\star_{(g,n)}$, and the generic edge η.

Proof. The factorisation system is exhibited from the one on Υ in [HRY20, Remark 4.17]. □

We will define the modular operad $\mathfrak{M}$ of (moduli stacks of) stacky curves by imposing the Segal conditions. That is, we will define the corresponding precosheaf $\Upsilon_g \to \mathfrak{dSt}$ by specifying its values on ${\Upsilon_g}^{\mathrm{op}\natural,\mathrm{el}}$, and extend it to the rest of ${\Upsilon_g}^{\mathrm{op}}$ by using the Segal decompositions of general graphs.

In the first place, we start by defining the restriction $\mathfrak{M}|_{{\Upsilon_g}^{\mathrm{op}\natural,\mathrm{inrt}}}$.

Construction 4.2.1.2.4. For any stable genus-g modular corolla $\star_{(g,n)}$, we set

$$\mathfrak{M}(\star_{(g,n)}) = \mathfrak{M}_{g,n+1} \tag{4.26}$$

(recall that the underlying graph $\star_n$ of $\star_{(g,n)}$ has $n+1$ leaves).

We set

$$\mathfrak{M}(\eta) = \coprod_r \mathcal{B}^2\,\mu_r \tag{4.27}$$

for the $\mathbb{Z}/(2)$-equivariant stack of colours; it is equipped with its involution inverting the band. For any inert morphism $\eta \to \star_n$, say the one picking the ith leaf, the associated $\mathfrak{M}_{g,n} \to \coprod_r \mathcal{B}^2\,\mu_r$ is the classifying morphism of the corresponding ith marked gerbe.

We also decreed that

$$\mathfrak{M}(\star_{(0,1)}) = \mathfrak{M}(\star_{(0,2)}) = \mathfrak{M}(\eta). \tag{4.28}$$

The image of the unique inert morphism $\eta \to \star_{(0,1)}$ is the identity. The images of the two morphisms $\eta \to \star_{(0,2)}$ are the identity and the involution.

We have defined the values of our putative Segal precosheaf on all elementaries, so we may now set, for a general modular graph T, $\mathfrak{M}(T) = \varprojlim_{T\to E} \mathfrak{M}(E)$. The image of any inert morphism $T \to E$ is then the canonical projection.

For the image of the active "contraction" morphisms, we shall need the following lemma.

Lemma 4.2.1.2.5 ([AGV08, Proposition 5.1.3])**.** *Let $g_1, g_2 \in \mathbb{N}$ be two genera, and let S_1 and S_2 be two set of marking indexes. There are representable gluing morphisms of classical stacks*

$$\mathfrak{M}_{g_1,S_1\amalg\{r_1\}} \overset{t}{\underset{\amalg_r \mathcal{B}^2\mu_r}{\times}} \mathfrak{M}_{g_2,S_2\amalg\{r_2\}} := t_0\Big(\mathfrak{M}_{g_1,S_1\amalg\{r_1\}} \underset{\amalg_r \mathcal{B}^2\mu_r}{\times} \mathfrak{M}_{g_2,S_2\amalg\{r_2\}}\Big) \to \mathfrak{M}_{g_1+g_2,S_1\amalg S_2}. \tag{4.29}$$

Proof. The fibre product is identified with an explicit stack describing a stratum of the boundary, whose value at a stack T is the category of objects the diagrams

$$\begin{array}{ccccc} \Gamma_1 & & \xrightarrow{\alpha} & & \Gamma_2 \\ \downarrow & & & & \downarrow \\ C_1 & & & & C_2 \\ & \searrow & & \swarrow & \\ & & T, & & \end{array} \tag{4.30}$$

where $C_1 \to T$ and $C_2 \to T$ are prestable stacky curves over T with markings respectively labelled by $S_1 \amalg \{\Gamma_1\}$ and $S_2 \amalg \{\Gamma_2\}$, and α is an isomorphism of gerbes inverting the band. The gluing morphism is then constructed by applying the gluing procedure of [AGV08, Proposition A.1.1] (see also [Lur19, Theorem 16.1.0.1]) for the closed immersions $\Gamma_1 \hookrightarrow C_1$ and $\Gamma_1 \xrightarrow[\alpha]{\simeq} \Gamma_2 \hookrightarrow C_2$. □

Corollary 4.2.1.2.6. *There is a modular ∞-operad $\mathfrak{M}$ of stacky curves, which is hapaxunital.*

Proof. Composition is given by the gluing morphisms of lemma 4.2.1.2.5. Since the stacks constituting $\mathfrak{M}$ are smooth by proposition 4.1.1.2.4, they are in particular flat and their truncated fibre products are also their fibre products as derived stacks, so exhibiting the operad structure in the $(\infty, 1)$-category of classical (higher) stacks suffices to deduce it in the $(\infty, 1)$-category of derived stacks.

This shows that we have a Segal $\Upsilon^{\mathrm{op}\natural}$-derived stack; to be able to interpret it properly as an ∞-operad, we also need to verify univalence. It follows from the definition of the spaces of colours and unary morphisms, and the fact that the underlying $(\infty, 1)$-category is an ∞-groupoid.

Finally, for the hapaxunital structure, we simply observe that the only unital colour is 1, the unique point of $\mathcal{B}^2 \mu_1 \simeq *$. □

By restriction along the Segal functor $\Omega^{\mathrm{op}\natural} \to \Upsilon^{\mathrm{op}\natural}$ (*i.e.* by restricting to the genus 0 part and forgetting the cyclic structure), there is also an ∞-operad $\mathfrak{M}_0$ of genus 0 stacky curves.

We also consider a variation adapted to a target for quasi-stable maps. Fix $(X, \mathcal{L})$ a polarised 1-Artin stack, and let $\mathrm{Eff}(X, \mathcal{L})$ denote the monoid of $\mathcal{L}$-effective classes in X.

Proposition 4.2.1.2.7. *There is a graded modular ∞-operad $\mathfrak{M}^X$ of $\mathrm{Eff}(X, \mathcal{L})$-decorated stacky curves.*

Proof. Given the previous discussion, this amounts to proving that the operadic composition is compatible with the grading. But this is clear, as composition is given by gluing, which corresponds to adding irreducible components, and the grading decomposition is additive with respect to components. □

4.2.2 The quasimap Geometric Field Theory

We can now apply the general operadic framework of subsection 2.3.2 to quasimap theory, as was done in [MR18] for the schematic (and thus, monochromatic) case. The operad $\mathfrak{M}_0^{X,\mathrm{sch}}$ considered in this work had three salient points:

- it is a monochromatic operad, which furthermore is unital (and in fact reduced),
- the maps $\mathfrak{M}_0^{X,\mathrm{sch}}(\star_{n+1}) \to \mathfrak{M}_0^{X,\mathrm{sch}}(\star_n)$ forgetting one input are given by the universal curves,

- the object $\mathfrak{M}_0^{X,\mathrm{sch}}(\star_3)$ of extensions of the identity (of the unique colour) is terminal, so that the mapping stack $\mathcal{M}or(\mathfrak{M}_0^{X,\mathrm{sch}}(\star_3), X)$ is just the moduli stack of points in X, which is X itself, ensuring that the eventual brane action takes place on X.

In our orbifold setting, the operad in play has some differences:

- it has several colours (so is in particular not reduced), only one of which is unital,
- due to the non-unitality, the object of extensions of the identity of a colour is not the whole of $\mathfrak{M}_0^X(\star_3)$ but only the part whose last colour is the unital one. This still ensures that extensions recover the unversal curve, by proposition 4.1.1.2.7.
- It is known from [AV02] that the stacky evaluation maps do not take values in X itself but its cyclotomic inertia stack; here the structure of $\mathfrak{M}_0^X$ will not produce a point but a collection of gerbes giving naturally rise to a derived thickening of the inertia stack.

4.2.2.1 The stable Gromov–Witten action

Applying corollary 2.3.2.0.4 to the $\mathrm{Eff}(X,\mathcal{L})$-graded ∞-operad $\mathfrak{M}_0^X$ produces a lax $\mathfrak{M}_0^X$-algebra in $\mathcal{S}pan(\mathfrak{dSt}_{/\_})^\times$ with carriers the mapping stacks $\mathcal{M}or(\mathrm{Ext}(\mathrm{id}_r), X)$. The original action $\mathfrak{M}_0^X \to \mathcal{C}ospan(\mathfrak{dSt}_{/\_})^{\amalg}$ is indeed lax, as the operad $\mathfrak{M}_0^X$ is not coherent: the gluing of curves is a colimit in the $(\infty,1)$-category of algebraic derived stacks, but not in the $(\infty,1)$-topos of all derived stacks as is needed for coherence. However it can be seen that the induced action becomes strong thanks to mapping into X, at least in the case where X is perfect in the sense of [BFN10, Definition 3.2]:

Proposition 4.2.2.1.1 ([MR18, Discussion before Corollary 3.1.8]). *Suppose* X *is a perfect derived stack. After applying* $\mathcal{M}or(-, X)$, *the brane action becomes a strong morphism.*

Proof. Let C_σ and C_τ be two curves with a gerbe of common order r. We have the comparison map $\theta\colon C_\sigma \amalg_{\mathcal{B}^2\mu_r} C_\tau \to C_{\tau\circ\sigma}$. By the universal property of the amalgamated sum in derived stacks, we have

$$\mathcal{M}or\left(C_\sigma \coprod_{\mathcal{B}\,\mu_r} C_\tau, X\right) \simeq \mathcal{M}or(C_\sigma, X) \underset{\mathcal{M}or(\mathcal{B}\,\mu_r, X)}{\times} \mathcal{M}or(C_\tau, X). \tag{4.31}$$

By the hypotheses on X and the properties of stacky curves, it ensues from theorem 3.2.1.1.4 that each of the mapping stacks appearing in equation (4.31) is algebraic. Thus, as the inclusion of algebraic derived stacks into derived stacks is fully faithful, the same equality also holds in the $(\infty,1)$-category of algebraic derived stacks. ☐

Construction 4.2.2.1.2 (Stable sub-action). Recall from remark 2.3.2.0.5 that the brane action on X is given by a discrete cocartesian fibration $\mathfrak{B}_{\mathfrak{dSt}}(\mathfrak{M}, X) \to \int \mathfrak{Env}^{\mathfrak{dSt}^{\mathrm{op}}}(\mathfrak{Tw}^{\mathfrak{dSt}^{\mathrm{op}}}(\mathcal{E}nv^{\mathfrak{dSt}^{\mathrm{op}}}(\mathfrak{M}))) \underset{\mathfrak{T}}{\times} \int \mathfrak{dSt}_{/\_}$ whose fibre at $(Z, \sigma, Y \to Z)$ is $\mathfrak{T}_{/Z}(\mathrm{Ext}(\sigma) \times_Z Y, X \times Z)$.

Fix a degree-stability condition as in example 3.2.1.1.8. We let $\mathfrak{B}_{\mathfrak{dSt}}^{\mathrm{stbl}}(\mathfrak{M}, X)$ denote the full subcategory on those families of maps which are representable and whose degree on each component of the source curve coincides with the degree specified by the grading.

Remark 4.2.2.1.3. Suppose, for ease of notation, that σ is an operation of a hapaxunital internal ∞-operad $\mathcal{M}$ lying over a corolla $\star_n$. The object of extensions $\mathrm{Ext}(\sigma)$ is, according to remark 2.2.1.2.10, the fibre product $Z \times_{\mathcal{M}(\star_n)} \mathcal{M}(\star_{n+1})_1$, where $\mathcal{M}(\star_{n+1})_1 = \mathcal{M}(\star_{n+1}) \times_{\mathcal{M}(\eta)} Z$ with the morphism

$\mathcal{M}(\star_{n+1}) \to \mathcal{M}(\star_n)$ the one using the hapaxunital structure to forget one input, and the morphism $Z \to \mathcal{M}(\eta)$ the one selecting the distinguished colour. In fact the whole diagram

$$\begin{array}{ccc} \mathrm{Ext}(\sigma) & \longrightarrow & X \times Z \\ & \searrow \quad \swarrow & \\ & Z & \end{array} \tag{4.32}$$

is the base change of

$$\begin{array}{ccc} \mathcal{M}(\star_{n+1})_1 & \longrightarrow & X \times \mathcal{M}(\star_n) \\ & \searrow \quad \swarrow & \\ & \mathcal{M}(\star_n) & \end{array} \tag{4.33}$$

along $\sigma\colon Z \to \mathcal{M}(\star_n)$.

From this we deduce that

$$\mathcal{M}or^{\mathrm{stbl}}_{/Z}(\mathrm{Ext}(\sigma), X \times Z) \simeq \mathcal{M}or^{\mathrm{stbl}}_{/\mathcal{M}(\star_n)}(\mathcal{M}(\star_{n+1})_1, X \times \mathcal{M}(\star_n)) \underset{\mathcal{M}(\star_n)}{\times} Z. \tag{4.34}$$

Example 4.2.2.1.4. In the case of the operad $\mathfrak{M}_0^X$ of degree-graded stacky curves, proposition 4.1.1.2.7 gives a geometric interpretation of extensions as obtained from the universal curve. In particular, the stack $\mathcal{M}or_{/\mathfrak{M}_0(\star_n)}(\mathfrak{M}_0(\star_{n+1})_1, X \times \mathfrak{M}_0(\star_n))$ is the derived moduli stack of maps containing $\mathbb{R}\mathcal{Q}^{\mathcal{L}}_{0,n}(X, \beta)$ as a stable open substack.

Proposition 4.2.2.1.5 ([MR18]). *The subfibration* $\mathfrak{B}^{\mathrm{stbl}}_{\mathfrak{dSt}}(\mathfrak{M}, X) \to \int \mathfrak{Env}^{\mathfrak{dSt}^{\mathrm{op}}}\left(\mathfrak{Tw}^{\mathfrak{dSt}^{\mathrm{op}}}\left(\mathcal{E}nv^{\mathfrak{dSt}^{\mathrm{op}}}(\mathfrak{M})\right)\right) \underset{\mathfrak{T}}{\times} \int \mathfrak{dSt}_{/-}$ *defines a morphism of* ∞*-operads* $\mathfrak{M}_0 \to \mathcal{S}pan(\mathfrak{dSt}_{/-})^{\times}$.

Proof. As in the proof of [MR18, Proposition 3.2.1]. □

4.2.2.2 Relative brane actions

Construction 4.2.2.2.1 (Internal (co)limits). By proposition 2.3.1.1.19, in an $(\infty,1)$-topos $\mathfrak{T}$, the assignment $Z \mapsto \mathfrak{T}_{/Z}$ defines a sheaf of $(\infty,1)$-categories on $\mathfrak{T}$, which by lemma 2.3.1.2.1 is equivalently an internal category in $\mathfrak{T}$. Let $\mathfrak{CAT}(\mathfrak{T})$ denote the $(\infty,2)$-category of internal categories of $\mathfrak{T}$. It is proved in [Mar21, Theorem 4.5.2] that, for every internal category $\mathcal{C}$ in $\mathfrak{T}$, there is an equivalence between the $(\infty,1)$-category of internal discrete opfibrations over $\mathcal{C}$ (*i.e.* discrete opfibrations over $\mathcal{C}$ in $\mathfrak{CAT}(\mathfrak{T})$) and the hom $(\infty,1)$-category $\mathfrak{CAT}(\mathfrak{T})(\mathcal{C}, \mathcal{U})$.

This shows that we can interpret morphisms $\mathcal{C} \to \mathcal{U}$ as **internal diagrams** indexed by $\mathcal{C}$. For our purposes, we shall only need the case where $\mathcal{C}$ is an internal groupoid; in this case matters simplify further as any morphism $\mathcal{C} \to \mathcal{U}$ factors uniquely through a morphism $\mathcal{C} \to \iota_0\mathcal{U}$. Since $\iota_0\mathcal{U}$ has a representing object $U \in \mathfrak{T}$, such a morphism corresponds to a map $C := \mathcal{B}\,\mathcal{C} \to U$, where we now also view $\mathcal{C}$ as a "discrete" internal groupoid on the object C.

Now let $C \to C'$ be a morphism in $\mathfrak{T}$, inducing the ∞-functor $\mathfrak{T}(C', U) \to \mathfrak{T}(C, U)$ between the ∞-groupoids of internal diagrams. A left-adjoint to this ∞-functor, which exists as a dependent sum by remark 2.3.1.1.13, is called an ∞-functor of **internal oplax extension** from C to C', while a right-adjoint, existing as a dependent product by remark 2.3.1.1.15, is called an ∞-functor of **internal lax extension**.

If $C' = *$ is the terminal object so that $C \to *$ is the unique morphism, an internal oplax extension of a C-indexed diagram is its **internal colimit**, while an internal lax extension is its **internal limit**.

Example 4.2.2.2.2 (Group actions). Let G be a group object in $\mathcal{T}$. An internal diagram $\mathcal{B}\,G \to \iota_0\mathcal{U}$ corresponds to a morphism $T \to \mathcal{B}\,G$, and thus an action of G on $X = T \times_{\mathcal{B}\,G} *$. The internal colimit of this internal diagram is the quotient $T = X/G$; this is a clear consequence of subsection 3.1.1.2. In addition, the internal limit is the object of G-invariants of the action.

Lemma 4.2.2.2.3. *Let $\mathcal{O}$ be a* B*-graded ∞-operad in an $(\infty,1)$-topos $\mathcal{T}$. Let $\mathcal{O} \to \mathcal{P}$ be a morphism to an ungraded monochromatic ∞-operad $\mathcal{P}$ in $\mathcal{T}$. Any morphism $b\colon \mathcal{O} \to \mathcal{S}pan(\mathcal{T}_{/-})^{\times}$ induces a lax $\mathcal{P}$-algebra structure on $\varinjlim_{\mathcal{O}(\eta)} b_\eta$ (where b_η is the component $\mathcal{O}(\eta) \to \mathcal{S}pan(\mathcal{T}_{/-})^{\times}(\eta)$ of the transformation* b*).*

Proof. We consider the Grothendieck constructions of the $(\infty,2)$-operads in play.

$$\begin{array}{ccc} \int \mathcal{O} & \longrightarrow & \int \mathcal{S}pan(\mathcal{T}_{/-})^{\times} \\ \downarrow & & \downarrow \\ \int \mathcal{P} & \longrightarrow & \mathcal{T} \times \Omega^{op} \end{array} \tag{4.35}$$

Following [MR18, Theorem 3.3.1], we can construct (the Grothendieck construction of) the desired morphism $\mathscr{P} \to \mathcal{S}pan(\mathcal{T}_{/-})^{\times}$ as a relative oplax extension.

For each tree T, we have an identification $\int \mathcal{O} \times_{\Omega^{op}} \{T\} \simeq \mathcal{T}_{/\mathcal{O}(T)}$. In particular, the map $\int \mathcal{O} \times_{\Omega^{op}} \{T\} \simeq \mathcal{T}_{/\mathcal{O}(T)} \to \mathcal{T}_{/\mathcal{P}(T)} \simeq \int \mathcal{P} \times_{\Omega^{op}} \{T\}$ is given by composition with $\mathcal{O}(T) \to \mathcal{P}(T)$. It thus admits a right-adjoint given by pullback, and these assemble into a right-adjoint to $\int \mathcal{O} \to \int \mathcal{P}$ relative to Ω^{op}.

Hence, for every σ in $\int \mathcal{P}$ lying over $Z \in \mathcal{T}$ and $T \in \Omega^{\natural,el}$, there is an equivalence of comma $(\infty,1)$-categories $\int \mathcal{O} \downarrow \{\sigma\} \simeq \mathcal{T}_{/Z \times_{\mathcal{P}(T)} \mathcal{O}(T)}$, so the formula for the relative oplax extension evaluated at σ becomes

$$\varinjlim_{Z' \to Z \times_{\mathcal{P}(T)} \coprod_\beta \mathcal{O}_\beta(T)} \mathrm{pr}(b(\tilde{\sigma})) \tag{4.36}$$

where pr denotes the functor of pushing down the base along $Z' \to Z$ and $\tilde{\sigma}$ denotes the operation of $\mathcal{O}$ classified by $Z' \to \coprod_\beta \mathcal{O}_\beta(T)$ as seen in the commuting square

$$\begin{array}{ccc} Z' & \xrightarrow{\ulcorner \tilde{\sigma} \urcorner} & \int \mathcal{O} \\ \downarrow & & \downarrow \\ Z & \xrightarrow[\ulcorner \sigma \urcorner]{} & \int \mathcal{P}. \end{array} \tag{4.37}$$

Since the projections to the $(\infty,1)$-topos $\mathcal{T}$ correspond to its self-indexing, we can interpret the relative oplax extension over $\mathcal{T} \times \Omega^{op}$ as an internal relative oplax extension over Ω^{op} in $\mathcal{T}$. In particular, evaluating at the edge η, since $\mathcal{P}$ is monochromatic meaning that $\mathcal{P}(\eta) = *$, we obtain the internal colimit

$$\mathrm{Opex}_{\mathcal{O}(\eta) \to *} b_\eta \simeq \varinjlim_{\mathcal{O}(\eta)} b \tag{4.38}$$

indexed by $\mathcal{O}(\eta)$. □

Example 4.2.2.2.4 ([MR18, Theorem 3.3.1]). For a stable subaction of a brane action, where

$b(\sigma\times_Z Z') = \mathcal{M}or^{\mathrm{stbl}}_{/Z}(\mathrm{Ext}(\sigma)\times_Z Z', X\times Z)$, we can expand on the expression of equation (4.36). Indeed, since the stability condition in construction 4.2.2.1.2 requires compatibility with the grading, the colimit can be taken over the ungraded operad (so we will write $\mathcal{O}(T) = \coprod_\beta \mathcal{O}_\beta(T)$) and, using remark 4.2.2.1.3, we compute over a corolla $\star_n$:

$$\begin{aligned}
&\varinjlim_{Z'\to Z \underset{\mathcal{P}(\star_n)}{\times} \coprod_\beta \mathcal{O}_\beta(\star_n)} \mathrm{pr}(\mathcal{M}or^{\mathrm{stbl}}_{/Z'}(\mathrm{Ext}(\tilde\sigma), X\times Z'))\\
=&\,\mathrm{pr} \varinjlim_{Z'\to Z \underset{\mathcal{P}(\star_n)}{\times} \coprod_\beta \mathcal{O}_\beta(\star_n)} \mathcal{M}or^{\mathrm{stbl}}_{/Z}(\mathrm{Ext}(\tilde\sigma), X\times Z')\\
=&\,\mathrm{pr} \varinjlim_{Z'\to Z \underset{\mathcal{P}(\star_n)}{\times} \coprod_\beta \mathcal{O}_\beta(\star_n)} \mathcal{M}or^{\mathrm{stbl}}_{/\mathcal{O}(\star_n)}\big(\mathcal{O}(\star_{n+1}), X\times \mathcal{O}(\star_n)\big) \underset{\mathcal{O}(\star_n)}{\times} Z'\\
=&\,\mathrm{pr} \varinjlim_{Z'\to Z \underset{\mathcal{P}(\star_n)}{\times} \mathcal{O}(\star_n)} \mathcal{M}or^{\mathrm{stbl}}_{/\mathcal{O}(\star_n)}\big(\mathcal{O}(\star_{n+1}), X\times \mathcal{O}(\star_n)\big) \underset{\mathcal{O}(\star_n)}{\times} \Big(\mathcal{O}(\star_n) \underset{\mathcal{P}(\star_n)}{\times} Z\Big) \underset{\mathcal{O}(\star_n) \underset{\mathcal{P}(\star_n)}{\times} Z}{\times} Z'
\end{aligned} \tag{4.39}$$

where pr still denotes the functor of pushing down the base along the morphism to Z.

We can then commute the colimit with the constant terms to obtain

$$\begin{aligned}
&\mathrm{pr}\,\mathcal{M}or^{\mathrm{stbl}}_{/\mathcal{O}(\star_n)}\big(\mathcal{O}(\star_{n+1}), X\times \mathcal{O}(\star_n)\big) \underset{\mathcal{O}(\star_n)}{\times} \mathcal{O}(\star_n) \underset{\mathcal{P}(\star_n)}{\times} Z \underset{\mathcal{O}(\star_n)\times_{\mathcal{P}(\star_n)} Z}{\times} \varinjlim_{Z'\to Z \underset{\mathcal{P}(\star_n)}{\times} \mathcal{O}(\star_n)} Z'\\
=&\,\mathrm{pr}\,\mathcal{M}or^{\mathrm{stbl}}_{/\mathcal{O}(\star_n)}\big(\mathcal{O}(\star_{n+1}), X\times \mathcal{O}(\star_n)\big) \underset{\mathcal{O}(\star_n)}{\times} \mathcal{O}(\star_n) \underset{\mathcal{P}(\star_n)}{\times} Z \underset{\mathcal{O}(\star_n)\times_{\mathcal{P}(\star_n)} Z}{\times} \mathcal{O}(\star_n) \underset{\mathcal{P}(\star_n)}{\times} Z\\
=&\,\mathrm{pr}\,\mathcal{M}or^{\mathrm{stbl}}_{/\mathcal{O}(\star_n)}\big(\mathcal{O}(\star_{n+1}), X\times \mathcal{O}(\star_n)\big) \underset{\mathcal{O}(\star_n)}{\times} \mathcal{O}(\star_n) \underset{\mathcal{P}(\star_n)}{\times} Z\\
=&\,\mathrm{pr}\,\mathcal{M}or^{\mathrm{stbl}}_{/\mathcal{O}(\star_n)}\big(\mathcal{O}(\star_{n+1}), X\times \mathcal{O}(\star_n)\big) \underset{\mathcal{P}(\star_n)}{\times} Z
\end{aligned} \tag{4.40}$$

Definition 4.2.2.2.5 (Cyclotomic loop stack)**.** Let X be a 1-Deligne–Mumford derived stack. Its **cyclotomic loop stack** is

$$\mathscr{L}_\mu X \coloneqq \coprod_{n\geqslant 1} \mathcal{M}or^{\mathrm{rep}}(\mathcal{B}\,\mu_n, X). \tag{4.41}$$

Its rigidified cyclotomic loop stack is

$$\overline{\mathscr{L}}_\mu X \coloneqq \coprod_{n\geqslant 1} \mathcal{M}or^{\mathrm{rep}}(\mathcal{B}\,\mu_n, X)/\mathcal{B}\,\mu_n. \tag{4.42}$$

Remark 4.2.2.2.6. The truncation of $\mathscr{L}_\mu X$ (respectively $\overline{\mathscr{L}}_\mu X$) is the cyclotomic inertia stack $\mathscr{I}_\mu(t_0X)$ (respectively rigidified cyclotomic inertia stack $\overline{\mathscr{I}}_\mu(t_0X)$) of [AGV08]. This is in analogy with the fact that the truncation of the loop stack $\mathscr{L}X = X\times_{X\times X} X$ is the inertia stack $\mathscr{I}(t_0X) = t_0X \overset{t}{\underset{t_0X\times t_0X}{\times}} t_0X$.

By applying our previous constructions to the stabilisation map $\mathfrak{M}_0 \to \overline{\mathscr{M}}_0$, we finally obtain our main result on the construction of the quasimap Geometric Field Theory.

Theorem 4.2.2.2.7. *Let* $(X, \mathcal{L})$ *be a rationnally polarised* 1*-Deligne–Mumford derived stack, and write* $X^{\mathcal{L}\text{-st}}$ *for its* $\mathcal{L}$*-stable locus. There is a lax* $\overline{\mathscr{M}}_0$*-algebra structure (in correspondences) on its rigidified cyclotomic loop stack, informally given by the spans*

$$\begin{array}{ccc} & \coprod_{\beta \in \mathrm{Eff}(X,\mathcal{L})} \mathbb{R}\mathcal{Q}^{\mathcal{L}}_{0,n+1}(X,\beta) & \\ \swarrow & & \searrow \\ \overline{\mathscr{M}}_{0,n+1} \times (\overline{\mathscr{L}}_\mu X^{\mathcal{L}\text{-st}})^n & & \overline{\mathscr{L}}_\mu X^{\mathcal{L}\text{-st}}. \end{array} \tag{4.43}$$

Proof. We apply equation (4.38) to the morphism Stab: $\mathfrak{M}_0^X \to \overline{\mathscr{M}}_0$, taking into account that

$$\mathrm{Ext}(\mathrm{id}_r) = \{\mathrm{id}_r\} \underset{\mathfrak{M}_{0,2,(r,r)}}{\times} \mathfrak{M}_{0,3,(r,1,r)} = \{\mathrm{id}_r\} \underset{\mathcal{B}^2\mu_r}{\times} * = \Omega\,\mathcal{B}^2\mu_r = \mathcal{B}\,\mu_r \tag{4.44}$$

to obtain that the unique colour of $\overline{\mathscr{M}}_0$ is mapped to $\overline{\mathscr{L}}_\mu X$: since $\mathfrak{M}_0(\eta) \simeq \coprod_r \mathcal{B}^2\mu_r$, the colimit over $\mathfrak{M}_0(\eta)$ computes term-wise $\mathcal{B}\,\mu_r$-coinvariants (by example 4.2.2.2.2) then takes the coproduct. In addition, the quasi-stability condition requires that the special points of curves, so the marked gerbes, be sent to the $\mathcal{L}$-stable locus, so that the evaluation maps can be astricted to $\overline{\mathscr{L}}_\mu X^{\mathcal{L}\text{-st}}$.

The fact that the spans are given by quasimap moduli stacks is a direct consequence of equation (4.39). □

Construction 4.2.2.2.8. As explained in [MR18, §4.1], the assignment a derived stack Z of the $(\infty,2)$-category $\mathfrak{QCoh}(Z)^{\otimes}$-$\mathfrak{Mod}(\mathfrak{St}_{\Bbbk})$ (of $\mathfrak{QCoh}(Z)^{\otimes}$-modules in the symmetric monoidal $(\infty,2)$-category of $\Bbbk$-linear stable $(\infty,1)$-categories) defines a categorical ∞-operad $\mathfrak{QMod}$ in $\mathfrak{dSt}$, and there is a morphism of such $\mathcal{S}pan(\mathfrak{dSt}_{/-})^{\times} \to \mathfrak{QMod}$, whose component at Z sends $Y \xrightarrow{\varpi} Z$ to $\mathfrak{QCoh}(Y)$ (with $\mathfrak{QCoh}(Z)$-module structure given by ϖ^*).

If Z is a *perfect* algebraic derived stack, the $(\infty,1)$-category $\mathfrak{QCoh}(Z)$ is compactly generated, and so the restriction of $\mathfrak{QMod}$ to perfect algebraic derived $\Bbbk$-stacks factors through the assignment $Z \mapsto \mathfrak{QCoh}(Z)^{\otimes}$-$\mathfrak{Mod}(\mathfrak{St}_{\Bbbk}^{\mathrm{cpct}})$ (where $\mathfrak{St}_{\Bbbk}^{\mathrm{cpct}}$ denotes compactly generated $\Bbbk$-linear stable $(\infty,1)$-categories), and extends to define a categorical ∞-operad $\mathfrak{QMod}^{\mathrm{cpct}}$ in $\mathfrak{dSt}$, with a morphism of such $\mathcal{S}pan(\mathfrak{dSt}^{\mathrm{perf}}_{/-})^{\times} \to \mathfrak{QMod}^{\mathrm{cpct}}$.

Corollary 4.2.2.2.9 ([MR18, Corollary 4.1.1, Proposition 4.1.2])**.** *There is a lax morphism* $\overline{\mathscr{M}}_0 \to \mathfrak{QMod}^{\mathrm{cpct}}$ *of categorical* ∞*-operads in* $\mathfrak{dSt}$ *whose components* $\mathfrak{QCoh}(\overline{\mathscr{L}}_\mu X^{\mathcal{L}\text{-st}})^{\otimes n} \otimes \mathfrak{QCoh}(\overline{\mathscr{M}}_{0,n+1}) \to \mathfrak{QCoh}(\overline{\mathscr{L}}_\mu X^{\mathcal{L}\text{-st}})$ *are given by pullback-pushforward along the correspondence equation* (4.43).

Remark 4.2.2.2.10 (Motivic variant). By [Kha16, Theorem 5.1.2], the assignment to Z of the stable motivic homotopy $(\infty,1)$-category $\mathfrak{SH}(Z)$-$\mathfrak{Mod}(\mathfrak{St}_{\Bbbk})$ defines a categorical ∞-operad $\mathfrak{SH}$ in $\mathfrak{dSt}$ with a morphism $\mathcal{S}pan(\mathfrak{dSt}_{/-})^{\times} \to \mathfrak{SH}$. We thus obtain in the same manner a lax morphism $\overline{\mathscr{M}}_0 \to \mathfrak{SH}$ whose components $\mathfrak{SH}(\overline{\mathscr{L}}_\mu X^{\mathcal{L}\text{-st}})^{\otimes n} \otimes \mathfrak{SH}(\overline{\mathscr{M}}_{0,n+1}) \to \mathfrak{SH}(\overline{\mathscr{L}}_\mu X^{\mathcal{L}\text{-st}})$ are given by the same pullback-pushforward procedure.

Lemma 4.2.2.2.11 ([MR18, Proposition 4.1.3])**.** *The categorical Gromov–Witten action is compatible with the sub-*$(\infty,1)$*-categories* $\mathfrak{Coh}^{\mathrm{b}}$ *and* $\mathfrak{Perf}$.

Remark 4.2.2.2.12 (Uncategorified quasimap invariants). By applying the K-theory functor to the categorical action on $\mathfrak{Coh}^b$, we obtain a $G(\overline{\mathscr{M}}_0)$-action on $G(\overline{\mathscr{L}}_\mu X^{\mathcal{L}\text{-st}})$. Recall (from the discussion after remark 3.1.3.1.1) that the G-theory of an Artin derived stack is isomorphic to that of its truncation; it ensues that the action can be seen as one on the G-theory of the rigidified cycltomic *inertia* stack of $t_0(X^{\mathcal{L}\text{-st}})$, and by lemma 4.2.1.1.10 coincides with (a G-theoretic lift of) the cohomological field theory constructed by [AGV08].

Since motivic homology theories are also insensitive to derived structures by [Kha19a, Corollary 3.2.9], the action at the level of motivic homotopy ∞-categories also gives rise to actions on the homology of $\overline{\mathscr{I}}_\mu(t_0(X^{\mathcal{L}\text{-st}}))$ with coefficient in any motivic spectrum.

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CONTENTS

Categorified quasimap theory of derived Deligne–Mumford stacks

Abstract

This thesis extends the results of [MR18] on the categorification of Gromov–Witten invariants to stack targets. This requires constructing a brane action for certain coloured ∞-operads, for which we develop a language for lax morphisms as well as a dendroidal version of monoidal envelopes. We finally obtain an action on a cyclotomic loop stack, given by moduli stacks of stable quasimaps. An application to the categorification of the quantum Lefschetz principle is also provided.

Keywords: derived geometry,operads,Gromov–Witten theory

Théorie de quasi-applications catégorifiée des champs de Deligne–Mumford dérivés

Résumé

Nous étendons les résultats de[MR18] sur la catégorification des invariants de Gromov–Witten aux cibles champêtres. Cela implique de construire une action de membranes pour certaines ∞-opérades colorées, ce pour quoi nous développons un langage pour les morphismes laxes ainsi qu'une version dendroïdale des enveloppes monoïdales. Nous obtenons finalement une action sur un champ de lacets cyclotomique, donnée par des champs de modules de quasi-applications. Nous décrivons également une application à la catégorification du principe de Lefschetz quantique.

Mots clés : géométrie dérivée,opérades,théorie de Gromov–Witten

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Titre : Théorie de quasi-applications catégorifiée des champs de Deligne–Mumford dérivés

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Résumé : Nous étendons les résultats de Mann–Robalo sur la catégorification des invariants de Gromov–Witten aux cibles champêtres. Cela implique de construire une action de membranes pour certaines ∞-opérades colorées, ce pour quoi nous développons un langage pour les morphismes laxes ainsi qu'une version dendroïdale des enveloppes monoïdales. Nous obtenons finalement une action sur un champ de lacets cyclotomique, donnée par des champs de modules de quasi-applications. Nous décrivons également une application à la catégorification du principe de Lefschetz quantique.

Title : Categorified quasimap theory of derived Deligne–Mumford stacks

Keywords : derived geometry, operads, Gromov–Witten theory

Abstract : This thesis extends the results of Mann–Robalo on the categorification of Gromov–Witten invariants to stack targets. This requires constructing a brane action for certain coloured ∞-operads, for which we develop a language for lax morphisms as well as a dendroidal version of monoidal envelopes. We finally obtain an action on a cyclotomic loop stack, given by moduli stacks of stable quasimaps. An application to the categorification of the quantum Lefschetz principle is also provided.